Differential Geometry of Curves and Surfaces

Project by Vishwajeet Bhoite

Supervised by Dr. Tejas Kalelkar

Fall-2017

IISER PUNE

IISER Pune
Date: 20th August 2017

Contents

Introduction This is a report of semester long project on Differential Geometry of Curves and Surfaces. This report closely follows the book by Manferdo P, Docarmo on Differential Geometry of Curves and Surfaces. All the figures, examples and exercises in this report are from the same book. This report covers first four chapters of the book. All the theorems studied during the project are mentioned in this report.

Chapter 1

Curves

1.1 Parametrized Curves

Definition 1.1.1. A **parametrized differentiable curve** is a differential map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line R to $\mathbb{R}^3$.

Throughout this report we assume that a real function of real variable is differentiable (or smooth) if it has, at all points, derivatives of all orders.

Definition 1.1.2. If $\alpha(t) = (x(t), y(t), z(t))$ $t \epsilon I$ then the vector $\alpha'(t) = (x'(t), y'(t), z'(t))$ ϵ $\mathbb{R}^3$ is called the **tangent vector** (or velocity vector) of the curve α at t. The image $\alpha(I) \subset \mathbb{R}^3$ is called the **trace of** α.

Example 1.1. $\alpha(t) = (a\cos(t), a\sin(t), bt)$ $t \epsilon \mathbb{R}$ *has a trace in* $\mathbb{R}^3$ *a helix of pitch* $2\pi b$ *on the cylinder* $x^2 + y^2 = a^2$.

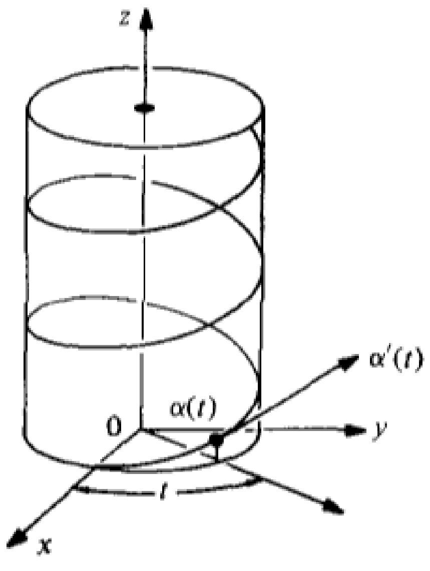


Figure 1.1: Helix

1.2 Regular Curves

Definition 1.2.1. A parametrized differential curve $\alpha : I \to \mathbb{R}^3$ is said to be regular if $\alpha'(t) \neq 0$ for all $t \epsilon I$.

Definition 1.2.2. The **arc length** of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ from a point t_0 is

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt.$$

A curve is said to be parametrized by arc length if $|\alpha'(t)| = 1$. Given a curve α parameterized by arc length $s \in (a, b)$, we may consider the curve β defined in $(-b, -a)$ by $\beta(-s) = \alpha(s)$ which has the same trace as the first one but is described in the opposite direction. The two curves are said to differ by change of orientation.

1.3 Orientation of a vector space

Orientation is a geometric notion that in two dimensions allows one to say when a cycle goes around clockwise or counterclockwise, and in three dimensions when a figure is left-handed or right-handed. The orientation on a real vector space is the arbitrary choice of which ordered bases are "positively" oriented and which are "negatively" oriented.
Two ordered bases $e = \{e_i\}$ and $f = \{f_i\}$, $i = 1, ..., n$, of an n-dimensional vector space V have the same orientation if the matrix of change of basis has a positive determinant. We denote this relation by $e \sim f$. This relation is an equivalence relation. The set of all ordered bases of V is thus decomposed into 2 equivalence clases since the determinant of change of basis is either positive or negative. Each of the equivalence classes determined is called an orientation of V. If $V = \mathbb{R}^3$ we call the orientation corresponding to the ordered basis $\mathbb{B} = \{e_1, e_2, e_3\}$ the positive orientation of $\mathbb{R}^3$, the other one being the negative orientation.
The vector product of two vectors $u, v \in \mathbb{R}^3$ is denoted by $u \wedge v$ and their product is denoted by $u \cdot v$. We have following identities

1. $\frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$.
2. $\frac{d}{dt}(u(t) \wedge v(t)) = u'(t) \wedge v(t) + u(t) \wedge v'(t)$.

1.4 The Local Theory of Curves Parametrized by Arc Length

Definition 1.4.1. Let $\alpha : I \to \mathbb{R}^3$ be a curve parameterized by arc length $s \in I$. The number $|\alpha''(s)| = k(s)$ is called the curvature of α at s.

Since the tangent vector has a unit length the curvature measures the rate of change of the angle which neighbouring tangents make with the tangents at s. Curvature therefore gives the measure of how rapidly the curve pulls away from the tangent line at s, is a neighbourhood of s.(see Fig 1.2). Also, note that the curvature remains invariant under a change of orientation.

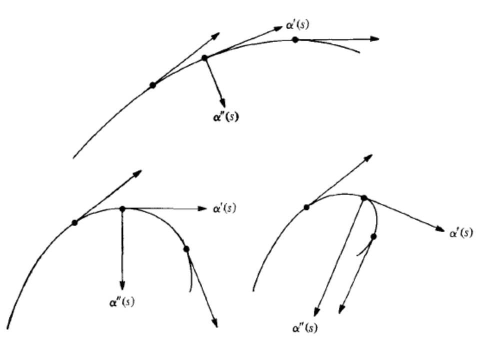


Figure 1.2: Curvature

Definition 1.4.2. At points where $k(s) \neq 0$, a unit vector $n(s)$ in the direction $\alpha''(s)$ is well defined by the equation $\alpha''(s) = k(s)n(s)$. $\alpha''(s)$ is normal to $\alpha'(s)$, because by differentiating $\alpha'(s) \cdot \alpha'(s) = 1$

we obtain $\alpha''(s) \cdot \alpha'(s) = 0$. Thus $n(s)$ is normal to $\alpha'(s)$ and is called the **normal vector** at s. The plane determined by $\alpha'(s)$ and $n(s)$, is called the **osculating plane** at s.

At points where $k(s) = 0$, the osculating plane is not defined. The points $s \epsilon I$ is a singular point of order 0 if $\alpha'(s) = 0$ and the points where $\alpha''(s) = 0$ is called singular point of order 1. We shall restrict ourselves to curves parameterized by arc length without singular points of order 1.

Notation: We shall denote $t(s) = \alpha'(s)$ the unit tangent vector of α at s. Thus $t'(s) = k(s)n(s)$.

Definition 1.4.3. The unit vector $b(s) = t(s) \wedge n(s)$ is normal to the osculating plane and is called **binormal vector** at s.

Since $b(s)$ is a unit vector, the length $|b'(s)|$ measures the rate of change of the neighbouring osculating planes with the osculating plane at s; that is $|b'(s)|$ measures how rapidly the cuve pulls away from the osculating plane at s, in a neighbourhood of s.

To compute $b'(s)$ we observe that, on the one hand, $b'(s)$ is normal to $b(s)$ and that, on the other hand,

$$b'(s) = t'(s) \wedge n(s) + t(s) \wedge n'(s) = t(s) \wedge n'(s);$$

that is, $b'(s)$ is parallel to t(s). It follows that $b'(s)$ is normal to $n(s)$, and we write

$$b'(s) = \tau(s)n(s)$$

for some function $\tau(s)$.

Also note that since $n = b \wedge t$, we have

$$n'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s) = -\tau b - kt.$$

Definition 1.4.4. Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length s such that $\alpha''(s) \neq 0$, $s \epsilon I$. The number $\tau(s)$ defined $b'(s) = \tau(s)n(s)$ is called the **torsion** of α at s.

Definition 1.4.5. To each value of parameter s, we have associated three orthogonal vectors $t(s), n(s), b(s)$. The trihedron thus formed is referred to as the **Fernet trihedron** at s.

We have the following equations called the $Frenet\ formulas$

1. $t' = kn$
2. $n' = -kt - \tau b$
3. $b' = \tau n.$

Definition 1.4.6. The tb plane is called is called the **rectifying plane**, and the nb plane the **normal plane** the lines which contain $n(s)$ and $b(s)$ and pass through $\alpha(s)$ are called the **principal normal** and the **binormal**, respectively. The inverse $R = 1/k$ of the curvature is called the radius of curvature at s.

Theorem 1.4.1. ***Fundamental Theorem of Local theory of Curves*** *Given differentiable functions $k(s) > 0$ and $\tau(s)$, $s \epsilon I$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature and $\tau(s)$ is the torsion of α. Moreover any other curve $\bar{\alpha}$, satisfying the same conditions, differs from α by a rigid motion; that is there exists a orthognal linear map ρ of $\mathbb{R}^3$, with positive determinant, and a vector c such that $\bar{\alpha} = \rho \circ \alpha + c$.*

We were able to only see the uniqueness part of the proof.

1.5 Global Properties of Plane Curves

Definition 1.5.1. A **differentiable function** on a closed interval $[a, b]$ is the restriction of a differentiable function defined on a open interval containing $[a, b]$.

Definition 1.5.2. A **closed plane curve** is a regular parametrized curve $\alpha : [a, b] \to \mathbb{R}^2$ such that α and all its derivatives agree on a and b. The curve is said to be **simple** if it has no further intersections.

The Isoperimetric Inequality

Remark. The isoperimetric problem is to determine a plane figure of the largest possible area whose boundary has a specified length. The closely related Dido's problem asks for a region of the maximal area bounded by a straight line and a curvilinear arc whose endpoints belong to that line. It is named after Dido, the legendary founder and first queen of Carthage. The solution to the isoperimetric problem is given by a circle and was known already in Ancient Greece. However, the first mathematically rigorous proof of this fact was obtained only in the 19th century. Since then, many other proofs have been found. The proof studied during this project was due to E. Schmidt(1939).

Theorem 1.5.1. *Let C be a simple closed curve with length l, and let A be the area of the region bounded by C. Then*

$$l^2 - 4\pi A \geq 0,$$

and the equality holds if and only if C is a circle.

Definition 1.5.3. A regular plane curve $\alpha : [a, b] \to \mathbb{R}^2$ is convex if, for all $t \in [a, b]$, the trace $\alpha([a, b])$ of α lies entirely on one side of the tangent line at t.

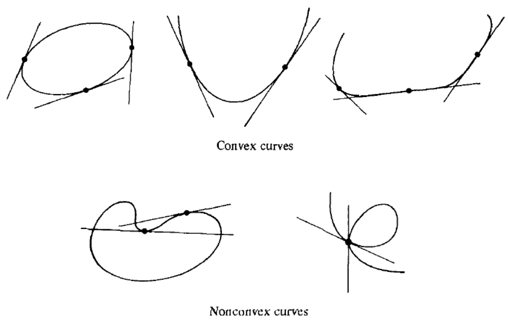


Figure 1.3: Convex and Non-Convex curves.

Definition 1.5.4. A **vertex** of a regular plane curve $\alpha : [a, b] \to \mathbb{R}^2$ is a point $t \in [a, b]$ where $k'(t) = 0$

The Four Vertex Theorem.

Remark. The four-vertex theorem states that the curvature function of a simple, closed, smooth plane curve has at least four local extrema (specifically, at least two local maxima and at least two local minima). The name of the theorem derives from the convention of calling an extreme point of the curvature function a vertex. This theorem has many generalizations, including a version for space curves where a vertex is defined as a point of vanishing torsion.

Theorem 1.5.2. *A simple closed convex curve has at least four vertices.*

Chapter 2

Regular Surfaces

In this chapter we first define regular surface in $\mathbb{R}^3$ and see some criteria to decide whether a given subset of $\mathbb{R}^3$ is regular surface. Then we define the notion of differentiability of a map whose domain is a Regular surface and we show that the usual notion of differential in $\mathbb{R}^2$ can be extended to such a function. Later we introduce the first fundamental form, to treat metric questions on a regular surface.

2.1 Regular Surfaces

Definition 2.1.1. A subset S of $\mathbb{R}^3$ is a **regular surface** if, for each $p \epsilon S$, there exists a neighbourhood V in $\mathbb{R}^3$ and a map $f : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. f is differentiable.
2. f is homeomorphism.
3. (The regularity condition) For each $q \epsilon U$, the differential $df_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-one. (cf. Def. 2)

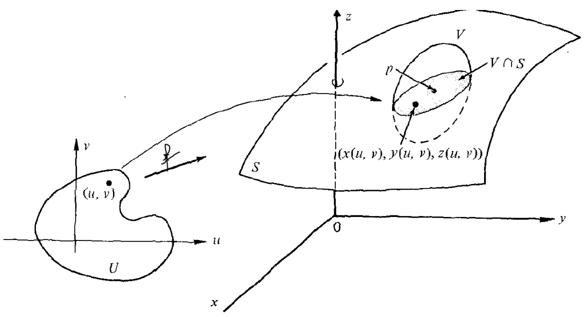


Figure 2.1: parametrization.

The mapping f is called a **parametrization or system of local coordinates** in p the neighbourhood $V \cap S$ of p in S is called a **coordinate neighbourhood.**

Definition 2.1.2. Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. To each $p \epsilon U$ we associate a linear map $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ which is called the **differential of** F **at** p and is defined as follows. Let $w \epsilon \mathbb{R}^n$ and let $\alpha : (-\epsilon, \epsilon) \to U$ be a differential curve such that $\alpha(0) = p$ and $\alpha'(0) = w$. By the chain rule the curve $\beta = F \circ \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m$ is also differentiable. Then

$$dF_p(w) = \beta'(0)$$

.

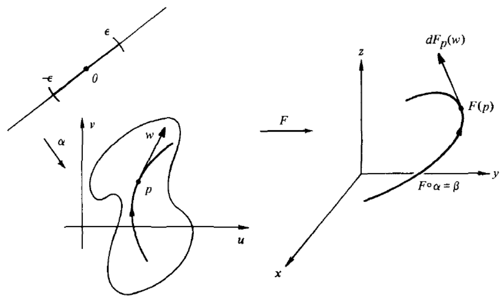


Figure 2.2: Differential of F at p.

The matrix of the linear map df_q in Definition 1 is:

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Condition 3 of Def.1 may be expressed by requiring the two coloumn vectors of the matrix to be linearly independent.

Remark. Condition 1 is natural if we expect to do some differential geometry on S. The one-to-oneness in condition 2 has a purpose in preventing self intersections in the regular surfaces. This is necessary to define tangent plane at a point $p \in S$(see Fig 2.3(a)). The continuity of inverse in condition 2 is necessary to define differentiable functions on a regular surface. Condition 3 will gaurantee the existance of tangent plane at all points of S (seen Fig. 2-3(b)).

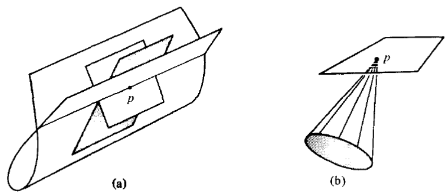


Figure 2.3: Some situations to be avoided in the definition of a regular surface.

Inverse Function Theorem

Theorem 2.1.1. *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping and suppose that at $p \in U$ the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Then there exists a neighbourhood V of p in U and a neighbourhood W of $F(p)$ in $\mathbb{R}^n$ such that $F : V \to W$ has a differentiable inverse $F^{-1} : W \to V$.*

The proof that the unit sphere is a Regular surface is detailed in **[Doc], pg. 55**, we instead will see a couple of exercise problems. (cf.Example 1 below)
Ex. — Show that the cylinder $S = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\}$ is a regular surface?

Answer (Ex.) — Let $p = (x, y, z) \in S$, note that atleast one coordinate of p is not zero. Wlog, assume $x \neq 0$. Also, assume $x > 0$, choose $U = \{(y, z)| -1 \leq y \leq 1\}$. Define $f : U \to \mathbb{R}^3$ *as*, $f(y, z) = (\sqrt{1 - y^2}, y, z)$. Choose $V = \{(x, y, z)|\ x \geq 0\}$. The inverse of f is the projection map π_{23}. 1. We see that f is differentiable since each of its components is differentiable. 2. f is homeomorphism since the projection map is continous.
3.

$$df_p(y, z) = \begin{pmatrix} \frac{y}{\sqrt{1-y^2}} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This matrix has full rank. For $x < 0$ we define $f : U \to \mathbb{R}^3$ *as*, $f(y, z) = (-\sqrt{1 - y^2}, y, z)$.

Ex. — Is the closed unit disk and open unit disk in the $xy-plane \subset \mathbb{R}^3$ regular surface?

Answer (Ex.) — The open unit disc is a regular surface. Choose V interior of open ball in $\mathbb{R}^3$ and U to be open unit disk and f to be the identity map on U. While the closed unit disk is not a regular surface since if we choose p to be a point on the boundary of the unit disk it will pose a problem as follows:
If x is in the boundary of the closed disk, and V is a contractible open neighborhood of x, then $V - \{x\}$ is again contractible. As $\mathbb{R}^2 - \{pt\}$ deformation retracts to S1 (which is not contractible), $U - \{x\}$ cannot be homeomorphic to $\mathbb{R}^2 - \{pt\}$, and therefore U is not homeomorphic to $\mathbb{R}^2$.

We now see that some special subsets of $\mathbb{R}^3$ are regular surfaces.

Theorem 2.1.2. *If $f : U \to \mathbb{R}$ is a differentiable function in an open subset U of $\mathbb{R}^2$, then the graph of f is a regular surface.*

Definition 2.1.3. Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open subset U of $\mathbb{R}^n$ we say that $p \epsilon U$ is a critical point of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective mapping. The image $F(p) \epsilon \mathbb{R}^m$ of a critical point is called a critical value of F. A point of $\mathbb{R}^m$ which is not a critical value is called a regular value. Notice that any point $a \notin F(U)$ is trivially a regular value of F.

Theorem 2.1.3. *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \epsilon f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.*

Example 2.1. *The ellisoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface. In fact it is the set $f^{-1}(0)$ where $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$ is a differentiable function and 0 is regular value of f. This example includes sphere as a particular case $(a = b = c = 1)$.

The converse of the above theorem is not true which is illustrated in the following exercise problem.

Ex. — 1. Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value of f and yet $f^{-1}(0)$ is a regular surface.

Answer (Ex.) — Note that $f'(0, 0, 0) = (0, 0, 0)$ but $f^{-1}(0) = xy-\ plane$

The following Theorem provides a local converse of Theorem 2.1.2.

Theorem 2.1.4. *Let $S \subset \mathbb{R}^3$ be a regular surface and $p \epsilon S$. Then there exists a neighbourhood V of p in S such that V is is graph of a differentiable function which has one of the following three forms: $z = f(x, y)$, $y = g(y, z)$, $x = h(y, z)$.*

Theorem 2.1.5. *Let $p \epsilon S$ be a point of a regular surface S and let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map $p \epsilon f(U) \subset S$ such that condition one of Def. 1 hold. Assume that f is one-one. Then f^{-1} is a continous.*

2.2 Differential Functions on Surfaces

In this section we will define what it means for a function $f : S \to R$ to be differentiable at a point p of a regular surface S. A natural way to proceed is to choose a coordinates of p with coordinates u, v, and say that f is differentiable at p if its expression in the coordinates u and v admits continous partial derivatives of all orders. The same point of S can belong to various coordinate neighbourhoods and the following theorem says that, it is possible to pass from one of the pairs of coordinates to the other by means of differentiable transformations.

Theorem 2.2.1. ***(Change of parameters)*** *Let p be a point on the regular surface S, and let $f : U \subset \mathbb{R}^2 \to S$, $g : V \subset \mathbb{R}^2 \to S$ be two parametrizations of S such that $p \epsilon f(U) \cap g(V) = W$. Then the change of coordinates $h = f^{-1} \circ g : g^{-1}(W) \to f^{-1}(W)$ is a diffeomorphism; that is h is differentiable and has a differentiable inverse h^{-1}.(Fig.2.4)*

We now define differentiable functions on regular surface.

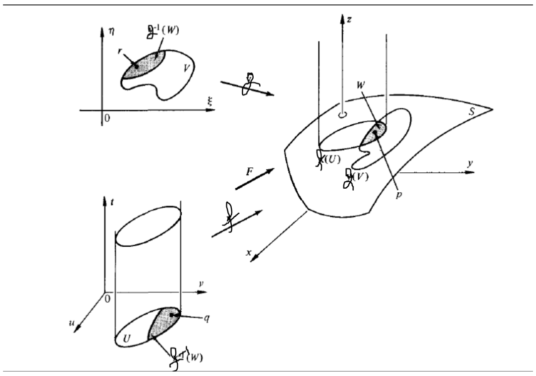


Figure 2.4: Change of parameters.

Definition 2.2.1. Let $F : V \subset S \to R$ be a map defined in an open subset V of a regular surface S. Then F is said to be differentiable at $p \epsilon V$ if, for some parametrization $f : U \subset \mathbb{R}^2 \to S$ with $p \epsilon f(U) \subset V$, the composition $F \circ f : U \subset \mathbb{R}^2 \to R$ is differentiable at $f^{-1}(p)$. F is said to be differentiable in V if it is differentiable at all points of V.

By theorem 2.2.1 it immediately follows that the definition is independent of the choice of parametrization f.

The definition of differentiability of a map can be easily extended to maps between surfaces. A continous map $F : V_1 \subset S_1 \to S_2$ of an open set V_1 of a regular surface S_1 to a regular surface S_2 is said to be differentiable at $p \epsilon V_1$ if, given parametrizations

$$f : U_1 \subset \mathbb{R}^2 \to S_1 \quad g : U_2 \subset \mathbb{R}^2 \to S_2$$

with $p \epsilon f(U)$ and $F(f(U_1)) \subset g(U_2)$, the map

$$g^{-1} \circ F \circ f : U_1 \to U_2$$

is differentiable at $q = f^{-1}(p)$.

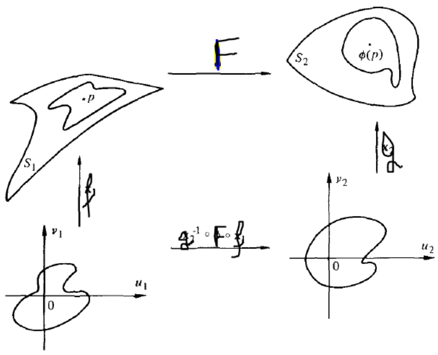


Figure 2.5: Differentiability of a function between surfaces.

Definition 2.2.2. Two regular surfaces S_1 and S_2 are diffeomorphic if there exists a differentiable map $F : S_1 \to S_2$ with a differentiable inverse $F^{-1} : S_2 \to S_1$.

Example 2.2. *Let S_1 and S_2 be regular surfaces Assume that $S_1 \subset V \subset \mathbb{R}^3$, where V is open set of $\mathbb{R}^3$, and that $F : V \to \mathbb{R}^3$ is a differentiable map such that $F(S_1) \subset S_2$. then the restriction $F|_{S_1} : S_1 \to S_2$ is a differentiable map.*

Example 2.3. *Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(x, y, z) = (xa, yb, zc)$, where a, b and c are non zero real numbers. F is clearly differentiable, and the restriction $F|_{\mathbb{S}^2}$ is a differentiable map from the sphere into an ellipsoid.*

Example 2.4. *(Surface of Revolution). Let $S \subset \mathbb{R}^3$ be the set obtained by rotating plane curve C about an axis in the plane which does not meet the curve: we shall take xz-plane as the plane of the curve and the z-axis as the rotation axis. Let*

$$x = f(v),\ z = g(v),\ a < v < b,\ f(v) > 0,$$

be the parametrization for C and denote by u the rotation angle about the z-axis. The curve C is called the generating curve of S, and the z-axis is called the axis of rotation.

2.3 The Tangent Plane

The tangent line (or simply tangent) to a plane curve at a given point is the straight line that "just touches" the curve at that point. Similarly, the tangent plane to a surface at a given point is the plane that "just touches" the surface at that point.
By a tangent vector at a point p of a regular surface S, we mean a tangent vector $\alpha'(0)$ of the differentiable parametrized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

Theorem 2.3.1. *Let $f : U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,*

$$df_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the tangent vectors to S at $f(q)$.

Definition 2.3.1. The plane $df_q(\mathbb{R}^2)$, in the above theorem is called the tangent plane at the point $f(q) = p$ to S and is denoted by $T_p(S)$. From the above theorem it is also observed that the tangent plane is independent of the parametrization

The choice of parametrization f determines a basis $\{f_u(q) = \partial f/\partial u, f_v(q) = \partial f/\partial v\}$ of $T_p(S)$, called the basis associated to f.

Theorem 2.3.2. *Let $F : U \subset S_1 \to S_2$ be a differentiable map. To each $p \in S_1$ we associate a map $dF_p : T_p(S_1) \to T_{F(p)}(S_2)$ which is called the* ***differential of*** F ***at*** *p and is defined as follows. Let $w \in T_p(S_1)$ and let $\alpha : (-\epsilon, \epsilon) \to U$ be a differential curve such that $\alpha(0) = p$, $\alpha'(0) = w$. By the chain rule the curve $\beta = F \circ \alpha : (-\epsilon, \epsilon) \to S_2$ is also differentiable. Then*

$$dF_p(w) = \beta'(0).$$

The map $dF_p : T_p(S_1) \to T_p(S_2)$ is linear.

The following theorem says that the inverse function theorem can be extended to differentiable maps between surfaces. We say that a mapping $f : U \subset S_1 \to S_2$ is a local diffeomorphism at $p \in U$ if there exists a neighbourhood $V \subset U$ of p such that F restricted to V is a diffeomorphism onto an open set $f(V) \subset S_2$.

Theorem 2.3.3. *If S_1 and S_2 are regular surfaces and $F : U \subset S_1 \to S_2$ is a differentiable mapping of an open set $U \subset S_1$ such that the differential dF_p of F at $p \in U$ is an isomorphism, then F is a local diffeomorphism at p.*

The unit normal vector at a point $q \in f(U)$ is given by the rule

$$N(q) = \frac{f_u \wedge f_v}{|f_u \wedge f_v|}(q),$$

where $f : U \subset \mathbb{R}^2 \to S$ is a parametrization around p.
The **angle** of two intersecting surfaces at an point of intersection p is the angle of their tangent plane at p.
We observe that the existance of tangent plane depends only on the existance and the continuity of the first partial derivatives. The following example admits a tangent plane at each point but is not sufficiently differentiable to satisfy the definition of regular surface.

Example 2.5. *Consider the graph of the function $z = (x^2 + y^2)^{1/3}$, generated by rotating the curve $z = x^{4/3}$ about the z-axis. At the origin the graph admits the xy-plane as the tangent plane. However, the partial derivative z_{xx} does not exist at the origin, and the graph considered is not a regular surface as defined above.*

2.4 The First Fundamental Form; Area

The inner product of $\mathbb{R}^3$ induces on each tangent plane $T_p(S)$ of a regular surface S an inner product, to be denotedd by $\langle , \rangle_p$

Definition 2.4.1. The quadratic form $I_p : T_p(S) \to R$ given by

$$I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0$$

is called the first fundamental form of a regular surface $S \subset \mathbb{R}^3$ at $p \,\epsilon\, S$.

The first fundamental form is expressed in the basis $\{f_u, f_v\}$ associated with the parametrization $f(u, v)$ at p. Let $w \,\epsilon\, T_p(S)$ is a tangent vector to the parametrized curve $\alpha(t) = f(u(t), v(t))$, $t \,\epsilon\, (-\epsilon, \epsilon)$, with $p = \alpha(0) = f(u_0, v_0)$, we obtain

$$I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle_p = \langle f_u u' + f_v v', f_u u' + f_v v' \rangle_p = E(u')^2 + 2Fu'v' + G(v')^2,$$

where the values of three functions involved are computed for $t = 0$, and

$$E(u_0, v_0) = \langle f_u, f_u \rangle_p,$$

$$F(u_0, v_0) = \langle f_u, f_v \rangle_p,$$

$$G(u_0, v_0) = \langle f_v, f_v \rangle_p,$$

are the coefficients of the first fundamental form in the basis $\{f_u, f_v\}$ of $T_p(S)$

Example 2.6. *A coordinate system for a plane $P \subset \mathbb{R}^3$ passing through $p = (x, y, z)$ and containing the orthonormal vectors $w_1 = (a_1, a_2, a_3)$, and $w_2 = (b_1, b_2, b_3)$ is given as follows:*

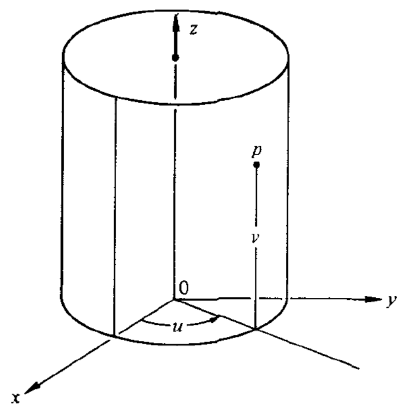
$$f(u, v) = p_0 + uw_1 + vw_2, \quad (u, v) \,\epsilon\, \mathbb{R}^2.$$

here $f_u = w_1$, $f_v = w_2$; since w_1 and w_2 are unit orthogonal vectors, the functions E, F, G are constant and given by

$$E = 1, \; F = 0 \; G = 1.$$

In this trivial case, the first fundamental form is essentially the Pythagorean theorem on P; i.e., the square of length of the vector w which has coordinates a, b in the basis (f_u, f_v) is equal to $a^2 + b^2$.

Example 2.7. *The right cylinder over the circle admits the parametrization $f : U \to \mathbb{R}^3$, where $f(u, v) = (\cos u, \sin u, v)$, $U = \{(u, v) \,\epsilon\, \mathbb{R}^2; 0 < u < 2\pi, \; -\infty < v < \infty\}$ $f_u = (-\sin u, \cos u, 0)$*



$f_v = (0, 0, 1)$ and therefore $E = 1$, $F = 0$, $G = 1$.

We now see how the first fundamental form is used to treat metric questions on a regular surface. Let $\alpha : I \to S$ be a parametrized curve then the arc length s is given by

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt = \int_{t_0}^{t} \sqrt{I(\alpha'(t))} dt.$$

In particular, if $\alpha(t) = f(u(t), v(t))$ is contained in a coordinate neighbourhood corresponding to the parametrization $f(u, v)$, we can compute the arc length of α between say 0 and t by

$$s(t) = \int_{t_0}^{t} \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}dt.$$

Another metric question that can be treated by the first fundamental form is the computation of the area of a bounded region of a regular surface S. A (regular) domain of S is open and connected subset of S such that its boundary is the image of a circle by a differentiable homeomorphism which is regular (that is, its differential is non zero) except at finite number of points. A region of S is the union of a domain with its boundary. A region of $S \subset \mathbb{R}^3$ is bounded if it is contained in some ball of $\mathbb{R}^3$.

Definition 2.4.2. Let $R \subset S$ be a bounded region of a regular surface contained in a coordinate neighbourhood of the parametrization $f : U \subset \mathbb{R}^2 \to S$. The positive number

$$\int\int_{Q} |f_u \wedge f_v| dudv = \int\int_{Q} \sqrt{EG - F^2} dudv = A(R),$$

where $Q = f^{-1}(R)$, is called the area of R.

2.5 Orientation on Regular Surfaces.

Definition 2.5.1. A regular surface S is called orientable if it is possible to cover it with a family of coordinate neighbourhoods in such a way that if a point $p \, \epsilon \, S$ belongs to two neighbourhoods of this family, then the change of coordinates has a positive Jacobian at p. The choice of such a family is called an orientation of S and S, in this case, is called orientated. If such a choice is not possible, the surface is called nonorientable.

Example 2.8. *A surface which is graph of a differentiable function is an orientable.*

Definition 2.5.2. A **differentiable field** of unit normal vectors on an open set $U \subset S$ is a differentiable map $N : U \to \mathbb{R}^3$ which associates to each $q \, \epsilon \, U$ a unit normal vector $N(q) \, \epsilon \, \mathbb{R}^3$ to S at q.

Theorem 2.5.1. *A regular surface $S \subset \mathbb{R}^3$ is orientable iff there exists a differentiable field of unit normal vectors $N : S \to \mathbb{R}^3$ on S.*

Example 2.9. *Mobius strip is non-orientable.*

Theorem 2.5.2. *If a regular surface is given by $S = \{(x, y, z) \, \epsilon \, \mathbb{R}^3; f(x, y, z) = a\}$, where $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable and a is a regular value of f, the S is orientable.*

Chapter 3

The Geometry of the Gauss Map

Throughout this chapter S will denote regular orientable surface in which an orientation N has been chosen. N is differentiable field of unit normal vectors on S.

3.1 Gauss Map

Definition 3.1.1. The map $N : S \to \mathbb{R}^3$ takes its values on the unit sphere. The map $N : S \to S^2$, thus defined, is called the **Gauss Map** of S.

The differential $dN_p : T_p(S) \to T_{N(p)}(S^2)$ can be looked as a linear map on $T_p(S)$ since $T_p(S)$ and $T_{N(p)}(S^2)$, are parallel planes.
Let $\alpha(t)$ be a parametrized curve on S with $\alpha(0) = p$, we get a parametrized curve $N \circ \alpha(t) = N(t)$ in the sphere.The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p(S)$ and measures the rate of change of the normal vector N, restricted to the curve $\alpha(t)$, at $t = 0$.

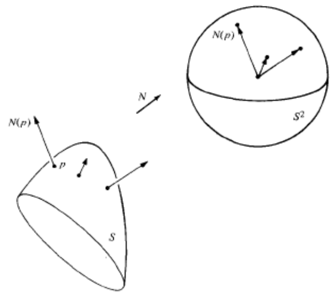


Figure 3.1: The Gauss map.

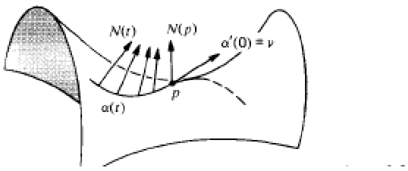


Figure 3.2: .

Example 3.1. *For a plane P given by $ax+by+cz+d = 0$, the unit normal vector $N = (a, b, c)/\sqrt{a^2 + b^2 + c^2}$ is a constant, and therefore $dN = 0$.*

Example 3.2. *Consider the unit sphere. If $\alpha(t) = (x(t), y(t), z(t))$ is a curve on S^2. Choose $N = (x, y, z)$, and N restricted to $\alpha(t)$ is $N(t) = (x(t), y(t), z(t))$ is vector function of t, and therefore we have*

$$dN(x'(t), y'(t), z'(t)) = N'(t) = (x'(t), y'(t), z'(t)).$$

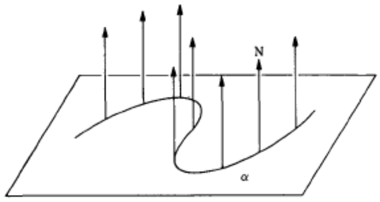


Figure 3.3: Plane $dN_p = 0$.

Theorem 3.1.1. *The differential $dN_p : T_p(S) \to T_p(S)$ of a Gauss map is a self-adjoint linear map.*

Definition 3.1.2. The quadratic form II_p, defined in $T_p(S)$ by $II_p(v) = -\langle dN_p(v), v\rangle$ is called the **second fundamental** form of S at p.

Definition 3.1.3. Let C be a regular curve in S passing through $p \in S$, k the curvature of C at p, and $\cos\theta = \langle n, N\rangle$, where n is the normal vector to C and N is the normal vector to S at p. the **number** $k_n = k\cos\theta$ is called the **normal curvature** of $C \subset S$ at p. In other words, k_n is the length of the projection of the vector kn over the normal to the surace at p.

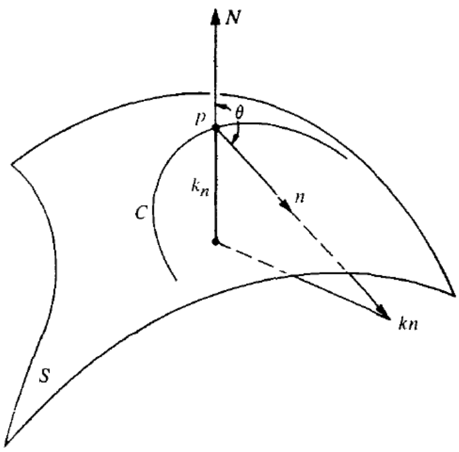


Figure 3.4: Normal curvature of C at p.

Remark. The normal curvature is independent of the orientation of the curve but changes its sign with the change in orientation of the surface.

Interpretation of second fundamental form II_p. Let $C \subset S$ be parametrized by $\alpha(s)$, where s is the arc length of C, and $\alpha(0) = p$. Let $N(s)$ be the restriction of the normal vector N to $\alpha(s)$, we have $\langle N(s), \alpha'(s)\rangle = 0$. Hence,

$$\langle N(s), \alpha''(s)\rangle = -\langle N'(s), \alpha'(s)\rangle.$$

Therefore,

$$II_p(\alpha'(0)) = \langle dN_p(\alpha'(0)), \alpha'(0)\rangle = -\langle N'(0), \alpha'(0)\rangle = \langle N(0), \alpha''(0)\rangle = \langle N, kn\rangle(p) = k_n(p).$$

In other words the value of the second fundamental form II_p for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v.

Theorem 3.1.2. ***(Meusnier)*** *All curves lying on surface S and having at a given point the same tangent line have at this point the same normal curvatures.*

Definition 3.1.4. Given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the **normal section** of S at p along v. In a neighbourhood of p, a normal section of S at p is a regular plane curve on S whose normal vector n at p is $\pm N(p)$ or zero; its cuvature is therefore equal to absolute value of the normal curvature along v at p.

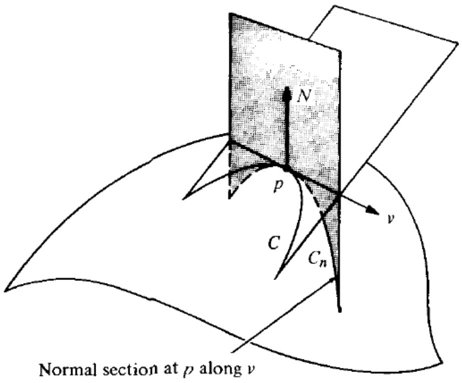


Figure 3.5: Meusnier theorem: C and C_n have the same normal curvature at p along v.

Example 3.3. *In the plane, all normal sections are straight lines; hence, all normal curvatures are zero. Thus, the second fundamental form is identically zero at all points. That is* $dN = 0$. *In the sphere* S^2*, the normal sections through a point* $p \in S^2$ *are circles with radius 1. Thus, all normal curvatures are equal to 1, and the second fundamental form is* $II_p(v) = 1$, *for all* $p \in S^2$ *and all* $v \in T_p(S)$ *with* $|v| = 1$.

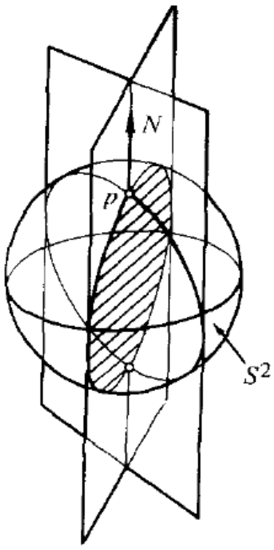


Figure 3.6: Normal sections on a sphere.

In the cylinder, the normal sections at a point p vary from a circle perpendicular to the axis of the cylinder to a straight line parallel to the axis of the cylinder, passing through a family of ellipses. Thus, the normal curvatures varies from 1 to 0.
For each p, since dN_p is a linear map there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1e_1$, $dN_p(e_2) = -k_2e_2$. Moreover, k_1 and k_2 ($k_1 \geq k_2$) are the maximum and minimum of the second fundamental form II_p restricted to the unit circle of $T_p(S)$; that is, they are the extreme values of the normal curvature at p.

Definition 3.1.5. The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the **principal curvatures** at p; the corresponding directions, that is, the directions given by the eigenvectors e_1, e_2, are called principal directions at p.

In the plane and the sphere all directions at all points are principal directions.

Definition 3.1.6. If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is the principal direction at p, then C is said to be a **line of curvature** of S.

Theorem 3.1.3. ***(Olinde Rodrigues)*** *A necessary and sufficient condition for a connected regular curve C on S to be a line of curvature of S is that*

$$N'(t) = \lambda(t)\alpha'(t)$$

for any parametrization $\alpha(t)$ of C, where $N(t) = N \circ \alpha(t)$ and $\lambda(t)$ is a differentiable function of t. In this case, $-\lambda(t)$ is the (principal)curvature along $\alpha(t)$.

The knowledge of principal curvatures at p allows us to compute easily the normal curvature along a given direction of $T_p(S)$. In fact, let $v \in T_p(S)$ with $|v| = 1$. Let e_1 and e_2 form an orthonormal basis of $T_p(S)$, we have

$$v = e_1 \cos\theta + e_2 \sin\theta,$$

where θ is the angle from e_1 to v in the orientation of $T_p(S)$. The normal curvature k_n along v is given by

$$\begin{aligned} k_n &= II_p(v) \\ &= -\langle dN_p(v), v\rangle \\ &= \langle e_1 k_1 \cos\theta + e_2 k_2 \sin\theta, e_1 \cos\theta + e_2 \sin\theta\rangle \\ &= k_1 \cos^2\theta + k_2 \sin^2\theta\rangle \end{aligned}$$

The last expression is known as the *Euler formula*.

Definition 3.1.7. Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the *Gauss map*. The determinant of dN_p is the **Gaussian curvature** K of S at p. The negative of half of the trace of dN_p is called the **mean curvature** H of S at p.

In terms of the principal curvatures we can write

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}.$$

If we change the orientation of the surface, the determinant does not change; the trace, however, changes sign.

Definition 3.1.8. A point of a surface S is called

1. **Elliptic** if $det(dN_p) > 0$.
2. **Hyperbolic** if $det(dN_p) < 0$.
3. **Parabolic** if $det(dN_p) = 0$, with $dN_p \neq 0$.
4. **Planar** if $dN_p = 0$.

We observe that the classification does not depend on the choice of the orientation.

The following table gives example of each type of point on a surface.

Point	Gaussian Curvature	Example
Elliptic	positive(the principal curvatures have same sign)	The point $(0,0,0)$ of the paraboloid $z = x^2 + ky^2, k > 0$
Hyperbolic	negative(the principal curvatures have opposite sign)	The point $(0,0,0)$ of the hyperbolic paraboloid $z = y^2 - x^2$
Parabolic	0(one of the principal curvature is non zero)	The points of cylinder
Planar	0(all principal curvatures are 0)	The points of a plane

Definition 3.1.9. If at $p \in S$, $k_1 = k_2$ then p is called an **umbilical point** of S.

Example 3.4. *The planar points are umbilical points ($k_1 = k_2 = 0$), all the points on the sphere are umbilical points.*

The following theorem states that the only surfaces made up entirely of umbilical points are essentially sphere and planes.

Theorem 3.1.4. *If all the points of a connected surface S are umbilical points, then S is either contained in a sphere or in a plane.*

Definition 3.1.10. Let p be a point in S. An **asymptotic direction** of S at p is the direction of $T_p(S)$ for which the normal curvature is zero. An **asymptotic curve** of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is an asymptotic direction.

Example 3.5. *At an elliptic point there are no asymptotic directions.*

Geometric interpretation of asymptotic directions.

Definition 3.1.11 (Dupin indicatrix)**.** Let $p \in S$, the **Dupin indicatrix** at p is the set of vectors w of $T_p(S)$ such that $II_p(w) = \pm 1$.

Let $w = ae_1 + be_2 \in T_p(S)$ where e_1 and e_2 are eigen vectors of dN_p and $\{e_1, e_2\}$ is the orthonormal basis of $T_p(S)$. Let ρ and θ be polar coordinates defined by $w = \rho v$, with $|v| = 1$ and $v = e_1 \cos\theta + e_2 \sin\theta$, if $\rho \neq 0$. By Euler's formula,

$$\begin{aligned} \pm 1 = II_p(w) &= \rho^2 II_p(v) \\ &= k_1 \rho^2 \cos^2\theta + k_2 \rho^2 \sin^2\theta \\ &= k_1 a^2 + k_2 b^2 \end{aligned}$$

Hence the Dupin indicatrix is the union of conics in $T_p(S)$.

Point	Dupin indicatrix	Reason
Elliptic	ellipse	k_1 and k_2 have same sign
Hyperbolic	two hyperbolas with a common pair of asymptotes	the principal curvatures have opposite signs
Parabolic	pair of parallel lines	one of the principal curvatures is zero

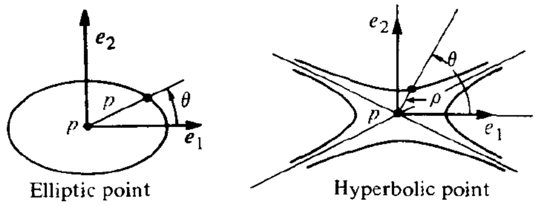


Figure 3.7: The Dupin Indicatrix.

Definition 3.1.12. Let p be a point on a surface S. Two nonzero vectors w_1, $w_2 \in T_p(S)$ are **conjugate** if $\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle = 0$. Two directions r_1 and r_2 at p are conjugate if a pair of nonzero vectors w_1, w_2 parallel to r_1 and r_2, respectively, are conjugates.

It follows that the principal directions are conjugate and that asymptotic directions are conjugates to itself. Furthermore, at a nonplanar umbilic, every orthogonal pair of directions, is a pair of conjugate directions, and at planar umbilic each direction is conjugate to any other direction.

Let $p \in S$ is not an umbilical point, and let $\{e_1, e_2\}$ be the orthonormal basis of $T_p(S)$ determined by the eigenvectors of dN_p. Let θ and ϕ be the angles that a pair of directions r_1 and r_2 make with e_1 then r_1 and r_2 are conjugates iff

$$k_1 \cos\theta \cos\phi = -k_2 \sin\theta \sin\phi.$$

3.2 The Gauss Map in Local Coordinates.

All parametrization considered in this section are considered to be compatible with the orientation. Let $h(u,v)$ be a parametrization at a point $p \in S$ of a surface S, and let $\alpha(t) = h(u(t), v(t))$ be a parametrized curve on S, with $\alpha(0) = p$. We have $\alpha' = h_u u' + h_v v'$ and $dN(\alpha') = N_u u' + N_v v'$, we write

$$N_u = a_{11} h_u + a_{21} h_v,$$

$$N_v = a_{12} h_u + a_{22} h_v,$$

and therefore,

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

The expression of second fundamental form in the basis $\{h_u, h_v\}$ is given by

$$II_p(\alpha') = -\langle dN(\alpha'), \alpha' \rangle = -\langle N_u u' + N_v v', h_u u' + h_v v' \rangle = e(u')^2 + 2fu'v' + g(v')^2,$$

where, since $\langle N, h_u \rangle = \langle N, h_v \rangle = 0$,

$$e = -\langle N_u, h_u \rangle = \langle N, h_{uu} \rangle, \tag{3.2.1}$$

$$f = -\langle N_v, h_u \rangle = \langle N, h_{uv} \rangle = \langle N, h_{vu} \rangle = -\langle N_u, h_v \rangle, \tag{3.2.2}$$

$$g = -\langle N_v, h_v \rangle = \langle N, h_{vv} \rangle, \tag{3.2.3}$$

If E, F *and* G are the coefficients of the first fundamental form in the basis $\{h_u, h_v\}$ we have:

$$a_{11} = \frac{fF - eG}{EG - F^2} \tag{3.2.4}$$

$$a_{12} = \frac{gF - fG}{EG - F^2} \tag{3.2.5}$$

$$a_{21} = \frac{eF - fE}{EG - F^2} \tag{3.2.6}$$

$$a_{22} = \frac{fF - gE}{EG - F^2} \tag{3.2.7}$$

The above relations are known as **the equations of Weingarten**.
Also upon calculations we get the following formulas,

$$K = det(a_{ij}) = \frac{eg - f^2}{EG - F^2}. \tag{3.2.8}$$

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} \tag{3.2.9}$$

$$k_1 = H + \sqrt{H^2 - K} \tag{3.2.10}$$

$$k_2 = H - \sqrt{H^2 - K}. \tag{3.2.11}$$

Example 3.6. *The Gaussian curvature of the points of the torus covered by parametrization*

$$h(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u), \;\; 0 < u, v < 2\pi,$$

is

$$K = \frac{\cos u}{r(a + r\cos u)}.$$

From this expression, it follows that $K = 0$ along the parallels $u = \pi/2$ and $u = 3\pi/2$; the points os such parallels are therefore parabolic poins. In the region of the torus given by $\pi/2 < u < 3\pi/2$, K is negative; the points in this region are therefore hyperbolic points. In the region given by $0 < u < \pi/2$ or $3\pi/2 < u < 2\pi$, the curvature is positive and the points are elliptic points.(see fig below)

The following theorem gives information about the position of a surface in the neighbourhood of an elliptic or an hyperbolic point, relative to the tangent plane at this point.

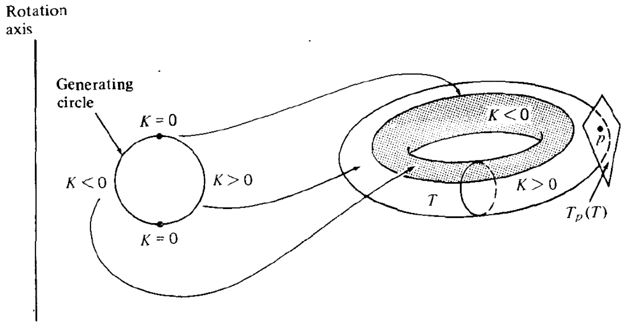


Figure 3.8: The Dupin Indicatrix.

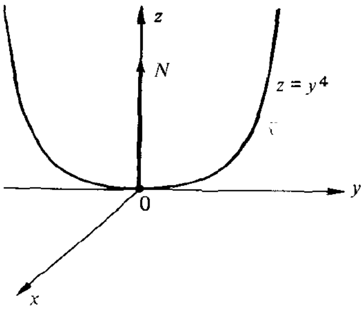


Figure 3.9: The surface lies on one side of the tangent plane at the planar point $(0, 0, 0)$.

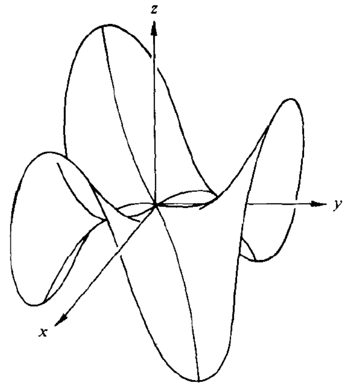


Figure 3.10: The surface lies on both sides of the tangent plane in any neighbourhood of the planar point $(0, 0, 0)$.

Theorem 3.2.1. *Let $p \in S$ be an elliptic point of the surface S. Then there exists a neighbourhood V of p in S such that all points in V belong to same side of $T_p(S)$. Let $p \in S$ be an hyperbolic point of the surface S. Then in each neighbourhood of p there exists points of S in both sides of $T_p(S)$.*

No such statement can be made in the neighbourhood of planar and parabolic point. For instance in the the surface of revolution obtained by rotating the curve $z = y^4$ about the $z - axis$ the point $p = (0, 0, 0)$ has $dN_p = 0$. and the surface lies on one side of tangent plane at p. On the other hand the surface given by $x = u, \quad y = v \quad z = u^3 - 3v^2u$, the point $(0, 0, 0)$ is a planar point. However, in the neighbourhood of this point there are points in both sides of its tangent plane.

The expression of the second fundamental form in the local coordinates is useful for the study of the asympotic and principal directions. Let $C := \alpha(t) = h(u(t), v(t)),\ t \in I$ be a asymptotic curve in the coordinate neighbourhood of the parametrization h. We have $II(\alpha'(t)) = 0$, for all $t \in I$. So we have

$$e(u')^2 + 2fu'v' + g(v')^2 = 0, \quad t \in I.$$

which says that, A necessary and sufficient condition for a parametrization in a neighbourhood of a hyperbolic point $(eg - f^2 < 0)$ to be such that the coordinate curves of the parametrization are asymptotic curves is that $e = g = 0$.

Also we have seen that a necessary and sufficient condition for a curve to be a line of curvature is that

$$dN(\alpha'(t) = \lambda(t)\alpha'(t)).$$

which is true if and only if

$$\begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

Using this fact we observe that a necessary and sufficient condition for the coordinate curves of the parametrization to be the lines of curvature in the neighbourhood of a nonumbilical point is that $F = f = 0$.

Example 3.7. *(Surface of Revolution). Consider the surface of revolution parametrized by*

$$h(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)),$$

$$0 < u < 2\pi, \quad a < v < b, \quad \varphi(v) \neq 0.$$

$$E = \varphi^2, \quad F = 0, \quad G = (\varphi')^2 + (\psi')^2 = 1.$$

Since we are assuming that rotating curve is parametrized by arc length. The computations of the coefficients of the second fundamental form yeilds

$$e = -\varphi\psi', \quad f = 0, \quad g = \psi'\varphi'' - \psi''\varphi'.$$

Since $F = f = 0$, we conclude that the parallels (v = constant.) and the meridians (u = constant.) of a surface of revolution are the lines of curvature. Because

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi}$$

and φ is always positive, it follows that the parabolic points are given by either $\psi' = 0$ or $\varphi'\psi'' - \psi'\varphi'' = 0$. A point which satisfies both conditions is a planar point, since these conditions imply that $e = f = g = 0$. Furthur simplification gives

$$K = -\frac{\varphi''}{\varphi}$$

The principal curvatures of a surface of revolution are given by

$$k_1 = -\frac{\psi'}{\varphi}, \quad k_2 = \psi'\varphi'' - \varphi''\psi'.$$

Definition 3.2.1. Let S and $\bar{S}$ be two orientated regular surfaces. Let $\varphi : S \to \bar{S}$ be a differentiable map and assume that for some $p \,\epsilon\, S$, $d\varphi_p$ is nonsingular. We say that φ is **orientation-preserving** at p if given a positive basis $\{w_1, w_2\}$ in $T_p(S)$, then $\{d\varphi_p(w_1), d\varphi_p(w_2)\}$ is a positive basis in $T_{\varphi(p)}(\bar{S})$. If $\{d\varphi_p(w_1), d\varphi_p(w_2)\}$ is not a positive basis, we say φ is **orientation-reversing** st p.

Let N be a Gauss map and $p \,\epsilon\, S$ be such that dN_p is nonsingular. Since for a basis $\{w_1, w_2\}$ in $T_p(S)$

$$dN_p(w_1) \wedge dN_p(w_2) = det(dN_p)(w_1 \wedge w_2) = K w_1 \wedge w_2,$$

the Gauss map will be orientation-preserving at $p \,\epsilon\, S$ if $K(p) > 0$ and orientation-reversing at $p \,\epsilon\, S$ if $K(p) < 0$.

Theorem 3.2.2. *Let p be a point of a surface S such that the Gaussian curvature $K(p) \neq 0$, and let V be a connected neighbourhood of p where K does not change sign. Then*

$$K(p) = \lim_{A \to 0} \frac{A'}{A},$$

where A is the area of a region $B \subset V$ containing p, A' is the area of the image of B by the Gauss map $N : S \to S^2$, and the limit is taken through a sequence of regions B_n that converges to p, in the sense that any sphere around p contains all B_n, for n sufficienty large.

Chapter 4

The Intrinsic Geometry of Surfaces

Many local properties of the surface can be expressed only in terms of the first fundamental form. The study of such properties is called the intrinsic geometry of surfaces. The intrinsic geometry in some sense is the geometry which two dimensional beings can recognize.

4.1 Isometries; Conformal Maps

In this section S and S' will denote regular surfaces.

Definition 4.1.1. A **diffeomorphism** $\varphi : S \to S'$ is an **isometry** if for all $p \,\epsilon\, S$ and all pairs $w_1, w_2 \,\epsilon\, T_p(S)$ we have

$$\langle w_1, w_2\rangle_p = \langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle_{\varphi(p)}.$$

The surfaces S and S' are then said to be isometric.

Lemma 4.1.1. *A diffeomorphism φ is an isometry iff φ preserves the first fundamental form.*

Proof. If φ is an isometry then clearly

$$I_p(w) = I_{\varphi(p)}(d\varphi_p(w)).$$

Conversely, if

$$I_p(w) = I_{\varphi(p)}(d\varphi_p(w)) \quad for\ all\ w \,\epsilon\, T_p(S),$$

then

$$\begin{aligned} 2\langle w_1, w_2\rangle &= I_p(w_1 + w_2) - I_p(w_1) - I_p(w_2) \\ &= I_{\varphi(p)}(d\varphi_p(w_1 + w_2)) - I_{\varphi(p)}(d\varphi_p(w_1)) - I_{\varphi(p)}(d\varphi_p(w_2)). \\ &= 2\langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle, \end{aligned}$$

and hence φ is an isometry. □

Definition 4.1.2. A map $\varphi : V \to S'$ of a neighbourhood V of $p \,\epsilon\, S$ is an **local isometry** at p if there exists a neighbourhood V' of $\varphi(p) \,\epsilon\, S'$ such that $\varphi : V \to V'$ is an isometry. If there exists a local isomtry into S' at every $p \,\epsilon\, S$, the surface S is said to be locally isometric to S'. S and S' are said to locally isometric if S is locally isometric to S' and S' is locally isometric to S.

It is clear that if $\varphi : S \to S'$ is a diffeomorphism and a local isometry for every $p \,\epsilon\, S$, then φ is an isometry globally. However two locally isometric surfaces need not be globally isometric. For example cylinder is locally isometric to a plane but is not even homeomorphic to it.

Theorem 4.1.2. *Assume the existence of parametrizations $f : U \to S$ and $g : U \to S'$ such that $E = E',\ F = F',\ G = G'$ in U. Then the map $\varphi = g \circ f^{-1} : f(U) \to S'$ is a local isometry.*

Ex. — Show that a diffeomorphism $\varphi : S \to S'$ is an isometry iff the arc length of any parametrized curve in S is equal to arc length of the image curve by φ.

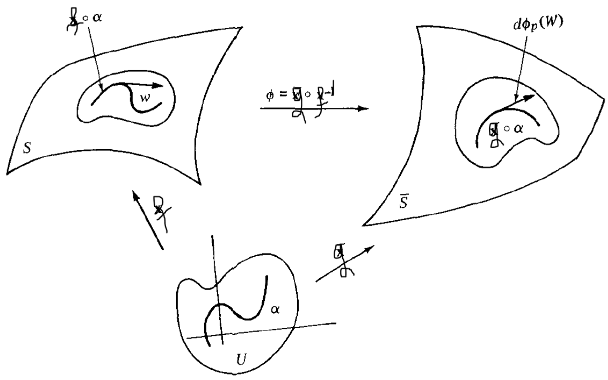


Figure 4.1: .

Answer (Ex.) — The only if part is immediate. To prove the "if" part, let $p \epsilon S$ and $v \epsilon T_p(S)$, $v \neq 0$. Consider a curve $\alpha : (-\in, \in) \to S$, with $\alpha'(0) = v$. We claim that $|d\varphi_p(\alpha'(0))| = |\alpha'(0)|$. Otherwise, say, $|d\varphi_p(\alpha'(0))| > |\alpha'(0)|$, and in a neighbourhood J of 0 in $(-\in, \in)$, we have $|d\varphi_p(\alpha'(t))| > |\alpha'(t)|$. This implies that the length of $\varphi \circ \alpha(J)$ is greater that length of $\alpha(J)$, a contradiction.

Definition 4.1.3. A $diffeomorphism$ $\varphi : S \to S'$ is called a **conformal map** if for all $p \epsilon S$ and all $v_1, v_2 \epsilon T_p(S)$ we have $\langle d\varphi_p(v_1), d\varphi_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle_p$, where λ^2 is a nowhere-zero differentiable function on S; the surface S and S' are then said to be conformal. A map $\varphi : V \to S'$ of a neighbourhood V of $p \epsilon S$ into S' is a **locally conformal map** at p if there exists a neighbourhood V' of $\varphi(p)$ such that $\varphi : V \to V'$ is a conformal map. If for each $p \epsilon S$, there exists a local conformal map at p, the surface is said to be **locally conformal** to S'.

Theorem 4.1.3. *Let $f : U \to S$ and $g : U \to S'$ be parametrization such that $E = \lambda^2 E'$, $F = \lambda^2 F'$, $G = \lambda^2 G'$ in U, where λ^2 is nowhere-zero differentiable function in U. Then the map $\varphi = g \circ f^{-1} : f(U) \to S'$ is a local conformal map.*

Theorem 4.1.4. *Any two regular surfaces are locally conformal.*

4.2 The Gauss Theorem

Let S be a regular oriented surface. Let $f : U \subset \mathbb{R}^2 \to S$ be a parametrization, to each point in $f(U)$ we attach a trihedron given by the vectors $\{f_u, f_v, N\}$ and express the derivatives of these vectors in the basis.

$$
\begin{aligned}
f_{uu} &= \Gamma_{11}^1 f_u + \Gamma_{11}^2 f_v + L_1 N,\\
f_{uv} &= \Gamma_{12}^1 f_u + \Gamma_{12}^2 f_v + L_2 N,\\
f_{vu} &= \Gamma_{21}^1 f_u + \Gamma_{21}^2 f_v + \bar{L}_2 N,\\
f_{vv} &= \Gamma_{22}^1 f_u + \Gamma_{22}^2 f_v + L_3 N,\\
N_u &= a_{11} f_u + a_{21} f_v,\\
N_v &= a_{12} f_u + a_{22} f_v.
\end{aligned}
\tag{4.2.1}
$$

The coefficients $\Gamma_{ij}^k, i, j, k = 1, 2$ are called the **Christoffel symbols** of S in the parametrization f. Since $f_{uv} = fvu$, we concllude that $\Gamma_{12}^1 = \Gamma_{21}^1$ and $\Gamma_{12}^2 = \Gamma_{21}^2$; that is, $Christoffel\ symbols$ are symmetric relative to lower indices. By taking the inner product of the first four relations in (1) with N we obtain $L_1 = e$, $L_2 = \bar{L}_2 = f$, $L_3 = g$, by taking the inner product of first four relations with f_u and f_v, we obtain the system,

$$
\begin{cases}
\Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle f_{uu}, f_u\rangle = \frac{1}{2}E_u,\\
\Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle f_{uu}, f_v\rangle = F_u - \frac{1}{2}E_v,
\end{cases}
\tag{4.2.2}
$$

$$\begin{cases} \Gamma^1_{12}E + \Gamma^2_{12}G = \langle f_{uv}, f_u \rangle = \frac{1}{2}E_v, \\ \Gamma^1_{12}F + \Gamma^2_{12}G = \langle f_{uv}, f_v \rangle = \frac{1}{2}G_u, \end{cases} \tag{4.2.3}$$

$$\begin{cases} \Gamma^1_{22}E + \Gamma^2_{22}F = \langle f_{uu}, f_u \rangle = F_v - \frac{1}{2}G_u, \\ \Gamma^1_{22}F + \Gamma^2_{22}G = \langle f_{uu}, f_u \rangle = \frac{1}{2}G_v, \end{cases} \tag{4.2.4}$$

From the above equations we infer that it is possible to compute the *Christoffel symbols* in terms of the coefficients of the first fundamental form, E,F,G, and its derivatives which will imply that *All geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries.*

THEOREMA EGREGIUM

Theorem 4.2.1. *(Gauss) The Gaussian curvature K of a surface is invariant by local isometries.*

Remark. Theorema Egregium is Latin for Remarkable Theorem. The theorem is "remarkable" because the starting definition of Gaussian curvature makes direct use of position of the surface in space. So it is quite surprising that the result does not depend on its embedding in spite of all bending and twisting deformations undergone.

4.3 Parallel Transport. Geodesics.

The term "geodesic" comes from geodesy, the science of measuring the size and shape of Earth; in the original sense, a geodesic was the shortest route between two points on the Earth's surface, namely, a segment of a great circle. A geodesic is a generalization of the notion of a "straight line" to "curved spaces"

Definition 4.3.1. A **vector field** in an open set $U \subset S$ of a regular surface S is a correspondence w that assigns to each $p \epsilon U$ a vector $w(p) \epsilon T_p(S)$. The vector field w is differentiable at p if, for some parametrization $f(u,v)$ in p, the components a and b of $w = af_u + bf_v$ in the basis $\{f_u, f_v\}$ are differentiable functions at $p \epsilon U$.

Definition 4.3.2. Let w be a differentiable vector field in an open set $U \subset S$ and $p \epsilon U$. Let $y \epsilon T_p(S)$. Consider the parametrization curve

$$\alpha : (-\in, \in) \to U,$$

with $\alpha(0) = p$ and $\alpha'(0) = y$, and let $w(t)$, $t \epsilon (-\in, \in)$, be the restriction of the vector field w to the curve α. The vector obtained by the normal projection of $(dw/dt)(0)$ onto the plane $T_p(S)$ is called the **covariant derivative** at p of the vector field w relative to vector y. This covariant derivative is denoted by $(Dw/dt)(0)$.

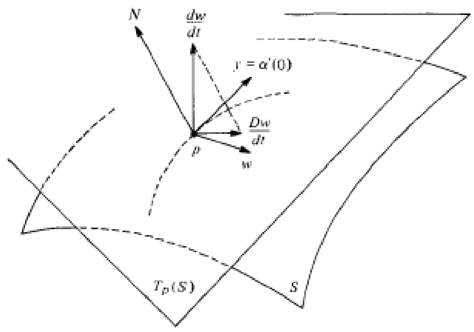


Figure 4.2: The covariant derivative.

The covariant differentiation is a concept of the intrinsic geometry and it does not depend on choice of the curve α. Let $f(u(t),v(t)) = \alpha(t)$, and let $w(t) = a(u(t),v(t))f_u + b(u(t),v(t))f_v$. Then we have

$$\frac{Dw}{dt} = (a' + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{12}bu' + \Gamma^1_{22}bv')f_u + (b' + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^2_{12}bu' + \Gamma^2_{22}bv')f_v. \quad (4.3.1)$$

Notation $[0,l] = I$.

Definition 4.3.3. Let $\alpha : I \to S$ be a parametrized curve in S. A vector field along α is a correspondence that assigns to each $t \in I$ a vector $w(t) \in T_{\alpha(t)}(S)$.

An example of a differentiable vector field along α is given by the field $\alpha'(t)$ of the tangent vectors of α.

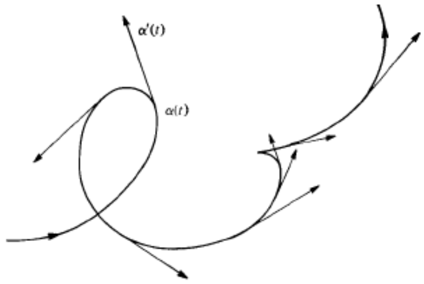


Figure 4.3: The field of tangent vectors along a curve α.

Definition 4.3.4. A vector field w along a parametrized curve $\alpha : I \to S$ is said to be **parallel** if $Dw/dt = 0$ for every $t \in I$.

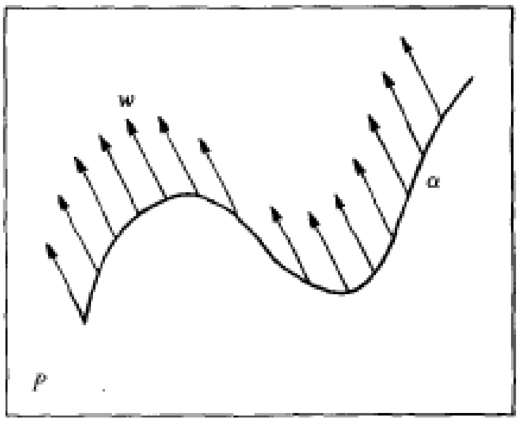


Figure 4.4: In case of plane, the notion of parallel field reduces to constant field along the curve.

Theorem 4.3.1. *Let w and v be parallel vector fields along $\alpha : I \to S$. Then $\langle w(t), v(t)\rangle$ is constant. In particular, $|w(t)|$ and $|v(t)|$ are constant, and the angle between $v(t)$ and $w(t)$ is constant.*

Theorem 4.3.2. *Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$ along $\alpha(t)$, with $w(t_0) = w_0$.*

Definition 4.3.5. Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Let w be the parallel vector field along α, with $w(t_0) = w_0$. The vector $w(t_1)$, $t_1 \in I$, is called the **parallel transport** of w_0 along α at the point t_1.

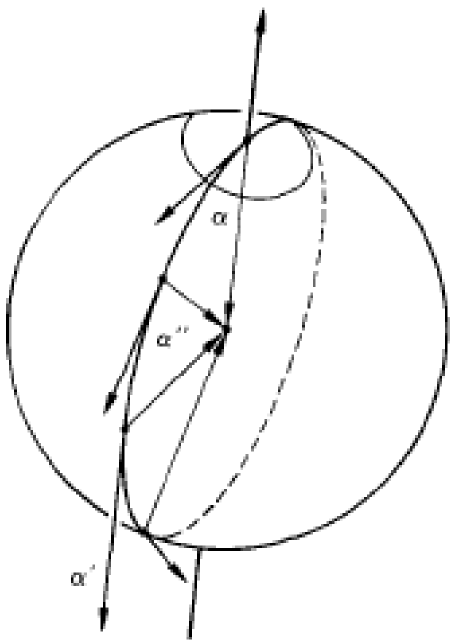


Figure 4.5: Parallel field on a sphere.

Definition 4.3.6. A nonconstant, parametrized curve $\gamma : I \to S$ is said to be **geodesic** at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t; that is,

$$\frac{D\gamma'(t)}{dt} = 0;$$

γ is a parametrized geodesic if it is geodesic for all $t \in I$.

In other words, a regular connected curve $C \subset S(k \neq 0)$ is a geodesic if and only if its principal normal at each $p \in C$ is parallel to the normal to S at p.

Example 4.1. *The great circles of sphere S^2 are geodesics.*

Definition 4.3.7. Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \to S$ on an oriented surface S. Since $w(t)$, $t \in I$, is a unit vector field, $(dw/dt)(t)$ is normal to $w(t)$, and therefore

$$\frac{Dw}{dt} = \lambda(N \wedge w(t)).$$

The real number $\lambda = \lambda(t)$, denoted by $[dw/dt]$, is called the **algebraic value** of the covariant derivative of w at t.

Observe that the sign of $[Dw/dt]$, depends on the orientation of S and that $[DW/dt] = \langle dw/dt, N \wedge w\rangle$.

Definition 4.3.8. Let C be a orientated regular curve contained on a oriented surface S, and let $\alpha(s)$ be a parametrization of C, in a neighbourhood of $p \in S$, by the arc length s. The algebraic value of the covariant derivative $[D\alpha'(s)/ds] = k_g$ of $\alpha'(s)$ at p is called the **geodesic curvature** of C at p.

The geodesics which are regular curves are thus characterized as curves whose geodesic curvature is zero. The absolute value of the geodesic curvature of C at p is the absolute value of the tangential component of the vector $\alpha''(s) = kn$, where k is the curvature of C at p and n is the normal vector of C at p. Since absolute value of the of the normal component of kn is normal curvature k_n of $C \subset S$ at p, we have

$$k^2 = k_g^2 + k_n^2.$$

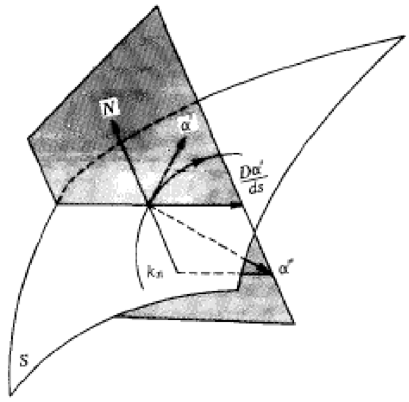


Figure 4.6: $k^2 = k_g^2 + k_n^2$.

Remark. The geodesic curvature of $C \subset S$ changes its sign with change in orientation of either C or S.

Lemma 4.3.3. *Let v and w be two differentiable vector fields along the curve $\alpha : I \to S$, with $|w(t)| = |v(t)| = 1,\ t \epsilon I$. Then $[\frac{Dw}{dt}] - [\frac{Dv}{dt}] = \frac{d\varphi}{dt}$, where φ is one of the differentiable determinations of angle from v to w.*

If we take $v(s)$ as parallel vector field along $\alpha(s)$. Then by taking $w(s) = \alpha'(s)$, we obtain

$$k_g(s) = \left[\frac{D\alpha'(s)}{ds}\right] = \frac{d\varphi}{ds}.$$

In other words, the geodesic curvature is the rate of change of the angle that the tangent to the curve makes with the parallel direction along the curve.

Theorem 4.3.4. *Let $f(u, v)$ be an orthogonal parametrization (that is $F = 0$) of a neighbourhood of an orientated surface S, and $w(t)$ be a differentiable field of unit vectors along the curve $f(u(t), v(t))$. Then*

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left\{G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right\} + \frac{d\varphi}{dt},$$

where $\varphi(t)$ is the angle from f_u to $w(t)$ in the given orientation.

Theorem 4.3.5 (Liouville)**.** *Let $\alpha(s)$ be a parametrization by arc length of a neighbourhood of a point $p \epsilon S$ of a regular orientated curve C on S. Let $f(u, v)$ be an orthogonal parametrization of S in p and $\varphi(s)$ be the angle that f_u makes with $\alpha'(s)$ in the given orientation. Then*

$$k_g = (k_g)_1 \cos\varphi + (k_g)_2 \sin\varphi + \frac{d\varphi}{ds},$$

where $(k_g)_1$ and $(k_g)_2$ are the geodesic curvatures of the coordinate curves $v = const.$ and $u = const.$ respectively.

Theorem 4.3.6. *Given a point $p \epsilon S$ and a vector $w \epsilon T_p(S)$, $w \neq 0$, there exist an $\in > 0$ and a unique parametrirized geodesic $\gamma : (-\in, \in) \to S$ such that $\gamma(0) = p$, $\gamma'(0) = w$.*

4.4 The Gauss-Bonnet Theorem and its Applications

The Gauss–Bonnet theorem or Gauss–Bonnet formula in differential geometry is an important statement about surfaces which connects their geometry (in the sense of curvature) to their topology (in the sense of the Euler characteristic).

Definition 4.4.1. Let $\alpha : [0, l] \to S$ be a continous map from the closed interval $[0, l]$ into the regular surface S. We say that α is a simple, closed, piecewise regular, parametrized curve if

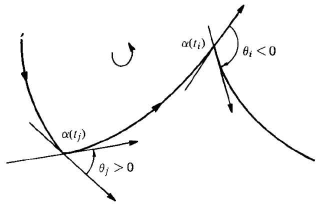


Figure 4.7: The sign of external angle.

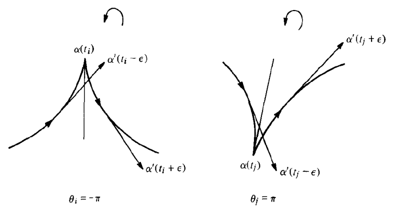


Figure 4.8: The sign of external angle in case of a cusp.

1. $\alpha(0) = \alpha(l)$
2. $t_1 \neq t_2, t_1, t_2 \,\epsilon\, [0, l)$, implies that $\alpha(t_1) \neq \alpha(t_2)$.
3. There exists a subdivision
$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$$
of $[0, l]$ such that α is differentiable and regular in each $[t_i, t_{i+1},]$ $i = 0, .., k$.

The points $\alpha(t_i)$, $i = 0, ..k$, are called the vertices of α and the traces $\alpha([t_i, t_{i+1}])$ are called the regular arcs of α. For each vertex $\alpha(t_i)$ we define

$$\lim_{t \to t_i} \alpha'(t) = \alpha'(t_i - 0) \neq 0 \ \ for\ t < t_i,$$

and

$$\lim_{t \to t_i} \alpha'(t) = \alpha'(t_i + 0) \neq 0 \ \ for\ t > t_i.$$

Assume now that S is orientated and let $|\theta_i|$, $0 < |\theta_i| \leq \pi$, be the smallest determination of the angle from $\alpha'(t_i - 0)$ to $\alpha'(t_i + 0)$. If $|\theta_i| \neq \pi$, we give θ_i the sign of the determinant $(\alpha'(t_i - 0), \alpha'(t_i + 0), N)$. The signed angle θ_i, $-\pi < \theta_i < \pi$, is called the external angle at the vertex $\alpha(t_i)$. In the case that the vertex $\alpha(t_i)$ is a cusp, that is, $|\theta_i| = \pi$, we choose the sign of θ_i as follows. By the condition of regularity, we can see that there exists a number $\in' > 0$ such that the determinant $(\alpha'(t_i - \in), \alpha'(t_i + \in), N)$ does not change sign for all $0 < \in < \in'$. We give θ_i the sign of this determinant. Let $f : U \subset \mathbb{R}^2 \to S$ be a parametrization compatiable with the orientation of S. Assume further that U is homeomorphic to an open disc in the plane. Let $\alpha : [0, l] \to f(U) \subset S$ be a simple closed, piecewise regular, parametrized curve, with vertices $\alpha(t_i)$ and external angles θ_i $i = 0, .., k$. Let $\varphi_i : [t_i, t_{i+1}] \to R$ be differentiable functions which measure at each $t \,\epsilon\, [t_i, t_{i+1}]$ the positive angle from f_u to $\alpha'(t)$.

Theorem 4.4.1. *(Theorem of Turning Tangents) With the above notation*

$$\sum_{i=0}^{k} (\varphi_i(t_{i+1}) - \varphi_i(t_i)) + \sum_{i=0}^{k} \theta_i = \pm 2\pi,$$

where the sign plus or minus depends on the orientation of α.

Let S be an orientated surface. A region $R \subset S$(union of connected open set with its boundary) is called a **simple region** if R is homeomorphic to a disk and the boundary δR of R is the trace of a simple, closed, piecewise regular, parametrized curve $\alpha; I \to S$. We say that α is positively orientated if for each $\alpha(t)$, belonging to regular arc, the positive orthogonal basis $\{\alpha'(t), h(t)\}$ satisfies the condition that $h(t)$ "points towards" R; more precisely, for any curve $\beta; I \to R$ with $\beta(0) = \alpha(t)$ and $\beta'(0) \neq \alpha'(t)$, we have that $\langle \beta'(0), h(t) \rangle > 0$.

Let $f : U \subset \mathbb{R}^2 \to S$ be a parametrization of S compatible with the orientation and let $R \subset f(U)$ be a bounded region of S. Let h be a differentiable function on S, then

$$\int \int_{f^{-1}(R)} h(u, v)\sqrt{EG - F^2}\ dudv$$

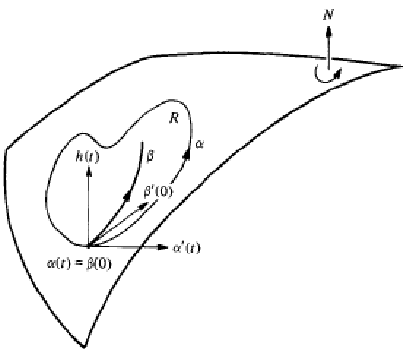


Figure 4.9: A positively oriented boundary curve.

is calLed the integral of h over the region R, and we denote it by

$$\iint_R h \, d\sigma.$$

Theorem 4.4.2. ***(Local Gauss-Bonnet Theorem.)*** *Let $f : U \to S$ be an orthogonal parametrization (that is, $F = 0$), of an orientated surface S, where $U \subset \mathbb{R}^2$ is homeomorphic to an open disk and f is compatible with the orientation of S. Let $R \subset f(U)$ be a simple region of S and let $\alpha : I \to S$ be such that $\delta R = \alpha(I)$. Assume that α is positively oriented, parametrized by arc length s, and let $\alpha(s_0), ..., \alpha(s_k)$ and $\theta_0, ..., \theta_k$ be, respectively, the vertices and the external angles of α. Then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) ds + \iint_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi$$

where $k_g(s)$ is the geodesic curvature of the regular arcs of α and K is the Gaussian curvature of S.

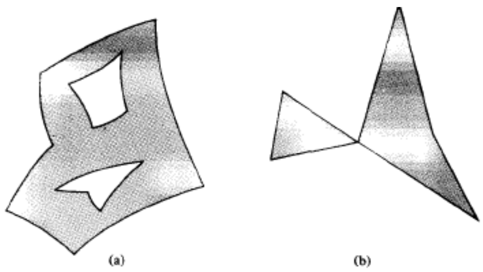


Figure 4.10: a)Regular region b)non-regular region.

Definition 4.4.2. A region $R \subset S$ is said to be **regular** if R is compact and its boundary δR is a finite union of (simple) closed piecewise regular curves which do not intersect. For convenience, we shall consider a compact surface as a regular region, the boundary of which is empty.

Definition 4.4.3. A simple region which has only three vertices with external angles $\alpha_i \neq 0$, $i = 1, 2, 3$, is called a **triangle.**

Definition 4.4.4. A **triangulation** of a regular region $R \subset S$ is a finite family τ of triangles $T_i, i = 1, 2, ..., n$ such that

1. $\cup_{i=1}^{n} T_i = R$
2. If $T_i \cap T_j \neq \phi$, then $T_i \cap T_j$ is either a common edge of T_i and T_j or a common vertex T_i and T_j.

Given a triangulation τ of a regular region $R \subset S$ of a surface S, we denote by F the number of triangles(faces), by E the number of sides(edges), and by V the number of vertices of the triangulation. The number

$$F - E + V = \chi$$

is called the Euler-Poincaré characteristic of the triangulation.

Theorem 4.4.3. *Every regular region of a regular surface admits a triangulation.*

Theorem 4.4.4. *Let S be an oriented surface and $\{f_\alpha\}$, $\alpha \epsilon A$, a family of parametrization compatible with the orientation of S. Let $R \subset S$ be a regular region of S. Then there is a triangulation τ of R such that every triangle $T \epsilon \tau$ is contained in some coordinate neighbourhood of the family $\{f_\alpha\}$. Furthermore, if the boundary of every triangle of τ is positively oriented, adjacent triangles determine opposite orientations in the common edge.*

Theorem 4.4.5. *If $R \subset S$ is a regular region of a surface S, the Euler Poincaré characteristic does not depend on the triangulation of R. It is denoted by $\chi(R)$.*

Theorem 4.4.6. *Let $S \subset \mathbb{R}^3$ be a compact connected surface; then one of the values $2, 0, -2, ..., -2n, ...$ is assumed by the Euler-Poincaré characteristic $\chi(S)$. Furthermore, if $S' \subset \mathbb{R}^3$ is another compact surface and $\chi(S) = \chi(S')$, then S is homeomorphic to S'.*

Definition 4.4.5. Let g be a differentiable function on S, the sum

$$\sum_{j=1}^{k} \int \int_{f^{-1}(T_i)} g(u_i, v_i)\sqrt{E_i G_i - F_i^2} du_j dv_j$$

does not depend on the triangulation τ or on the family $\{x_j\}$ of parametrizations of S. We denote it by

$$\int \int_R g \ d\sigma.$$

Theorem 4.4.7 (Global Gauss-Bonnet Theorem). *Let $R \subset S$ be a regular region of an oriented surface and let $C_1, ..., C_n$ be the closed, simple, piecewise regular curves which form the boundary δR of R. Suppose that each C_i is positively oriented and let $\theta_1, ..., \theta_p$ be the set of all external angles of the curves $C_1, ..., C_n$. Then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) ds + \int \int_R K d\sigma + \sum_{i=0}^{p} \theta_i = 2\pi\chi(R),$$

where s denotes the arc length of C_i, and the integral over C_i means the sum of integrals in every regular arc of C_i.

Corollary 4.4.7.1. *If R is a simple region of S, then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) ds + \int \int_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi,$$

Since a compact surface can be considered as a region with empty boundary, we have

Corollary 4.4.7.2. *Let S be a orientable compact surface; then*

$$\int \int_R K d\sigma = 2\pi\chi(S).$$

Applications of the Gauss-Bonnet theorem

1. A compact surface of positive curvature is homeomorphic to a sphere.

2. If T is a geodesic triangle in a oriented surface S. Let θ_1, θ_2, θ_3 be the external angles of T and let $\varphi_1 = \pi - \theta_1$, $\varphi_2 = \pi - \theta_2$, $\varphi_3 = \pi - \theta_3$ be its interior angles. By Gauss-Bonnet theorem,

$$\int\int_T K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi,$$

Thus,

$$\int\int_T K d\sigma = \sum_{i=0}^{k} \varphi_i - \pi,$$

It follows that the sum of interior angles, $\sum_{i=1}^{3} \varphi$, of a geodesic triangle is

(a) Equal to π if $K = 0$.

(b) Greater than π if $K > 0$.

(c) Smaller than π if $K < 0$.

References

1. [Doc] MANFREDO P. DO CARMO. Differential Geometry Of Curves and Surfaces.
2. [Dif] Victor Guillemin, Alan Pollack. Differential Topology.