Homology and Cohomology

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1 Introduction

The aim of this project is to get an understanding of basic concepts of homology and cohomology. The treatment of homology and cohomology in this report primarily follows Algebraic Topology by Allen Hatcher. All the figures used are also from the same book. To avoid overuse of the word 'continuous', we adopt the convention that maps between spaces are always assumed to be continuous unless stated otherwise.

1.1 Basic definitions

A **deformation retract** of a space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in I$, such that $f_0 = 1_X$ (the identity map), $f_1(X) = A$, and $f_t \mid A = 1_A$ for all $t \in I$. The family f_t should be continuous in the sense that the associated map $X \times I \to X$, $(x, t) \mapsto f_t(x)$, is continuous.

For a map $f : X \to Y$, the **mapping cylinder** M_f is the quotient space of the disjoint union $(X \times I) \coprod Y$ obtained by identifying each $(x, 1) \in X \times I$ with $f(x) \in Y$.

A deformation retraction $f_t : X \to X$ is a special case of the general notion of a **homotopy**, which is family of maps $f_t : X \to Y, t \in I$, such that the associated map $F : X \times I \to Y$ given by $F(x, t) = f_t(x)$ is continuous. Two maps $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy f_t connecting them. In these terms, a deformation retraction of X onto a subspace A is a homotopy from the identity map of X to a **retraction** of X onto A, that is a map $r : X \to X$ such that $r(X) = A$ and $r \mid A = 1_A$.

1.2 Operations on spaces

Cone The cone CX is the union of all line segments joining points of X to an external vertex, that is $CX = X \times I/X \times \{1\}$.

Suspension For space a X, the suspension SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point, that is $SX = CX/X \times \{0\}$

join Given topological spaces X and Y, join $X * Y$ is the quotient space of $X \times Y \times I$ under the identification $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. Thus we are joining each point of X to each point of Y by a line segment. In particular if $Y = \{p, q\}$ then $X * Y = SX$

Wedge sum Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \coprod Y$ obtained by identifying x_0 and y_0 to a single point.

Smash product Inside a product space $X \times Y$ there are copies of X and Y, namely $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $x_0 \in X$ $y_0 \in Y$. These two copies intersect only at the point (x_0, y_0), so their union can be identified with the wedge sum $X \vee Y$. The smash product $X \wedge Y$ is then defined to be quotient $X \times Y/X \vee Y$.

1.3 The Homotopy Extention property

Let $A \subset X$, then (X, A) has the **homotopy extenxion property** if given a map $f_0 : X \to Y$ and a homotopy $f_t : A \to Y$ of $f_0 \mid A$ we can give a homotopy $f_t : X \to Y$ of f_0. A pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is retract of $X \times I$. The homotopy extension property for (X, A) implies that the identity map $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ extends to a map $X \times I \to X \times \{0\} \cup A \times I$ so $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. Conversely, if there is a retraction $X \times I \to X \times \{0\} \cup A \times I$, then by composing with this retraction we can extend every map $X \times \{0\} \cup A \times I \to Y$ to a map $X \times I \to Y$.

2 Homology

2.1 CW complex, Δ-complex, Simplicial complex

All closed surfaces can be constructed from triangles by identifying edges. Using only triangles we can construct a large class of 2-dimensional spaces that are not surfaces in strict sense, by identifying more than two edges together. The idea of a Δ-complex is to generalise such consrtuctions in higher dimensions. The n-dimensional analogue of triangle is n-simplex. We first define a CW-complex, whose specific case is a Δ-complex.

An n-cell is a topological space homeomorphic to closed n-dimensional disk (D^n). A topological space X is said to be **CW complex** if it can be constructed in following manner.
(1) Start with a discrete set X^0, whose points are regarded as 0-cells.
(2) Inductively, form n-skeleton X^n from X^{n-1} by attaching n-cells D^n_α via maps ϕ_α:$\partial D^n \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union of X^{n-1} with a collection of $n-$disks D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$, i.e. $X^{n-1} \coprod_\alpha D^n_\alpha / x \sim \phi_\alpha(x)$ for all $x \in \partial D^n_\alpha$

This inductive process can stop at finite stage or can continue indefinitely. In latter case X is given weak topology: A set $A \subset X$ is open(or closed) if and only if $A \cap X^n$ is open(or closed) in X^n for each n.

Example The sphere S^n has the structure of a cell complex with just two cells, D^0 and D^n, the n-cell being attached by the constant map $\partial D^n \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$. An **n-simplex** is the smallest convex set in Euclidean space $\mathbb{R}^m$ containing n+1 points $v_0, ..., v_n$ that do not lie in a hyperplane of dimension less than n. Here hyperplane means set of solutions of linear equations. An equivalent condition is that the vectors $v_1 - v_0, ..., v_n - v_0$ are linearly independent.

An n-simplex will be denoted by $[v_0, ..., v_n]$.The points v_i are called the **vertices** of an n-simplex. If we delete one of the $n+1$ vertices of an n-simplex, then the remaining n vertices span an (n-1)-simplex, called a **face** of n-simplex. For example **standard n-simplex**

$$\triangle^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}; \sum_i x_i = 1 \ and \ x_i \leq 0 \ for \ all \ i\}$$

whose vertices are the unit vectors along the coordinate axes.
A **Δ-complex** is a quotient space of disjoint union of simplices under identifications of some of their faces via canonical linear homeomorphisms preserving ordering of vertices, i.e. a homeomorphism $\phi : [v_0, ..., v_n] \to [w_0, ..., w_n]$given by $\phi(\sum t_i v_i) = \sum t_i(w_i)$.
Δ complexes are also called **semi-simplicial complexes** and can be combinatorially defined

as follows- A semi-simplicial complex K is a collection of elements $\{\sigma\}$ called simplices together with two functions. The first function associates with each simplex σ an integer $q \geq 0$ called the dimension of σ; we then say that σ is a q-simplex. The second function associates with each q-simplex $\{\sigma\}$ $q > 0$ of K and with each integer $0 \leq i \leq q$ a $(q-1)$-simplex $\sigma^{(i)}$ called the i^{th} face of σ, subject to the condition

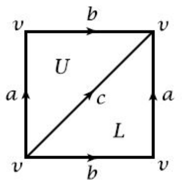
$$[\sigma^{(j)}]^{(i)} = [\sigma^{(i)}]^{(j-1)}$$

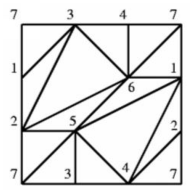
for $i < j$ and $q > 1$.

A **Simplicial Complex** K consists of a set $V = \{v_j\}$ of vertices and a set S=$\{s_j\}$ of finite non-empty subsets of $\{v\} = V$ called simplices such that
(1) Any set consisting of exactly one vertex is a simplex.
(2) Any non-empty subset of simplex is a simplex.
Simplex K containing $q+1$ vertices is a q simplex. We say that dimension of K is q and we write $dim(K) = q$. If $K' \subseteq K$, then K' is called **face** of K and if K' is p simplex, it is called a **p-face** of K.

Triangulation of X is a simplicial complex K with a homeomorphism $h : X \to K$.

Example: The figure below on the left shows the Δ-complex structure of torus and the figure on the right shows the simplicial complex structure of torus after appropriate identification of the edges of the square.





There are CW complexes which cannot be triangulated, (see [4])

2.2 Simplicial Homology

Now we shall define simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ be the free abelian group with basis the open $n-simplices$ e^n_α of X. Elements of $\Delta_n(X)$, called **n-chains** and can be written as finite formal sums $\sum_\alpha n_\alpha e^n_\alpha$ with $n_\alpha \in \mathbb{Z}$.
For a general Δ-complex X, a **boundary homomorphism** $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by specifying its value on basis elements, σ_α.

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha \mid [v_0, ..., \hat{v}_i, ..., v_n]$$

Lemma 1. *The composition* $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ *is zero.*

Proof. We have $\partial_n(\sigma) = \sum_i (-1)^i \sigma_\alpha \mid [v_0, ..., \hat{v}_i, ...v_n]$

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i(-1)^j \sigma_\alpha \mid [v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_n]$$
$$+ \sum_{j>i} (-1)^i(-1)^{j-1} \sigma_\alpha \mid [v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_n]$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. □

Now we have a sequence of homomorphisms of abelian groups

$$... \to \Delta_{n+1} \xrightarrow{\partial_{n+1}} \Delta_n \xrightarrow{\partial_n} \Delta_{n-1} \to ... \to \Delta_1 \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for each n. Such a sequence is called a **chain complex**. The n^{th}**homology group** of the chain complex is the quotient group
$H_n(X) = Ker \partial_n / Im\ \partial_{n+1}$. Elements of $Ker\ \partial_n$ are called **cycles** and elements of $Im\ \partial_{n+1}$ are called **boundaries**.

2.3 Singular Homology

A **singular n-simplex** in a space X is a map $\sigma : \Delta^n \to X$. Let $C_n(X)$ be the free abelian group with the basis the singular n-simplices in X. A boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined as above,

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \mid [v_0, ..., \hat{v}_i, ..., v_n]$$

Similarly we define singular chain complex

$$... \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to ... \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0$ for each n and n^{th} **singular homology group** of the singular chain complex to be the quotient group $H_n(X) = Ker \partial_n / Im \partial_{n+1}$.

Lemma 2. *Corresponding to the decomposition of a space X into its path-components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\bigoplus_\alpha H_n(X_\alpha)$.*

Proof. Since a simplex always has path-connected image, $C_n(X)$ splits in direct sum of $C_n(X_\alpha)$. The boundary maps ∂_n preserves this direct sum decomposition, taking $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, and similarly the $Ker \partial_n$ and $Im\ \partial_{n+1}$ split as direct sums, hence homology groups also split, $H_n(x) \cong \bigoplus_\alpha H_n(X_\alpha)$. □

Lemma 3. *If X is non empty and path-connected, then $H_0(X) \cong \mathbb{Z}$. Hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s,one for each path-component of X.*

Proof. Since $\partial_0 = 0$, $H_0(X) = C_0(X)/Im\ \partial_1$. Let

$$\varepsilon : C_0(X) \to \mathbb{Z}$$

$$\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$$

This is surjective if X is non empty. The claim is that $Im\ \partial_1 = Ker\ \varepsilon$, if X is path connected. Observe that $Im\ \partial_1 \subset Ker\ \varepsilon$, since for a singular 1-simplex $\sigma : \Delta^1 \to X$, $\varepsilon\partial_1(\sigma) = \varepsilon(\sigma \mid [v_1] - \sigma \mid [v_0]) = 1 - 1 = 0$. Suppose $\varepsilon(\sum_i n_i\sigma_i) = 0$, so $\sum_i n_i = 0$. The $\sigma_i s$ are singular 0-simplices, which are points of in X. Choose a path $\tau_i : I \to X$ from a basepoint x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0. Then τ_i can be viewed as a singular 1-simplex, a map $\tau_i : [v_0, v_1] \to X$, and then $\partial\tau_i = \sigma_i - \sigma_0$. Hence $\partial(\sum_i n_i\tau_i) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$ since $\sum_i n_i = 0$. Thus $\sum_i n_i\sigma_i$ is a boundary. Hence $Ker\ \varepsilon \subset Im\ \partial_1$. □

Lemma 4. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$.*

Proof. : In this case there is a unique singular n-simplex σ_n for each n and $\partial(\sigma_n) = \sum_i (-1)^i \sigma_{n-1}$, a sum of $n+1$ terms, which is therefore 0 for n odd and σ_{n-1} for n even, $n \neq 0$. Thus the chain complex with boundary maps alternately isomorphisms and trivial maps, except at last $\mathbb{Z}$.

$$... \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Clearly homology groups for this complex are trivial except for $H_0 \cong \mathbb{Z}$. □

There is a slightly modified version of homology, in which a point has trivial homology groups in all dimensions. This is done by defining **reduced homology groups** $\tilde{H}_n(X)$ to be homology groups of the augmented chain complex

$$... \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon(\sum_i n_i\sigma_i) = \sum_i n_i$. Here we want X to be non empty, to avoid having nontrivial homology group in dimension -1. Since $\varepsilon\partial_1 = 0$, ε vanishes on $Im\partial_1$ and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(X)$, so $H_0(X) \cong \tilde{H}_0(X) \bigoplus \mathbb{Z}$. Since the chain complex is same as augmented chain complex for $n > 0$, $H_n \cong \tilde{H}_n(X)$.

2.3.1 Functoriality

For a map $f : X \to Y$, an induced homomorphism $f' : C_n(X) \to C_n(Y)$ is defined by composing each singular n-simplex $\sigma : \Delta^n \to X$ with f to get a singular n-simplex $f'(\sigma) = f\sigma : \Delta^n \to Y$, and then extending it linearly via $f'(\sum_i n_i\sigma_i) = \sum_i n_i f\sigma_i$.

Lemma 5. *With f' defined as above, $f'\partial = \partial f'$.*

Proof.

$$\begin{aligned} f'\partial(\sigma) &= f'(\sum_i (-1)^i \sigma \mid [v_0, ..., \hat{v}_i, ..., v_n] \\ &= \sum_i (-1)^i f\sigma \mid [v_0, ..., \hat{v}_i, ..., v_n] = \partial f'(\sigma) \end{aligned}$$

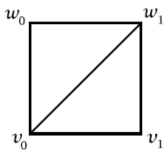
□

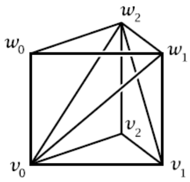
This implies that f' takes cycles to cycles and boundaries to boundaries. Hence f' induces a homomorphism $f_* : H_n(X) \to H_nY$. Two properties of this induced homomorphism are
(i) $(fg)_* = f_*g_*$ for a composed mapping $X \xrightarrow{g} Y \xrightarrow{f} Z$. This follows from associativity of composition $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$.
(ii) $id_* = id$, where id denotes identity map of a space or a group.

2.3.2 Homotopy Invariance

Theorem 1. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

Proof. The essential ingredient is a procedure for subdividing $\Delta_n \times I$ into $\Delta_{(n+1)}$ simplices. The figure shows the cases $n = 1, 2$.





In $\Delta^n \times I$, let $\Delta^n \times \{0\} = [v_0, \ \cdots, \ v_n]$ and $\Delta^n \times \{1\} = [w_0, \ \cdots, \ w_n]$, We can go from $[v_0, \ \cdots, \ v_n]$ to $[w_0, \ \cdots, \ w_n]$ by interpolating a sequence of n-simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. Thus the first step is to move $[v_0, \ \cdots, \ v_n]$ up to $[v_0, \ \cdots, \ v_{n-1}, \ w_n]$, then the second step is to move this up to $[v_0, \ \cdots, \ v_{n-2}, \ w_{n-1}, \ w_n]$, and so on. In the typical step $[v_0, \ \cdots, \ v_i, \ w_{i+1}, \ \cdots, \ w_n]$ moves up to $[v_0, \ \cdots, \ v_{i-1}, \ w_i, \ \cdots, \ w_n]$. The region between these two n-simplices is exactly $[v_0, \ \cdots, \ v_i, \ w_i, \ \cdots, \ w_n]$ which has $[v_0, \ \cdots, \ v_i, \ w_{i+1}, \ \cdots, \ w_n]$ as its lower face and $[v_0, \ \cdots, \ v_{i-1}, \ w_i, \ \cdots, \ w_n]$ as its upper face. Altogether, $\Delta^n \times I$ is the union of the $(n+1)$-simplices $[v_0, \ \cdots, \ v_i, \ w_i, \ \cdots, \ w_n]$, each intersecting the next in an n-simplex face. Given a homotopy $F : X \times I \to Y$ from f to g and a singular simplex $\sigma : \triangle^n \to X$, we can form the composition $F \circ (\sigma \times id) : \Delta^n \times I \to X \times I \to Y$. Using this, we can define *prism operators* $P : C_n(X) \to C_{n+1}(Y)$ by the following formula:

$$P(\sigma) = \sum_i F(\sigma \times id)|[v_0, \ \cdots, \ v_i, \ w_i, \ \cdots, \ w_n]$$

This prism operators satisfy the basic relation

$$\partial P = g' - f' - P\partial$$

This relationship is expressed by saying P is a **chain homotopy** between the chain maps f' and g'. Geometrically, the left side of this equation represents the boundary of the prism, and the three terms on the right side represent the top $\Delta^n \times \{1\}$, the bottom $\Delta^n \times \{0\}$, and the sides $\partial\Delta^n \times I$ of the prism. If $\alpha \in C_n(X)$ is a cycle, then we have $g'(\alpha) - f'(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$ since $\partial\alpha = 0$. Thus $g'(\alpha) - f'(\alpha)$ is a boundary, so $g'(\alpha)$ and $f'(\alpha)$ determine the same homology class, which means that g_* equals f_* on the homology class of α. □

2.3.3 Exactness

Given a space X and a subspace $A \subset X$. Let $C_n(X, A) = C_n(X)/C_n(A)$. Since the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$. Letting n vary we get a chain complex and homology groups of this chain complex are called relative homology groups, $H_n(X, A)$.

Theorem 2. *For any pair (X, A), we have a long exact sequence*

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \ldots \to H_0(X, A) \to 0$$

Proof. Here i_* is the map induced by inclusion map of A into X and j_* is the map induced by the quotient map from $C(X)$ to $C(X)/C(A)$. To define the boundary map $\partial : H_n(X, A) \to H_{n-1}(A)$, let $c \in C_n(X, A)$ be a cycle. Since j is onto, $c = j(b)$ for some $b \in C_n(X)$. The element $\partial b \in C_{n-1}(X)$ is in Ker$j$ since $j(\partial b) = \partial j(b) = \partial c = 0$ in $C_n(X, A)$. So $\partial b = i(a)$ for some $a \in C_{n-1}(A)$ since Ker$j$ = Imi. We define $\partial : H_n(X, A) \to H_{n-1}(A)$ by sending the homology class of c to the homology class of a, $\partial[c] = [a]$ Using these maps, one can check that the above sequence is indeed a long exact sequence. □

2.3.4 Excision

Lemma 6. *The inclusion $i : C_n^{\mathfrak{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho : C_n(X) \to C_n^{\mathfrak{U}}(X)$ such that $i\rho$ and ρi are chain homotopic to identity. Hence i induces isomorphisms $H_n^{\mathfrak{U}}(X) \approx H_n(X)$ for all n.*

This lemma can be proved using barycentric subdivision.

Theorem 3. *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X - Z, X - A) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \to H_n(X, A)$ for all n. Equivalently, for subspaces $A, B \subset X$ whose interiors covers X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$*

The translation between the two versions is obtained by setting $B = X - Z$. For a space X, let $\mathfrak{U} = \{U_\alpha\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $C_n^{\mathfrak{U}}(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i\sigma_i$ such that each σ_i has image contained in some set in the cover $\mathfrak{U}$. The boundary map $\partial : C_n(X) \to C_n(X)$ takes $C_n^{\mathfrak{U}}(X)$ to $C_{n-1}^{\mathfrak{U}}(X)$, so the groups $C_n^{\mathfrak{U}}(X)$ form a chain complex. We denote the homology groups of this chain complex by $H_n^{\mathfrak{U}}(X)$. Using this lemma we can prove the second equivalent condition by decomposing $X = A \cup B$ and applying the lemma for the cover $\mathfrak{U} = \{A, B\}$

2.3.5 The equivalence of simplicial and singular homology

Let X be a Δ-complex with $A \subset X$ a sub-complex. Thus A is the Δ-complex formed by any union of simplices of X. Relative groups $H_n^{\Delta}(X, A)$ can be defined in the same way as for singular homology, via relative chains $\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A)$, and this yields a long exact sequence of simplicial homology groups for the pair (X, A) by the same algebraic argument as for singular homology. There is a canonical homomorphism $H_n^{\Delta}(X, A) \to H_n(X, A)$ induced by the chain map $\Delta_n(X, A) \to C_n(X, A)$ sending each n-simplex of X to its characteristic map $\sigma : \Delta^n \to X$. The possibility $A = \emptyset$ is not excluded, in which case the relative groups reduce to absolute groups.

Theorem 4. *The homomorphisms $H_n^{\Delta}(X, A) \to H_n(X, A)$ are isomorphisms for all n and all Δ-complex pairs (X, A).*

Proof. : First we do the case that X is finite-dimensional and A is empty. For X^k the k-skeleton of X, consisting of all simplices of dimension k or less, we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
H_{n+1}^{\Delta}(X^k, X^{k-1}) & \longrightarrow & H_n^{\Delta}(X^{k-1}) & \longrightarrow & H_n^{\Delta}(X^k) & \longrightarrow & H_n^{\Delta}(X^k, X^{k-1}) & \longrightarrow & H_{n-1}^{\Delta}(X^{k-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

Let us first show that the first and fourth vertical maps are isomorphisms for all n. The simplicial chain group $\Delta_n(X^k, X^{k-1})$ is zero for $n \neq k$, and is free abelian with basis the k-simplices of X when $n = k$. Hence $H_n^{\Delta}(X^k, X^{k-1})$ has exactly the same description. The corresponding singular homology groups $H_n(X^k, X^{k-1})$ can be computed by considering the map $\Phi : \coprod_\alpha (\triangle_\alpha^k, \partial\triangle_\alpha^k) \to (X^k, X^{k-1})$ formed by the characteristic maps $\triangle^k \to X$ for all the k-simplices of X. Since Φ induces a homeomorphism of quotient spaces $\coprod_\alpha \triangle_\alpha^k / \coprod_\alpha \partial\triangle_\alpha^k \approx X^k/X^{k-1}$, it induces isomorphisms on all singular homology groups. Thus $H_n(X^k,\ X^{k-1})$ is zero for $n \neq k$, while for $n = k$ this group is free abelian with basis represented by the relative cycles given by the characteristic maps of all the k-simplices of X, in view of the fact that $H_k(\triangle^k,\ \partial\triangle^k)$ is generated by the identity map $\triangle^k \to \triangle^k$. Therefore the map $H_k^{\triangle}(X^k,\ X^{k-1}) \to H_k(X^k,\ X^{k-1})$ is an isomorphism.

By induction on k we may assume the second and fifth vertical maps in the preceding diagram are isomorphisms as well. Then by five lemma the middle vertical map is an isomorphism, finishing the proof when X is finite-dimensional and $A = \emptyset$. Infinite-dimensional part follows if we consider the fact: A compact set in X meets only finitely many open simplices of X, that is, simplices with their proper faces deleted, so each cycle lies in some X^k. □

2.4 Cellular homology

Cellular homology is a very efficient tool for computing the homology groups of CW complexes, based on degree calculations. Before giving the definition of cellular homology, we first establish a few preliminary facts. For a map $f : S^n \to S^n$ with $n > 0$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = d\alpha$ for some integer d depending only on f. This integer is called the degree of f, with the notation $\deg f$. Here are some basic properties of degree:

(a) $\deg id = 1$, since $id_* = id$.

(b) $\deg f = 0$ if f is not surjective. For if we choose a point $\mathrm{x}_0 \in S^n - f(S^n)$ then f can be factored as a composition $S^n \to S^n - \{\mathrm{x}_0\} \hookrightarrow S^n$ and $H_n(S^n - \{\mathrm{x}_0\}) = 0$ since $S^n - \{\mathrm{x}_0\}$ is contractible. Hence $f_* = 0$.

(c) If $f \simeq g$ then $\deg f = \deg g$ since $f_* = g_*$.

(d) $\deg fg = \deg f \deg g$, since $(fg)_* = f_* g_*$. As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence since $fg \simeq id$ implies $\deg f \deg g = \deg id = 1$.

(e) $\deg f = -1$ if f is a reflection of S^n, fixing the points in a subsphere S^{n-1} and interchanging the two complementary hemispheres. For we can give S^n a Δ-complex structure with these two hemispheres as its two n-simplices Δ_1^n and Δ_2^n, and the n-chain $\Delta_1^n - \Delta_2^n$ represents a generator of $H_n(S^n)$ so the reflection interchanging Δ_1^n and Δ_2^n sends this generator to its negative.

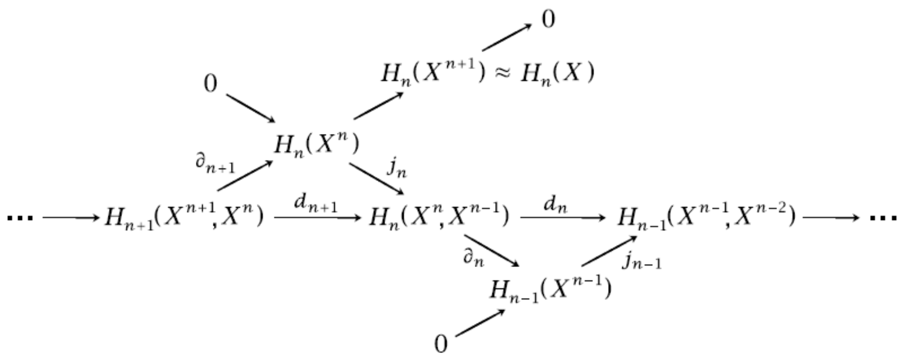
(f) The antipodal map $-id : S^n \to S^n, \mathrm{x} \mapsto -\mathrm{x}$, has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections in R^{n+1}, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$.

(g) If $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$. For if $f(\mathrm{x}) \neq \mathrm{x}$ then the line segment from $f(\mathrm{x})$ to $-\mathrm{x}$, defined by $t \mapsto (1-t)f(\mathrm{x}) - t\mathrm{x}$ for $0 \leq t \leq 1$, does not pass through the origin. Hence if f has no fixed points, the formula $f_t(\mathrm{x}) = [(1-t)f(\mathrm{x}) - t\mathrm{x}]/|(1-t)f(\mathrm{x}) - t\mathrm{x}|$ defines a homotopy from f to the antipodal map, which has degree $(-1)^{n+1}$

One can prove the following facts about a CW complex X

1. $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.
2. $H_k(X)^n$=0 for $k > n$. If X is finite-dimensional then $H_k(X)$=0 for $k > dim(X)$.
3. The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Let X be a CW complex. Using the above facts, portions of the long exact sequences for the pairs $(X^{n+1},\ X^n), (X^n,\ X^{n-1})$, and $(X^{n-1},\ X^{n-2})$ fit into a diagram



where d_{n+1} and d_n are defined as the compositions $j_n\prime\partial_{n+1}$ and $j_{n-1}\prime\partial_n$, which are just 'relativizations' of the boundary maps ∂_{n+1} and ∂_n. The composition $d_n d_{n+1}$ includes two successive maps in one of the exact sequences, hence is zero. Thus the horizontal row in the diagram is a chain complex, called the cellular chain complex of X since $H_n(X^n,\ X^{n-1})$ is free with basis in one-to-one correspondence with the n-cells of X, so one can think of elements of $H_n(X^n,\ X^{n-1})$ as linear combinations of n-cells of X. The homology groups of this cellular chain complex are called the cellular homology groups of X. We denote them by $H_n^{CW}(X)$.
Example: Considering the cell structure of S^n given above it is easy to check that $H_n(S^n) \cong H_0(S^n) \cong (Z)$ and $H_i(S^n) = 0$ otherwise.

Theorem 5. $H_n^{CW}(X) \cong H_n(X)$.

Proof. :From the diagram above, $H_n(X)$ can be identified with $H_n(X^n)/\mathrm{Im}\partial_{n+1}$. Since j_n is injective, it maps $\mathrm{Im}\partial_{n+1}$ isomorphically onto $\mathrm{Im}(j_n\partial_{n+1}) = \mathrm{Im}d_{n+1}$ and $H_n(X^n)$ isomorphically onto $Imj_n = Ker\partial_n$. Since j_{n-1} is injective, $Ker\partial_n = Kerd_n$. Thus j_n induces an isomorphism of the quotient $H_n(X^n)/Im\partial_{n+1}$ onto $Kerd_n/Imd_{n+1}$. □

Example : Real Projective Space $\mathbb{R}\mathrm{P}^n$ has a CW structure with one cell e^k in each dimension $k \leq n$, and the attaching map for e^k is the 2-sheeted covering projection $\varphi : S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$. To compute the boundary map d_k we compute the degree of the composition $S^{k-1} \xrightarrow{\varphi} \mathbb{R}\mathrm{P}^{k-1} \xrightarrow{q} \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} = S^{k-1}$, with q the quotient map. The map $q\varphi$ is a homeomorphism when restricted to each component of $S^{k-1} - S^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1}, which has degree $(-1)^k$. Hence $\deg q\varphi = \deg id + \deg(-id) = 1 + (-1)^k$, and so d_k is either 0 or multiplication by 2 according to whether k is odd or even. Thus the cellular chain complex for $\mathbb{R}\mathrm{P}^n$ is

$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \ldots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$ if n is even

$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \ldots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$ if n is odd

From this it follows that

$$H_k(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z} \text{ for } k\text{=0 and for } k = n \text{ odd} \\ \mathbb{Z}_2 \text{ for } k \text{ odd, } 0 < k < n \\ 0 \text{ otherwise} \end{cases}$$

Example : Lens Spaces. Given an integer $m > 1$ and integers $\ell_1, \cdots, \ell_n$ relatively prime to m, define the lens space $L = L_m(\ell_1,\ \cdots,\ \ell_n)$ to be the orbit space $S^{2n-1}/\mathbb{Z}_m$ of the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ with the action of $\mathbb{Z}_m$ generated by the rotation $\rho(z_1,\ \cdots,\ z_n) = (e^{2\pi\iota\ell_1/m}z_1,\ \cdots,\ e^{2\pi\iota\ell_n/m}z_n)$, rotating the j^{th} factor of $\mathbb{C}^n$ by the angle $2\pi\ell_j/m$. In particular, when $m = 2, \rho$ is the antipodal map, so $L = \mathbb{R}\mathrm{P}^{2n-1}$ in this case. In the general case, the projection $S^{2n-1} \to L$ is a covering space since the action of $\mathbb{Z}_m$ on S^{2n-1} is free: Only the identity element fixes any point of S^{2n-1} since each point of S^{2n-1} has some coordinate z_j nonzero and then $e^{2\pi ik\ell_j/m} z_j \neq z_j$ for $0 < k < m$, as a result of the assumption that ℓ_j is relatively prime to m.

We shall construct a CW structure on L with one cell e^k for each $k \leq 2n-1$ and show that the resulting cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with boundary maps alternately 0 and multiplication by m. Hence

$$H_k(L_m(\ell_1,\ \cdots,\ \ell_n)) = \begin{cases} \mathbb{Z} \text{ for } k\text{=0, 2n-1} \\ \mathbb{Z}_m \text{ for } k \text{ odd, } 0 < k < 2n-1 \\ 0 \text{ otherwise} \end{cases}$$

To obtain the CW structure, first subdivide the unit circle C in the $n^{th} factor of$ $\mathbb{C}^n$ by taking the points $e^{2\pi \imath j/m} \in C$ as vertices, $j = 1, \cdots, m$. Joining the j^{th} vertex of C to the unit sphere $S^{2n-3} \subset \mathbb{C}^{n-1}$ by arcs of great circles in S^{2n-1} yields a$(2n-2)$-dimensional ball $B^{2n-2}_{j^l}$ bounded by S^{2n-3}. Specifically, $B^{2n-2}_{j^l}$ consists of the points $\cos\theta(0,\ \cdots,\ 0,\ e^{2\pi ij/m})+$ $\sin\theta(z_1,\ \cdots,\ z_{n-1}, 0)$ for $0 \leq \theta \leq \pi/2$. Similarly, joining the j^{th} edge of C to S^{2n-3} gives a ball $B^{2n-1}_{j^l}$ bounded by $B^{2n-2}_{j^l}$ and $B^{2n-2}_{j^l+1}$, subscripts being taken mod m. The rotation ρ carries S^{2n-3} to itself and rotates C by the angle $2\pi\ell_n/m$, hence ρ permutes the $B^{2n-2}_{j^l}$s and the $B^{2n-1}_{j^l}$s. A suitable power of ρ, namely ρ^r where r$\ell_n \equiv 1$mod m, takes each $B^{2n-2}_{j^l}$ and $B^{2n-1}_{j^l}$ to the next one. Since ρ^r has order m, it is also a generator of the rotation group $\mathbb{Z}_m$, and hence we may obtain L as the quotient of one $B^{2n-1}_{j^l}$ by identifying its two faces $B^{2n-2}_{j^l}$ and $B^{2n-2}_{j^l+1}$ together via ρ^r

Observe that the $(2n-3)$-dimensional lens space $L_m(\ell_1,\ \cdots,\ \ell_{n-1})$ sits in $L_m(\ell_1,\ \cdots,\ \ell_n)$ as the quotient of S^{2n-3}, and $L_m(\ell_1,\ \cdots,\ \ell_n)$ is obtained from this subspace by attaching two cells, of dimensions $2n-2$ and $2n-1$, coming from the interiors of $B^{2n-1}_{j^l}$ and its two identified faces $B^{2n-2}_{j^l}$ and $B^{2n-2}_{j^l+1}$. Inductively this gives a CW structure on $L_m(\ell_1,\ \cdots,\ \ell_n)$ with one cell e^k in each dimension $k \leq 2n-1$.

The boundary maps in the associated cellular chain complex are computed as follows. The first one, d_{2n-1}, is zero since the identification of the two faces of $B^{2n-1}_{j^l}$ is via a reflection (degree -1) across $B^{2n-1}_{j^l}$ fixing S^{2n-3}, followed by a rotation (degree $+1$), so $d_{2n-1}(e^{2n-1}) = e^{2n-2} - e^{2n-2} = 0$. The next boundary map d_{2n-2} takes e^{2n-2} to me^{2n-3} since the attaching map for e^{2n-2} is the quotient map $S^{2n-3} \to L_m(\ell_1,\ \cdots,\ \ell_{n-1})$ and the balls $B^{2n-3}_{j^l}$ in S^{2n-3} which project down onto e^{2n-3} are permuted cyclically by the rotation ρ of degree $+1$. Inductively, the subsequent boundary maps d_k then alternate between 0 and multiplication by m.

Also of interest are the infinite-dimensional lens spaces $L_m(\ell_1,\ \ell_2,\ \cdots) = S^\infty/\mathbb{Z}_m$ defined in the same way as in the finite-dimensional case, starting from a sequence of integers $\ell_1, \ell_2, \cdots$ relatively prime to m. The space $L_m(\ell_1,\ \ell_2,\ \cdots)$ is the union of the increasing sequence of finite-dimensional lens spaces $L_m(\ell_1,\ \cdots,\ \ell_n)$ for $n = 1, 2, \cdots$, each of which is a subcomplex of the next in the cell structure we have just constructed, so $L_m(\ell_1,\ \ell_2,\ \cdots)$ is also a CW complex. Its cellular chain complex consists of a $\mathbb{Z}$ in each dimension with boundary maps alternately 0 and m, so its reduced homology consists of a $\mathbb{Z}_m$ in each odd dimension. The Mayer-Vietoris sequence is also applied frequently in induction arguments, where we know that a certain statement is true for A, B, and $A \cap B$ by induction and then

deduce that it is true for $A \cup B$ by the exact sequence.
Example Take $X = S^n$ with A and B the northern and southern hemispheres, so that $A \cap B = S^{n-1}$. Then in the reduced MayerVietoris sequence the terms $\tilde{H}_i(A) \oplus \tilde{H}_i(B)$ are zero, so we obtain isomorphisms $\tilde{H}_i(S^n) \approx \tilde{H}_{i-1}(S^{n-1})$. This gives another way of calculating the homology groups of S^n by induction.

2.5 Homology with coefficients

There is an generalization of the homology theory we have considered so far. The generalization consists of using chains of the form $\sum_i n_i\sigma_i$ where each σ_i is a singular n-simplex in X as before, but now the coefficients n_i lie in a fixed abelian group G rather than $\mathbb{Z}$. Such n-chains form an abelian group $C_n(X;G)$, and relative groups $C_n(X,A;G) = C_n(X;G)/C_n(A;G)$. The formula for the boundary maps ∂ for arbitrary G, is

$$\partial(\sum_i n_i\sigma_i) = \sum_{i,j}(-1)^j n_i\sigma_i|[v_0,\ \cdots\ ,\ \hat{v}_j,\ \cdots\ ,\ v_n]$$

A calculation shows that $\partial^2 = 0$, so the groups $C_n(X;G)$ and $C_n(X,A;G)$ form chain complexes. The resulting homology groups $H_n(X;G)$ and $H_n(X,A;G)$ are called
homology groups with coefficients in G. Reduced groups $\tilde{H}_n(X;G)$ are defined via the augmented chain complex $\ldots \to C_0(X;G) \xrightarrow{\epsilon} G \to 0$ with ϵ again defined by summing coefficients.
Example: Consider $G = \mathbb{Z}_2$, this is particularly simple since we only have sums of singular simplices with coefficients 0 or 1, so by discarding terms with coefficient 0 we can think of chains as just finite 'unions' of singular simplices. The boundary formulas also simplify since we no longer have to worry about signs. Since signs are an algebraic representation of orientation, we can also ignore orientations. This means that homology with $\mathbb{Z}_2$ coefficients is often the most natural tool in the absence of orientability.

All the theory we have for $\mathbb{Z}$ coefficients carries over directly to general coefficient groups G with no change in the proofs. Cellular homology also generalizes to homology with coefficients, with the cellular chain group $H_n(X^n,\ X^{n-1})$ replaced by $H_n(X^n,\ X^{n-1};G)$, which is a direct sum of G's, one for each n-cell. The proof that the cellular homology groups $H_n^{CW}(X)$ agree with singular homology $H(X)$ extends to give $H_n^{CW}(X;G) \approx H_n(X;G)$.

2.6 Axioms of Homology

A homology theory has to satisfy the following axioms.
1. **Functoriality** H_q is a functor from category of pairs of topological spaces to category of abelian groups.
2. **Homotopy axiom** If $f, g : (X,Y) \to (X',Y')$ are homotopic then they induce the same map between homology groups.
3. **Excision** Suppose (X,Y) is a pair and U is open in X and $\overline{U} \subset Y$ then the inclusion $i : (X-U, Y-U) \to (X,Y)$ induces isomorphism in all the $H_q s$.
4. **Exactness** $\forall (X,Y)$ $\exists$ a family of natural transformations
$\delta_q : H_q(X,Y) \to H_{q-1}(Y)$ such that the sequence shown below is exact

$$\ldots \to H_q(Y,\phi) \to H_q(X,\phi) \to H_q(X,Y) \xrightarrow{\delta_q} H_{q-1}(Y) \to \ldots$$

5. **Dimension axiom**

$$H_q(point, \phi) \begin{cases} = 0 \text{ if } q \neq 0 \\ \simeq \mathbb{Z} \text{ if } q = 0 \end{cases}$$

2.7 Relation between homology and homotopy groups

There is a close connection between $H_1(X)$ and $\pi_1(X)$, arising from the fact that a map $f : I \to X$ can be viewed as either a path or a singular 1-simplex. If f is a loop, with $f(0) = f(1)$, this singular 1-simplex is a cycle since $\partial f = f(1) - f(0)$.

Theorem 6. $H_1(X)$ *is abelianization of* $\pi_1(X)$

proof: The notation $f \simeq g$ is for the relation of homotopy, fixing endpoints, between paths f and g. Regarding f and g as chains, the notation $f \sim g$ will mean that f is homologous to g, that is, $f - g$ is the boundary of some 2-chain. Here are some facts about this relation.

(i) If f is a constant path, then $f \sim 0$. Namely, f is a cycle since it is a loop, and since $H_1(point) = 0, f$ must then be a boundary. Explicitly, f is the boundary of the constant singular 2-simplex σ having the same image as f since

$$\partial\sigma = \sigma|[v_1,\ v_2] - \sigma|[v_0,\ v_2] + \sigma|[v_0,\ v_1] = f - f + f = f$$

(ii) If $f \simeq g$ then $f \sim g$.
(iii) $f \cdot g \sim f + g$, where $f \cdot g$ denotes the product of the paths f and g. For if $\sigma : \Delta^2 \to X$ is the composition of orthogonal projection of $\Delta^2 = [v_0,\ v_1,\ v_2]$ onto the edge $[v_0,\ v_2]$ followed by $f \cdot g : [v_0,\ v_2] \to X$, then $\partial\sigma = g - f \cdot g + f$.
(iv) $\overline{f} \sim -f$, where $\overline{f}$ is the inverse path of f. This follows from the preceding three observations, which give $f + \overline{f} \sim f \cdot \overline{f} \sim 0$.

Applying (ii) and (iii) to loops, it follows that we have a well-defined homomorphism $h : \pi_1(X,\ \mathrm{x}_0) \to H_1(X)$ sending the homotopy class of a loop f to the homology class of the 1-cycle f. This h is surjective when X is path-connected and kernel is equal to commutator subgroup of $\pi_1(X)$

There is a similar theorem for higher homology and homotopy groups, Hurewicz theorem, which we will state without proof. A space X with base point x_0 is said to be **n-connected** if $\pi_i(X, x_0) = 0$ for $i \leq n$. Thus the first nonzero homotopy and homology groups of a simply-connected space occur in the same dimension and are isomorphic.

Theorem 7. *If a space* X *is (n-1)-connected,* $n > 1$*, then* $\tilde{H}_i(X) = 0$ *for* $i < n$ *and* $\pi_n(X) \approx H_n(X)$*. If a pair* (X, A) *is (n-1)-connected,* $n > 1$*, with* A *simply-connected and non-empty, then* $H_i(X, A) = 0$ *for* $i < n$ *and* $\pi(X, A) \approx H_n(X, A)$*.*

2.8 Simplicial approximation

Many spaces of interest in algebraic topology can be given the structure of simplicial complexes, which can be exploited later. One of the good features of simplicial complexes is that arbitrary continuous maps between them can always be deformed to maps that are linear on the simplices of some subdivision of the domain complex. This is the idea of simplicial approximation. Here is the relevant definition: If K and L are simplicial complexes, then a map $f : K \to L$ is **simplicial** if it sends each simplex of K to a simplex of L by a linear map taking vertices to vertices. Since a linear map from a simplex to a simplex is uniquely determined by its values on vertices, this means that a simplicial map is uniquely determined by its values on vertices. It is easy to see that a map from the vertices of K to the vertices of L extends to a simplicial map iff it sends the vertices of each simplex of K to the vertices of some simplex of L. Here is the most basic form of the Simplicial Approximation Theorem.

Theorem 8. *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f : K \to L$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.*

The simplicial approximation theorem allows arbitrary continuous maps to be replaced by homotopic simplicial maps in many situations, and one might wonder about the analogous question for spaces: Which spaces are homotopy equivalent to simplicial complexes? Following theorem answers this question.

Theorem 9. *Every CW complex X is homotopy equivalent to a simplicial complex, which can be chosen to be of the same dimension as X, finite if X is finite, and countable if X is countable.*

2.8.1 Lefschetz Fixed Point Theorem

This is generalisation of Brower's fixed point theorem. For a homomorphism $\phi : \mathbb{Z}^n \to \mathbb{Z}^n$ with matrix $[a_{ij}]$, the trace $tr\phi$ is defined to be $\Sigma_i a_{ii}$, the sum of the diagonal elements of $[a_{ij}]$. For a homomorphism $\phi : A \to A$ of a finitely generated abelian group A we can then define $tr\phi$ to be the trace of the induced homomorphism $\phi' : A/Torsion \to A/Torsion$. For a map $f : X \to X$ of a finite CW complex X, or more generally any space whose homology groups are finitely generated and vanish in high dimensions, the Lefschetz number $\tau(f)$ is defined to be $\Sigma_n(-1)^n tr(f_*) : H_n(X) \to H_n(X)$. In particular, if f is the identity, or is homotopic to the identity, then $\tau(f)$ is the Euler characteristic $\chi(X)$ since the trace of $n \times n$ identity matrix is n. Here is the Lefschetz fixed point theorem.

Theorem 10. *If X is a retract of a finite simplicial complex and $f : X \to X$ is a map with $\tau(f) \neq 0$, then f has a fixed point.*

Proof. : The general case easily reduces to the case of finite simplicial complexes, for suppose $r : K \to X$ is a retraction of the finite simplicial complex K onto X. For a map $f : X \to X$, the composition $fr : K \to X \subset K$ then has exactly the same fixed points as f. Since $r_* : H_n(K) \to H_n(X)$ is projection onto a direct summand, we clearly have $tr(f_*r_*) = trf_*$, so $\tau(f_*r_*) = \tau(f_*)$. For X a finite simplicial complex, suppose that $f : X \to X$ has no fixed points, then one can prove that there is a subdivision L of X, a further subdivision K of L, and a simplicial map $g : K \to L$ homotopic to f such that $g(\sigma) \cap \sigma = \emptyset$ for each simplex σ of K. The Lefschetz numbers $\tau(f)$ and $\tau(g)$ are equal since f and g are homotopic. The map g induces a chain map of the cellular chain complex $H_n(K^n, K^{n-1})$ to itself. This can be used to compute $\tau(g)$ according to the formula

$$\tau(g) = \Sigma_n(-1)^n tr(g_* : H_n(K^n, K^{n-1})) \to H_n(K^n, K^{n-1})$$

Using this one can calculate $\tau(g)$ which will come out to be 0, which is contradicting our hypothesis. □

3 Cohomology

Cohomology is the dualisation of homology. Homology groups $H_n(X)$ are the result of a two-stage process: First one forms a chain complex $... \to C_n \xrightarrow{\partial} C_{n-1} \to ...$ of singular, simplicial, or cellular chains, then one takes the homology groups of this chain complex $Ker\partial/Im\partial$. To obtain the cohomology groups $H^n(X;G)$ we interpolate an intermediate step, replacing the chain groups C_n by the dual groups $Hom(C_n, G)$ and the boundary maps ∂ by their dual maps δ, defined as $\delta(\sigma) = \sigma\partial$, before forming the cohomology groups $Ker\delta/Im\delta$.
A **free resolution** of an abelian group H is an exact sequence

$$... \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

with each F_n free. If we dualize this free resolution by applying $Hom(-, G)$, we may lose exactness, but at least we get a co-chain i.e. $f^*_{i+1} \circ f^*_i = 0$. This dual complex has the form

$$... \leftarrow F^*_2 \xleftarrow{f^*_2} F^*_1 \xleftarrow{f^*_1} F^*_0 \xleftarrow{f^*_0} H^* \leftarrow 0$$

Let $H^n(F;G)$ denote the homology group $\text{Ker}f^*_{n+1}/\text{Im}f^*_n$ of this dual complex.

Lemma 7. *(a)Given free resolutions F and F' of abelian groups H and H', then every homomorphism $\alpha : H \to H'$ can be extended to a chain map from F to F':*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow 0 \\
 & \downarrow{\alpha_2} & & \downarrow{\alpha_1} & & \downarrow{\alpha_0} & & \downarrow{\alpha} & \\
\cdots \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow 0
\end{array}$$

Any two such chain maps extending α are chain homotopic.
(b) For any two free resolutions F and F' of H, there are canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$ for all n.

We will use this lemma to prove universal coefficient theorem.

3.1 Universal coefficient Theorem

Universal Coefficient theorem or more specifically it's corollary gives us a method to calculate cohomology groups using homology groups. Foe which we need to know what $Ext(H, G)$ is. $Ext(H, G)$ has an interpretation as the set of isomorphism classes of extensions of G by H that is, short exact sequences $0 \to G \to J \to H \to 0$, with a natural definition of isomorphism between such exact sequences. Another interpretation is given after stating the theorem.

Theorem 11. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C_n, G)$ are determined by split exact sequences* $0 \to Ext(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} Hom(H_n(C), G) \to 0$

Consider the map $h : H^n(C;G) \to \text{Hom}(H_n(C),\ G)$, defined as follows. Denote the cycles and boundaries by $Z_n = \text{Ker}\partial \subset C_n$ and $B_n = \text{Im}\partial \subset C_n$. A class in $H^n(C;G)$ is represented by a homomorphism $\varphi : C_n \to G$ such that $\delta\varphi = 0$ ($\varphi\partial = 0$) or in other words φ vanishes on B_n. The restriction $\varphi_0 = \varphi|Z_n$ then induces a quotient homomorphism $\varphi_0 : Z_n/B_n \to G$, an element of $\text{Hom}(H_n(C), G)$. If φ is in $\text{Im}\delta$, say $\varphi = \delta\psi = \psi\partial$, then φ is zero on Z_n, so $\varphi_0 = 0$ and hence also $\varphi_0 = 0$. Thus there is a well-defined quotient map $h : H^n(C;G) \to \text{Hom}(H_n(C),\ G)$ sending the cohomology class of φ to $\overline{\varphi}_0$. One can check that h is a surjective homomorphism. Every abelian group H has a free resolution of the form $0 \to F_1 \to F_0 \to H \to 0$, with $F_i = 0$ for $i > 1$, obtainable in the following way. Choose a set of generators for H and let F_0 be a free abelian group with basis in one-to-one correspondence with these generators. Then we have a surjective homomorphism $f_0 : F_0 \to H$ sending the basis elements to the chosen generators. The kernel of f_0 is free, being a subgroup of a free abelian group, so we can let F_1 be this kernel with $f_1 : F_1 \to F_0$ the inclusion, and we can then take $F_i = 0$ for $i > 1$. For this free resolution we obviously have $H^n(F;G) = 0$ for $n > 1$, so this must also be true for all free resolutions by the above lemma. Thus the only interesting group $H^n(F;G)$ is $H^1(F;G)$. As we have seen, this group depends only on H and G, and the standard notation for it is $\text{Ext}(H,\ G)$.

Corollary 3.1. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(C;\mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$.*

Proof. One can calculate $Ext(H,G)$ for finitely generated H using the following three properties:
1.$Ext(H \oplus H', G) \cong Ext(H,G) \oplus Ext(H',G)$.
2.$Ext(H,G) = 0$ if H is free.
3.$Ext(\mathbb{Z}_n, G) \cong G/nG$.
These three properties imply that $Ext(H,\mathbb{Z})$ is isomorphic to the torsion subgroup of H if H is finitely generated. Also $Hom(H,\mathbb{Z})$ is isomorphic to the free part of H if H is finitely generated. The first can be obtained by considering direct sum of free resolutions of H and H' as free resolution for $H \oplus H'$. If H is free, the free resolution $0 \to H \to H \to 0$ yields the second property, and third will come from dualising the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}_n \to 0$. □

3.2 Cohomology of spaces

Given a space X and an abelian group G, we define the group $C^n(X;G)$ of singular n-cochains with coefficients in G to be the dual group $Hom(C_n(X),G)$ of the singular chain group $C_n(X)$. Thus an n-cochain $\varphi \in C^n(X;G)$ assigns to each singular n-simplex $\sigma : \Delta^n \to X$ a value $\varphi(\sigma) \in G$. Since the singular n-simplices form a basis for $C_n(X)$, these values can be chosen arbitrarily, hence n-cochains are exactly equivalent to functions from singular n-simplices to G.

The coboundary map $\delta : C^n(X;G) \to C^{n+1}(X;G)$ is the dual ∂^*, so for a cochain $\varphi \in C^n(X;G)$, its coboundary $\delta\varphi$ is the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G$. This means that for a singular $(n+1)$-simplex $\sigma : \triangle^{n+1} \to X$ we have

$$\delta\varphi(\sigma) = \sum(-1)^i \varphi(\sigma|[v_0,\ \cdots, \hat{v}_i,\ \cdots,\ v_{n+1}])$$

It is automatic that $\delta^2 = 0$ since δ^2 is the dual of $\partial^2 = 0$. Therefore we can define the cohomology group $H^n(X;G)$ with coefficients in G to be the quotient $Ker\delta/\text{Im}\delta$ at $C^n(X;G)$ in the cochain complex

$$\leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \xleftarrow{\delta} C^{n-1}(X;G) \leftarrow \cdots \leftarrow C^0(X;G) \leftarrow 0$$

Elements of $Ker\delta$ are cocycles, and elements of $\text{Im}\delta$ are coboundaries. For a cochain φ to be a cocycle means that $\delta\varphi = \varphi\partial = 0$, or in other words, φ vanishes on boundaries.

3.2.1 Reduced groups

Reduced cohomology groups $\tilde{H}^n(X;G)$ can be defined by dualizing the augmented chain complex $... \rightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$, (where epsilon is as defined before) and then taking Ker/Im. As with homology, this gives $\tilde{H}^n(X;G) = H^n(X;G)$ for $n > 0$, and the universal coefficient theorem identifies $\tilde{H}^0(X;G)$ with $Hom(\tilde{H}_0(X), G)$. We can describe the difference between $\tilde{H}^0(X;G)$ and $H^0(X;G)$ more explicitly by using the interpretation of $H^0(X;G)$ as functions $X \rightarrow G$ that are constant on path-components. Recall that the augmentation map $\epsilon : C_0(X) \rightarrow \mathbb{Z}$ sends each singular 0-simplex σ to 1, so the dual map ϵ^* sends a homomorphism $\varphi : \mathbb{Z} \rightarrow G$ to the composition $C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\varphi} G$, which is the function $\sigma \mapsto \varphi(1)$. This is a constant function $X \rightarrow G$, and since $\varphi(1)$ can be any element of G, the image of ϵ^* consists of precisely the constant functions. Thus $\tilde{H}^0(X;G)$ is all functions $X \rightarrow G$ that are constant on path-components modulo the functions that are constant on all of X.

3.2.2 Relative Groups and the Long Exact Sequence of a Pair

To define relative groups $H^n(X,A;G)$ for a pair (X,A) we first dualize the short exact sequence

$$0 \rightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \rightarrow 0$$

by applying $Hom(\_, G)$ to get

$$0 \leftarrow C^n(A;G) \xleftarrow{i^*} C^n(X;G) \xleftarrow{j^*} C^n(X,A;G) \leftarrow 0$$

where by definition $C^n(X,A;G) = Hom(C_n(X,A), G)$. This sequence is exact by the following direct argument. The map i^* restricts a cochain on X to a cochain on A. Thus for a function from singular n-simplices in X to G, the image of this function under i^* is obtained by restricting the domain of the function to singular n-simplices in A. Every function from singular n-simplices in A to G can be extended to be defined on all singular n-simplices in X, for example by assigning the value 0 to all singular n-simplices not in A, so i^* is surjective. The kernel of i^* consists of cochains taking the value 0 on singular n-simplices in A. Such cochains are the same as homomorphisms $C_n(X,A) = C_n(X)/C_n(A) \rightarrow G$, so the kernel of i^* is exactly $C^n(X,A;G) = Hom(C_n(X,A),\ G)$, giving the desired exactness. Notice that we can view $C^n(X,A;G)$ as the functions from singular n-simplices in X to G that vanish on simplices in A, since the basis for $C_n(X)$ consisting of singular n-simplices in X is the disjoint union of the simplices with image contained in A and the simplices with image not contained in A.

Relative coboundary maps $\delta : C^n(X,A;G) \rightarrow C^{n+1}(X,A;G)$ are obtained as restrictions of the absolute δ's, so relative cohomology groups $H^n(X,A;G)$ are defined. The maps i^* and j^* commute with δ since i and j commute with ∂, so the preceding displayed short exact sequence of cochain groups is part of a short exact sequence of cochain complexes, giving rise to an associated long exact sequence of cohomology groups

$$\cdots \rightarrow H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \rightarrow \cdots$$

More generally there is a long exact sequence for a triple (X,A,B) coming from the short exact sequences

$$0 \leftarrow C^n(A,B;G) \xleftarrow{i^*} C^n(X,B;G) \xleftarrow{j^*} C^n(X,A;G) \leftarrow 0$$

3.2.3 Functoriality

Dual to the chain maps $f_\# : C_n(X) \to C_n(Y)$ induced by $f : X \to Y$ are the cochain maps $f^\# : C^n(Y;G) \to C^n(X;G)$. The relation $f_\#\partial = \partial f_\#$ dualizes to $\delta f^\# = f^\#\delta$, so $f^\#$ induces homomorphisms $f^* : H^n(Y;G) \to H^n(X;G)$. In the relative case a map $f : (X,A) \to (Y,B)$ induces $f^* : H^n(\mathrm{Y},B;G) \to H^n(X,A;G)$ by the same reasoning, and in fact f induces a map between short exact sequences of cochain complexes, hence a map between long exact sequences of cohomology groups, with commuting squares. The properties $(fg)^\# = g^\# f^\#$ and $id^\# = id$ imply $(fg)^* = g^* f^*$ and $id^* = id$, so $X \mapsto H^n(X;G)$ and $(X,A) \mapsto H^n(X,A;G)$ are contravariant functors, the 'contra' indicating that induced maps go in the reverse direction.

3.2.4 Homotopy Invariance

If f is homotopic to g $f \simeq g : (X,A) \to (Y,B)$ then $f^* = g^* : H^n(Y,B) \to H^n(X,A)$. This is proved by direct dualization of the proof for homology. We have a chain homotopy P satisfying $g_\# - f_\# = \partial P + P\partial$. This relation dualizes to $g^\# - f^\# = P^*\delta + \delta P^*$, so P^* is a chain homotopy between the maps $f^\#, g^\# : C^n(Y;G) \to C^n(X;G)$. This restricts also to a chain homotopy between $f^\#$ and $g^\#$ on relative cochains, with cochains vanishing on singular simplices in the subspaces B and A. Since $f^\#$ and $g^\#$ are chain homotopic, they induce the same homomorphism $f^* = g^*$ on cohomology.

3.2.5 Excision

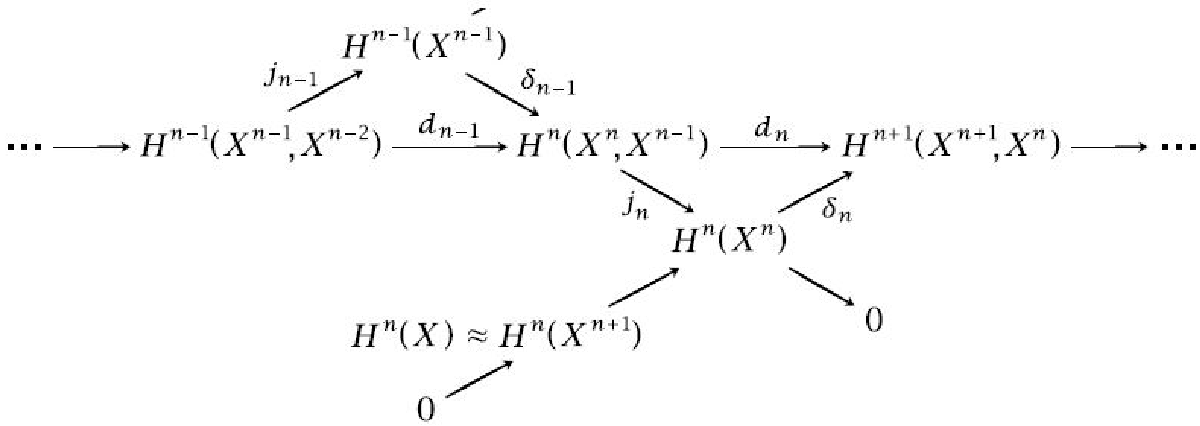
For cohomology this says that for subspaces $Z \subset A \subset X$ with the closure of Z contained in the interior of A, the inclusion $i : (X - Z, A - Z) \hookrightarrow (X,A)$ induces isomorphisms
$i^*\colon H^n(X,A;G) \to H^n(X-Z,A-Z;G)$ for all n. This follows from the corresponding result for homology by the naturality of the universal coefficient theorem and the five-lemma.

3.2.6 Simplicial Cohomology

If X is a Δ-complex and $A \subset X$ is a subcomplex, then the simplicial chain groups $\Delta_n(X,A)$ dualize to simplicial cochain groups $\Delta^n(X,A;G) = Hom(\Delta_n(X,A),G)$, and the resulting cohomology groups are by definition the simplicial cohomology groups $H^n_\Delta(X,A;G)$. Since the inclusions $\Delta_n(X,A) \subset C_n(X,A)$ induce isomorphisms $H^\Delta_n(X,A) \approx H_n(X,A)$, the dual maps $C^n(X,A;G) \to \Delta^n(X,A;G)$ also induce isomorphisms $H^n(X,A;G) \approx H^n_\Delta(X,A;G)$.

3.2.7 Cellular Cohomology

For a CW complex X this is defined via the cellular cochain complex formed by the horizontal sequence in the following diagram, where coefficients in a given group G are understood, and the cellular coboundary maps d_n are the composition $\delta_n j_n$, making the triangles commute.



3.2.8 Mayer-Vietoris Sequence

In the absolute case these take the form

$$\cdots \to H^n(X;G) \xrightarrow{\Psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\Phi} H^n(A \cap B;G) \to H^{n+1}(X;G) \to \cdots$$

where X is the union of the interiors of A and B. This is the long exact sequence associated to the short exact sequence of cochain complexes

$$0 \to C^n(A+B;G) \xrightarrow{\psi} C^n(A;G) \oplus C^n(B;G) \xrightarrow{\varphi} C^n(A \cap B;G) \to 0$$

Here $C^n(A+B;G)$ is the dual of the subgroup $C_n(A+B) \subset C(X)$ consisting of sums of singular n-simplices lying in A or in B.

3.2.9 Axioms of Cohomology

The axioms of cohomology theory are dual of the axioms oh homology theory.
1. **Functoriality** Cohomology group H^q is a contravariant functor from category of pairs of topological spaces to category of abelian groups.
2. **Homotopy axiom** If $f, g : (X, Y) \to (X', Y')$ are homotopic then they induce the same map between cohomology groups.
3. **Excision** Suppose (X, Y) is a pair and U is open in X and $\overline{U} \subset Y$ then the inclusion $i : (X - U, Y - U) \to (X, Y)$ induces isomorphism in all the $H^q s$.
4. **Exactness** For all (X, Y) $\exists$ a family of natural transformations
$\delta^q : H^q(X, Y) \to H^{q-1}(Y)$ such that the sequence

$$... \leftarrow H^q(Y;G) \xleftarrow{i^*} H^q(X;G) \xleftarrow{j^*} H_q(X,Y;G) \xleftarrow{\delta^q} H^{q-1}(Y;G) \leftarrow ...$$

is exact.
5. **Dimension axiom**

$$H^q(point;G) \begin{cases} = 0 \text{ if } q \neq 0 \\ \simeq G \text{ if } q = 0 \end{cases}$$

3.3 Cup product and Cohomology Ring

To define the cup product we consider cohomology with coefficients in a ring R, the most common choices being $\mathbb{Z}, \mathbb{Z}_n$, and $\mathbb{Q}$. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, the cup product $\varphi \cup \psi \in C^{k+\ell}(X;R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ is given by the formula

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|[v_0, \cdots, v_k])\psi(\sigma|[v_k, \cdots, v_{k+\ell}])$$

where the right-hand side is a product in R.

Lemma 8. *$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$ for $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$.*

Proof. : For $\sigma : \Delta^{k+\ell+1} \to X$ we have

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{k+1}])\psi(\sigma|[v_{k+1}, \cdots, v_{k+\ell+1}])$$

$$(-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|[v_0, \cdots, v_k])\psi(\sigma|[v_k, \cdots, \hat{v}_i, \cdots, v_{k+\ell+1}])$$

When we add these two expressions, the last term of the first sum cancels the first term of the second sum, and the remaining terms are exactly $\delta(\varphi \cup \psi)(\sigma) = (\varphi \cup \psi)(\partial\sigma)$ since $\partial\sigma = \sum_{i=0}^{k+\ell+1} (-1)^i \sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{k+\ell+1}]$. □

From the formula $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi \pm \varphi \cup \delta\psi$ it is apparent that the cup product of two cocycles is again a cocycle. Also, the cup product of a cocycle and a coboundary, in either order, is a coboundary since $\varphi \cup \delta\psi = \pm\delta(\varphi \cup \psi)$ if $\delta\varphi = 0$, and $\delta\varphi \cup \psi = \delta(\varphi \cup \psi)$ if $\delta\psi = 0$. It follows that there is an induced cup product

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R)$$

This is associative and distributive since at the level of cochains the cup product has these properties.

Proposition 3.2. *For a map $f : X \to Y$ the induced maps $f^* : H^n(\mathrm{Y};R) \to H^n(X;R)$ satisfy $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$.*

Proof. This comes from the cochain formula $f^\#(\varphi) \cup f^\#(\psi) = f^\#(\varphi \cup \psi)$:

$$\begin{aligned}(f^\#\varphi \cup f^\#\psi)(\sigma) &= f^\#\varphi(\sigma|[v_0, \cdots, v_k])f^\#\psi(\sigma|[v_k, \cdots, v_{k+\ell}]) \\ &= \varphi(f\sigma|[v_0, \cdots, v_k])\psi(f\sigma|[v_k, \cdots, v_{k+\ell}]) \\ &= (\varphi \cup \psi)(f\sigma) = f^\#(\varphi \cup \psi)(\sigma)\end{aligned}$$

□

Since cup product is associative and distributive it is natural to try to make it the multiplication in a ring structure on the cohomology groups of a space. Let $H^*(X;R)$ be the direct sum of groups $H^n(X;R)$. Elements of $H^*(X;R)$ are finite sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X;R)$, and the product of two such sums is defined to be $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i\beta_j$. One can check that this makes $H^*(X;R)$ into a ring. One always regards the cohomology ring as a graded ring i.e. a ring A with a decomposition as a sum $\oplus_{k\geq 0} A_k$ of additive subgroups A_k such that the multiplication takes $A_k \times A_\ell$ to $A_{k+\ell}$. To indicate that an element $a \in A$ lies in A_k we write $|a| = k$.

3.4 Kunneth Formula

Kunneth Formula allows us to calculate the cohomology ring of a product space from the cohomology rings of the original spaces. Let us first define a map

$$H^*(X; R) \times H^*(Y; R) \xrightarrow{\times} H^*(X \times Y; R)$$

given by $a \times b = p_1^*(a) \cup p_2^*(b)$ where p_1 and p_2 are the projections of $X \times Y$ onto X and Y. Using this we define **cross product**

$$H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(X \times Y; R)$$

given by $a \otimes b \mapsto a \times b$. We define the multiplication in a tensor product of graded rings by $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ where $|x|$ denotes the dimension of x. This makes the cross product map a ring homomorphism.

Theorem 12. *The cross product $H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y; R)$ is a finitely generated free R module for all k.*

To prove this theorem we will need the following lemma.

Lemma 9. *If a natural transformation between unreduced cohomology theories on the category of CW pairs is an isomorphism when the CW pairs is $(point, \phi)$, then it is an isomorphism for all CW pairs.*

. Idea of the proof of the theorem is to consider, for a fixed CW complex Y, the functors $h^n(X, A) = \bigoplus_i (H^i(X, A; R)) \oplus_R H^{n-i}(Y; R)$ and $k^n(X, A) = H^n(X \times Y, A \times Y; R)$. The cross product defines a map $\mu : h^n(X, A) \to k^n(X, A)$ which we will show is an isomorphism when X is a CW complex and $A = \emptyset$, in two steps

(1) h^* and k^* are cohomology theories on the category of CW pairs.

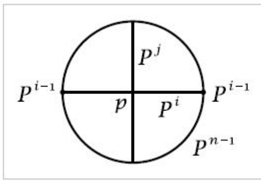
(2) μ is a natural transformation: It commutes with induced homomorphisms and with coboundary homomorphisms in long exact sequences of pairs.

The map $\mu : h^n(X) \to k^n(X)$ is an isomorphism when X is a point since it is just the scalar multiplication map $R \otimes_R H^n(\mathrm{Y}; R) \to H^n(\mathrm{Y}; R)$. Then the above lemma will then imply the theorem.

Example We will prove that $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1}$. To simplify notation we abbreviate $\mathbb{R}\mathrm{P}^n$ to P^n and we let the coefficient group $\mathbb{Z}_2$ be implicit. Since the inclusion $P^{n-1} \hookrightarrow P^n$ induces an isomorphism on H^i for $iln - 1$, it suffices by induction on n to show that the cup product of a generator of $H^{n-1}(P^n)$ with a generator of $H^1(P^n)$ is a generator of $H^n(P^n)$. Then we will show that the cup product of a generator of $H^i(P^n)$ with a generator of $H^{n-i}(P^n)$ is a generator of $H^n(P^n)$. As a further notational aid, let $j = n - i$, so $i + j = n$.

The space P^n consists of nonzero vectors $(\mathrm{x}_0, \cdots, \mathrm{x}_n) \in \mathbb{R}^{n+1}$ modulo multiplication by nonzero scalars. Inside P^n is a copy of P^i represented by vectors whose last j coordinates $\mathrm{x}_{i+1}, \cdots, \mathrm{x}_n$ are zero. We also have a copy of P^j represented by points whose first i coordinates $\mathrm{x}_0, \cdots, \mathrm{x}_{i-1}$ are zero. The intersection $P^i \cap P^j$ is a single point p, represented by vectors whose only nonzero coordinate is x_i. Let U be the subspace of P^n represented by vectors with nonzero coordinate x_i. Each point in U may be represented by a unique vector with $\mathrm{x}_i = 1$ and the other n coordinates arbitrary, so U is homeomorphic to $\mathbb{R}^n$, with p corresponding to 0 under this homeomorphism.

We can write this $\mathbb{R}^n$ as $\mathbb{R}^i \times \mathbb{R}^j$, with $\mathbb{R}^i$ as the coordinates $\mathrm{x}_0, , \mathrm{x}_{i-1}$ and $\mathbb{R}^j$ as the coordinates $\mathrm{x}_{i+1}, \cdots, \mathrm{x}_n$. In the figure P^n is represented as a disk with antipodal points of its boundary sphere identified to form a $P^{n-1} \subset P^n$ with $U = P^n - P^{n-1}$ the interior of the disk.



Consider the diagram

$$\text{(i)}\qquad \begin{array}{ccc} H^i(P^n) \times H^j(P^n) & \xrightarrow{\quad\smile\quad} & H^n(P^n) \\ \uparrow & & \uparrow \\ H^i(P^n, P^n - P^j) \times H^j(P^n, P^n - P^i) & \xrightarrow{\smile} & H^n(P^n, P^n - \{p\}) \\ \downarrow & & \downarrow \\ H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \xrightarrow{\smile} & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \end{array}$$

which commutes by naturality of cup product. We will show that the four vertical maps are isomorphisms and that the lower cup product map takes generator cross generator to generator. Commutativity of the diagram will then imply that the upper cup product map also takes generator cross generator to generator.

The lower map in the right column is an isomorphism by excision. For the upper map in this column, the fact that $P^n - \{p\}$ deformation retracts to a P^{n-1} gives an isomorphism $H^n(P^n,\ P^n - \{p\}) \approx H^n(P^n,\ P^{n-1})$ via the five-lemma applied to the long exact sequences for these pairs. And $H^n(P^n,\ P^{n-1}) \approx H^n(P^n)$ by cellular cohomology. To see that the vertical maps in the left column of (i) are isomorphisms we will use the following commutative diagram:

$$\text{(ii)}\qquad \begin{array}{ccccccc} H^i(P^n) & \longleftarrow & H^i(P^n, P^{i-1}) & \longleftarrow & H^i(P^n, P^n - P^j) & \longrightarrow & H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^i(P^i) & \longleftarrow & H^i(P^i, P^{i-1}) & \longleftarrow & H^i(P^i, P^i - \{p\}) & \longrightarrow & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \end{array}$$

The left-hand square in (ii) consists of isomorphisms by cellular cohomology. The right-hand vertical map is obviously an isomorphism. The lower right horizontal map is an isomorphism by excision, and the map to the left of this is an isomorphism since $P^i - \{p\}$ deformation retracts onto P^{i-1}. The remaining maps will be isomorphisms if the middle map in the upper row is an isomorphism. And this map is in fact an isomorphism because $P^n - P^j$ deformation retracts onto P^{i-1} by the following argument. The subspace $P^n - P^j \subset P^n$ consists of points represented by vectors $v = (\mathrm{x}_0,\ \cdots,\ \mathrm{x}_n)$ with at least one of the coordinates $\mathrm{x}_0, \cdots, \mathrm{x}_{i-1}$ nonzero. The formula $f_t(v) = (\mathrm{x}_0, \cdots, \mathrm{X}, tX, \cdots, tX$ for t decreasing from 1 to 0 gives a well-defined deformation retraction of $P^n - P^j$ onto P^{i-1} since $f_t(\lambda v) = \lambda f(v)$ for scalars $\lambda \in \mathrm{R}$.

The cup product map in the bottom row of (i) is equivalent to the cross product $H^i(I^i,\ \partial I^i) \times H^j(I^j,\ \partial I^j) \to H^n(I^n,\ \partial I^n)$.

3.5 Poincare Duality

Poicare duality gives a symmetric relationship between homology and cohomology groups of special class of topological spaces, namely manifolds. A **manifold** of dimension n is a Hausdorff second countable space M in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$. We need the concept of orientation of manifold to define Poincare duality. An **orientation of** $\mathbb{R}^n$ at a point x is a choice of generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n - x)$. A **local orientation** of M at a point x is a choice of generator μ_x of the infinite cyclic group $H_n(M, M - x)$. To simplify notation we will write $H_n(X, X - A)$ as $H_n(X|A)$. An **orientation of an n dimensional manifold M** is a function $x \to \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M|x)$, satisfying the 'local consistency' condition that each $x \in M$ has a neighborhood $\mathbb{R}^n \subset M$ containing an open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of one generator μ_B of $H_n(M|B) \cong H_n(\mathbb{R}^n|B)$ under the natural maps $H_n(M|B) \to H_n(M|y)$. If an orientation exists for M, then M is called orientable. One can generalize the definition of orientation by replacing the coefficient group $\mathbb{Z}$ by any commutative ring R with identity. Then an **R-orientation of M** assigns to each $x \in M$ a generator of $H_n(M|x; R) \cong R$, subject to the corresponding local consistency condition, where a generator of R is an element u such that $Ru = R$. An element of $H_n(M; R)$ whose image in $H_n(M|x; R)$ is a generator for all x is called a **fundamental class for M** with coefficients in R. The form of Poincare duality we will prove asserts that for an R-orientable closed n-manifold, a certain naturally defined map $H^k(M; R) \to H_{n-k}(M; R)$ is an isomorphism. The definition of this map will be in terms of a more general construction called **cap product**, which has close connections with cup product.

For an arbitrary space X and coefficient ring R, define an R-bilinear cap product $\cap$: $C_k(X; R) \times C^\ell(X; R) \to C_{k-\ell}(X; R)$ for $k \geq \ell$ by setting

$$\sigma \cap \varphi = \varphi(\sigma|[v_0, \cdots, v_\ell])\sigma|[v_\ell, \cdots v_k]$$

for $\sigma : \triangle^k \to X$ and $\varphi \in C^\ell(X; R)$. This induces a cap product in homology and cohomology as follows

$$\partial(\sigma \cap \varphi) = (-1)^\ell(\partial\sigma \cap \varphi\sigma \cap \delta\varphi)$$

which is checked by a calculation:

$$\partial\sigma \cap \varphi = \sum_{i=0}^{\ell}(-1)^i\varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{\ell+1}])\sigma|[v_{\ell+1}, \cdots v_k]$$

$$+ \sum_{i=l+1}^{k}(-1)^i\varphi(\sigma|[v_0, \cdots, v_\ell])\sigma|[v_\ell, \cdots \hat{v}_i, \cdots v_k]$$

$$\sigma \cap \delta\varphi = \sum_{\iota'=0}^{\ell+1}(-1)^i\varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{\ell+1}])\sigma|[v_{\ell+1}, \cdots v_k]$$

$$\partial(\sigma \cap \varphi) = \sum_{i=\ell}^{k}(-1)^{i-\ell}\varphi(\sigma|[v_0, \cdots, v_\ell])\sigma|[v_\ell, \cdots \hat{v}_i, \cdots v_k]$$

From the relation $\partial(\sigma \cap \varphi) = \pm(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$ it follows that the cap product of a cycle σ and a cocycle φ is a cycle. Further, if $\partial\sigma = 0$ then $\partial(\sigma \cap \varphi) = \pm(\sigma \cap \delta\varphi)$, so the cap product of a cycle and a coboundary is a boundary. And if $\delta\varphi = 0$ then $\partial(\sigma \cap \varphi) = \pm(\partial\sigma \cap \varphi)$, so the cap product of a boundary and a cocycle is a boundary. These facts imply that there is an induced cap product

$$H_k(X; R) \times H^\ell(X; R) \xrightarrow{\cap} H_{k-\ell}(X; R)$$

which is R-linear in each variable.

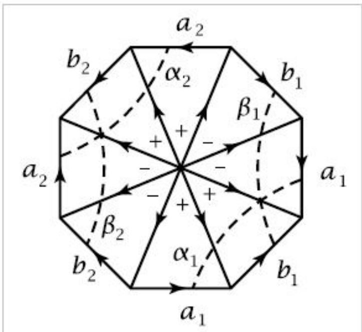
Given a map $f : X \to$ Y, the relevant induced maps on homology and cohomology fit into the diagram shown below.

$$\begin{array}{ccccc} H_k(X) & \times & H^\ell(X) & \xrightarrow{\frown} & H_{k-\ell}(X) \\ \downarrow f_* & & \uparrow f^* & & \downarrow f_* \\ H_k(Y) & \times & H^\ell(Y) & \xrightarrow{\frown} & H_{k-\ell}(Y) \end{array}$$

If we substitute $f\sigma$ for σ in the definition of cap product: $f\sigma \cap \varphi = \varphi(f\sigma|[v_0, \cdots, v_\ell])f\sigma|[v_\ell, \cdots, v_k]$, we get

$$f_*(\alpha) \cap \phi = f_*(\alpha \cap f^*(\phi))$$

Example: Let M be the closed orientable surface of genus g, obtained as usual from a $4g$-gon by identifying pairs of edges according to the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. A $\triangle$-complex structure on M is obtained by coning off the $4g$-gon to its center, as indicated in the figure for the case $g = 2$.



We can compute cap products using simplicial homology and cohomology since cap products are defined for simplicial homology and cohomology by exactly the same formula as for singular homology and cohomology, so the isomorphism between the simplicial and singular theories respects cap products. A fundamental class $[M]$ generating $H_2(M)$ is represented by the 2-cycle formed by the sum of all $4g$ 2-simplices with the signs indicated. The edges a_i and b_i form a basis for $H_1(M)$. Under the isomorphism $H^1(M) \approx \mathrm{Hom}(H_1(M), \mathbb{Z})$, the cohomology class α_i corresponding to a_i assigns the value 1 to a_i and 0 to the other basis elements. This class α_i is represented by the cocycle φ_i assigning the value 1 to the 1-simplices meeting the arc labeled α_i in the figure and 0 to the other 1-simplices. Similarly we have a class β_i corresponding to b_i, represented by the cocycle ψ_i assigning the value 1 to the 1-simplices meeting the arc β_i and 0 to the other 1-simplices. Applying the definition of cap product, we have $[M] \cap \varphi_i = b_i$ and $[M] \cap \psi_i = -a_i$ since in both cases there is just one 2-simplex $[v_0, v_1, v_2]$ where φ_i or ψ_i is nonzero on the edge $[v_0, v_1]$. Thus b_i is the Poincare dual of α_i and $-a_i$ is the Poincare dual of β_i. If we interpret Poincare duality entirely in terms of homology, identifying α_i with its Hom-dual a_i and β_i with b_i, then the classes a_i and b_i are Poincare duals of each other, up to sign at least. Geometrically, Poincare duality is reflected in the fact that the loops α_i and b_i are homotopic, as are the loops β_i and a_i.

Proof of Poincare Duality requires a version of Poincare duality for noncompact manifolds and can satisfy Poincare duality only when different form of cohomology, called cohomology of compact support is used. Let $C_c^i(X; G)$ be the subgroup of $C^i(X; G)$ consisting of cochains $\varphi : C_i(X) \to G$ for which there exists a compact set $K = K_\varphi \subset X$ such that φ is zero on all chains in $X - K$. Note that $\delta\varphi$ is then also zero on chains in $X - K$, so $\delta\varphi$ lies in $C_c^{i+1}(X; G)$ and the $C_c^i(X; G)$'s for varying i form a subcomplex of the singular cochain complex of X. The cohomology groups $H_c^i(X; G)$ of this subcomplex are the **cohomology groups with compact supports**.For M an R-orientable n-manifold, possibly noncompact, we can define a duality map $D_M : H_c^k(M; R) \to H_{n-k}(M; R)$ by a limiting process in the following way. For compact sets $K \subset L \subset M$ we have a diagram

$$\begin{array}{ccc} H_n(M|L;R) \times H^k(M|L;R) & \overset{\frown}{\searrow} & \\ \downarrow i_* \qquad\qquad \uparrow i^* & & H_{n-k}(M;R) \\ H_n(M|K;R) \times H^k(M|K;R) & \underset{\frown}{\nearrow} & \end{array}$$

where $H_n(M|A;R) = H_n(M, M-A;R)$ and $H^k(M|A;R) = H^k(M, M-A;R)$. One can check that there are unique elements $\mu_K \in H_n(M|K;R)$ and $\mu_L \in H_n(M|L;R)$ restricting to a given orientation of M at each point of K and L, respectively. From the uniqueness we have $i_*(\mu_L) = \mu_K$. The naturality of cap product implies that $i_*(\mu_L) \cap x = \mu_L \cap i^*(x)$ for all x $\in H^k(M|K;R)$, so $\mu_K \cap x = \mu_L \cap i^*(x)$. Therefore, letting K vary over compact sets in M, the homomorphisms $H^k(M|K;R) \to H_{n-k}(M;R)$, x $\mapsto \mu_K \cap x$, induce in the limit a duality homomorphism $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$. Since $H_c^*(M;R) = H^*(M;R)$ if M is compact, the following theorem generalizes Poincare duality for closed manifolds:

Theorem 13. *The duality map $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k whenever M is an R-orientable n-manifold.*

We will use this to prove poincare duality, stated below.

Theorem 14. ***(Poincare duality)*** *If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D : H^k(M;R) \to H_{n-k}(M;R)$ defined by $D(\alpha) = [M] \cap \alpha$ is an isomorphism for all k.*

Proof. The coefficient ring R will be fixed throughout the proof, and for simplicity we will omit it from the notation for homology and cohomology.

If M is the union of a sequence of open sets $U_1 \subset U_2 \subset \cdots$ and each duality map $D_{U_i} : H_c^k(U_i) \to H(U)$ is an isomorphism, then so is D_M. By excision, $H_c^k(U)$ can be regarded as the limit of the groups $H^k(M|K)$ as K ranges over compact subsets of U_i. Then there are natural maps $H_c^k(U_i) \to H_c^k(U)$ since the second of these groups is a limit over a larger collection of K's. Thus we can form $\lim\limits_{\to} H_c^k(U)$ which is obviously isomorphic to $H_c^k(M)$ since the compact sets in M are just the compact sets in all the U_i's. $H_{n-k}(M) \cong \lim_{\to} H_{n-k}(U_i)$. The map D_M is thus the limit of the isomorphisms D_{U_i}, hence is an isomorphism. Now after all these preliminaries we can prove the theorem in three easy steps:

(1) The case $M = \mathbb{R}^n$ can be proved by regarding $\mathbb{R}^n$ as the interior of $\triangle^n$, and then the map D_M can be identified with the map $H^k(\triangle^n, \partial\triangle^n) \to H_{n-k}(\triangle^n)$ given by cap product with a unit times the generator $[\triangle^n] \in H_n(\triangle^n, \partial\triangle^n)$ defined by the identity map of $\triangle^n$, which is a relative cycle. The only nontrivial value of k is $k = n$, when the cap product map is an isomorphism since a generator of $H^n(\triangle^n, \partial\triangle^n) \cong \mathrm{Hom}(H_n(\triangle^n, \partial\triangle^n), R)$ is represented by a cocycle φ taking the value 1 on $\triangle^n$, so by the definition of cap product, $\triangle^n \cap \varphi$ is the last vertex of $\triangle^n$, representing a generator of $H_0(\triangle^n)$.

(2)More generally, D_M is an isomorphism for M an arbitrary open set in $\mathbb{R}^n$. Write M as a countable union of nonempty bounded convex open sets U_i, for example open balls, and let $V_i = \cup_{j<i} U_j$. Both V_i and $U_i \cap V_i$ are unions of $i-1$ bounded convex open sets, so by induction on the number of such sets in a cover we may assume that D_{V_i} and $D_{U_i \cap \mathrm{V}_i}$ are isomorphisms. D_{U_i} is an isomorphism since U_i is homeomorphic to $\mathbb{R}^n$. Hence $D_{U_i \cup \mathrm{V}_i}$ is an isomorphism. Since M is the increasing union of the V_i's and each D_{V_i} is an isomorphism, so is D_M by above argument.

(3)If M is a finite or countably infinite union of open sets U_i homeomorphic to $\mathbb{R}^n$, the theorem now follows by the argument above, with each appearance of the words bounded convex open set replaced by open set in $\mathbb{R}^n$. Thus the proof is finished for closed manifolds, as well as for all the noncompact manifolds.

To handle a completely general noncompact manifold M we use a Zorn's Lemma argument. Consider the collection of open sets $U \subset M$ for which the duality maps D_U are isomorphisms. This collection is partially ordered by inclusion, and the union of every totally ordered subcollection is again in the collection by the argument above, which did not really use the hypothesis that the collection $\{U\}$ was indexed by the positive integers. Zorn's Lemma then implies that there exists a maximal open set U for which the theorem holds. If $U \neq M$, choose a point $x \in M - U$ and an open neighborhood V of x homeomorphic to $\mathbb{R}^n$. The theorem holds for V and $U \cap V$, and it holds for U by assumption, so it holds for $U \cup V$, contradicting the maximality of U. □

4 Conclusion

We first looked at an homotopy invariant, the homology groups. We started by studying the most geometric homology theory, the simplicial homology and then studied singular homology. Next we abstracted out its properties and get a generalised homology theory. Then we looked at cellular homology and Mayer Vietoris sequence, the two most useful tools to compute homology groups. Then we defined cohomology groups of a topological space, which are dual of homology groups. Next we studied universal coefficient theorem which gives us a criteria to compute cohomology groups using homology groups. Then we looked at kunneth formula which is a useful tool to compute cohomology groups. Then we studied a nice symmetric relation between homology and cohomology groups, namely the Poincare Duality.

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