STRONGLY IRREDUCIBLE HEEGAARD SPLITTINGS OF HYPERBOLIC 3-MANIFOLDS

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Abstract. Colding and Gabai have given an effective version of Li's theorem that non-Haken hyperbolic 3-manifolds have finitely many irreducible Heegaard splittings. As a corollary of their work, we show that Haken hyperbolic 3-manifolds have a finite collection of strongly irreducible Heegaard surfaces S_i and incompressible surfaces K_j such that any strongly irreducible Heegaard surface is a Haken sum $S_i + \sum_j n_j K_j$, up to one-sided associates of the Heegaard surfaces.

Colding and Gabai[1] used negatively curved branched surfaces that carry all index ≤ 1 surfaces to show that non-Haken hyperbolic 3-manifolds have finitely many irreducible Heegaard surfaces. They end with a conjecture for Heegaard splittings of Haken manifolds (Question 7.10 of [1]). In this short note we show that in fact their proof essentially resolves this conjecture for strongly irreducible splittings of hyperbolic manifolds, stated below as a theorem:

Theorem 1. *Let M be a closed hyperbolic manifold. There exist finitely many surfaces $S_1, ..., S_n$ which are either strongly irreducible Heegaard surfaces or the one-sided associates of strongly irreducible Heegaard surfaces and finitely many incompressible surfaces $K_1, ..., K_m$ such that if S is any strongly irreducible Heegaard surface then for some S_i either $S = S_i + \sum_j n_j K_j$ or it has a one-sided associate S' and $S' = S_i + \sum_j n_j K_j$. Furthermore, if the surface $K = \sum_j n_j K_j$ is non-empty then it is incompressible.*

A restatement of the Claim in the proof of Theorem 6.3 of [1] is the following:

Theorem 2. *Let M be a closed hyperbolic manifold. There exists $K^* > 0$ and finitely many pairs of branched surfaces (E_i, L_i) with L_i a (possibly empty) sub-branch surface of E_i such that for each strongly irreducible Heegaard surface S of M there exists a pair (E, L) from this finite collection such that*

1. *S or an associated one-sided surface S' is fully carried by E.*
2. *$S = S_1 + S_2$ or $S' = S_1 + S_2$, where S_2 is fully carried by L and $S_1 = \sum n_i F_i$ where F_i are the fundamental surfaces carried by E and $0 \leq n_i < K^*$.*
3. *If $L \neq \emptyset$ then L is incompressible, carries no torus, has no complementary monogons and has no disks of contact.*

Proof. Take the finite collection (E_i, L_i) of branch surfaces as in the Claim in the proof of Theorem 6.3 of [1]. As L_i are sub-branch surfaces of splittings of a $-1/2$-negatively curved branch surface so they carry no tori. As no component of $\partial_h N(L_i)$ is a disk or annulus so if a complementary component of $N(L_i)$ has a monogon,

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it has a compression disk (Lemma 5.5 of [1]). If any E carries infinitely many strongly irreducible Heegaard surfaces H_n and the corresponding L is compressible then the rest of the proof of Theorem 6.3 of [1] shows that some H_n is in fact weakly reducible. □

In the following lemma, we use the notation $\mathbb{N}$ to denote non-negative integers.

Lemma 3. *Given a sequence $\{H_n\}$ of $\mathbb{N}^k$, there exists a finite subset $\{S_i\}$ of $\{H_n\}$ such that for each $n \in \mathbb{N}$ there exists some S_i with $H_n - S_i \geq 0$.*

Proof. We proceed by induction on k. When $k = 1$, take S as the minimum of H_n so that $H_n - S \geq 0$. Assuming the statement for $k-1$, we ignore the k-th coordinate of the sequence $\{H_n\}$, so that there exist finitely many S_i in the sequence such that for any H_n there exists S_i such that $H_n(j) \geq S_i(j)$ for $j = 1...(k-1)$ where $H_n(j)$ and $S_i(j)$ denote the j-th component of H_n and S_i respectively. Let M be the maximum of the finite set $\{S_i(k)\}$. For each $m < M$, consider the subsequence $\{H_n^{(m)}\}$ of $\{H_n\}$ where $H_n^{(m)}(k) = m$. Again ignoring the last coordinate we get finitely many $S_i^{(m)}$'s in the sequence such that for all $n \in \mathbb{N}$ there exists some $S_i^{(m)}$ such that $H_n^{(m)}(j) \geq S_i^{(m)}(j)$ for $j = 1...(k-1)$ and $H_n^{(m)}(k) = S_i^{(m)}(k) = m$.

We claim that the union of the sets $\{S_i\}$, $\{S_i^{(1)}\}$, ..., $\{S_i^{(M-1)}\}$ is the required finite collection for k. For any H_n, ignoring the k-th coordinate there exist S_i such that $H_n(j) \geq S_i(j)$ for $j = 1...(k-1)$. If $H_n(k) \geq S_i(k)$ then $H_n - S_i \geq 0$ as required. If $H_n(k) = m < S_i(k) \leq M$ then H_n is a point in the subsequence $\{H_n^{(m)}\}$ and hence there exists $S_i^{(m)}$ such that $H_n - S_i^{(m)} \geq 0$ as required. □

We can now prove the main theorem of this article:

Proof of Theorem 1. For each pair (E, L) of branched surfaces in Theorem 2, if $S = S_1 + S_2$ is a strongly irreducible Heegaard surface carried by E then there are only finitely many possibilities for S_1. Let $\{H^n\}$ be a sequence of all strongly irreducible Heegaard surfaces carried by E with $H_1^n = S_1$. Applying Lemma 3 to the coefficients of H^n with respect the fundamental surfaces of E, there are finitely many $\{S^i\}$ in this sequence such that for any H^n there exists some S^i with $H^n - S^i = K$ a closed surface carried by L. By Theorem 2 of [4], K is incompressible. Let K_j be the collection of fundamental surfaces of the finitely many sub-branched surfaces L. Expressing K in terms of these K_j we get the required result. □

Remark 4. As the bound K^* and the branched surfaces (E, L) used in Theorem 2 are all effectively constructible so the finitely many strongly irreducible surfaces S_i and incompressible surfaces K_j in Theorem 1 are also effectively constructible.

In all known examples of manifolds which have infinitely many irreducible Heegaard surfaces, the Heegaard surfaces are of the form $S + nK$ [2, 3]. As a corollary of Theorem 1, it is easy to observe a weak version of this phenomenon.

Corollary 5. *Let M be a closed hyperbolic manifold. If M has infinitely many strongly irreducible Heegaard surfaces then it has infinitely many strongly irreducible Heegaard surfaces or their one-sided associates which are the Haken sum $S + K^i$ where S is a strongly irreducible Heegaard surface or its one-sided associate and K^i are incompressible surfaces all fully carried by a fixed branched surface.*

Proof. Given infinitely many strongly irreducible Heegaard surfaces or their one-sided associates carried by a fixed branched surface E, by ignoring the fundamental surfaces of L with zero coefficients we can pass to a subsequence $S^i = S + \sum_j n^i_j K_j$ where for each j, n^i_j is a sequence of positive numbers. Let L' be the smallest sub-branched surface of E that carries all these K_j. Then each $K^i = \sum_j n^i_j K_j$ is an incompressible surface fully carried by L'. □

With respect to a triangulation, strongly irreducible splitting surfaces can be put in almost normal form and with respect to a hyperbolic metric, they can be isotoped to either index ≤ 1 minimal surfaces or to the double cover of an index-0 surface with an unknotted tube connecting the sheets. There is no such nice form for weakly reducible splitting surfaces, so Question 7.10 of [1] is still open in its full generality.

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