Homology and Cohomology

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Preface

This report is based on the contents of [1], with §2 and §3 based on the respective chapters in [1]. §4 contains sketches of solutions to some exercises from [1]. In §2, the order in which topics are covered differs significantly from chapter 2 of [1]. The most prominent example of this is that the Mayer-Vietoris sequence is discussed before relative homology and the excision theorem (contrary to the approach taken in [1]). Furthermore, the present proof of the Mayer-Vietoris sequence is a polished version of the proof I produced early on in the project. My proof of the equivalence of simplicial and singular homology using the Mayer-Vietoris sequence is furnished in §2.3.1.

In certain places, some simple but important results not covered in [1] have been added. For instance, the introduction to §2 includes a discussion regarding how the signs in the formula for the boundary of a singular simplex relate to orientation. Another instance is **proposition 2.21**, which indicates how rotations of the S^n can aid in calculating local degrees of maps $S^n \to S^n$. In §2.6, great care has been taken to minimise abstract identifications which can be tricky to unpack at the level of concrete examples, particularly in the cellular boundary formula (**theorem 2.26**).

The proof of the universal coefficient theorem for cohomology is my own, although it is mostly similar to that in [1]. Once again, significant emphasis has been placed on minimising identifications so as to keep the discussion grounded. The discussion of free resolutions has been greatly simplified by omitting algebraic details which are not necessary for a topology-centeric viewpoint.

My philosophy while writing this report has largely been to cover only those topics regarding which I have something non-trivial to add — a different proof, a discussion of the underlying intuition or an exposition which I believe to be better. Consequently I have omitted proofs of several algebraic lemmas and the discussion of the axioms for homology, where I have nothing to add to the discussion already present in [1].

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Contents

1 Introduction

'Homology' and 'cohomology' are ways of associating algebraic homotopy invariants (abelian groups, rings and modules) to topological spaces and, loosely speaking, capture information about 'holes' in these spaces. From an algebraic point of view, the broad idea for homology is the following. Given a topological space X, we first associate an abelian group, say C_n, to X in each 'dimension' $n \geq 0$. For $n < 0$ we set $C_n = 0$. These groups generally have a simple structure, and are not homotopy invariants. A 'boundary map' $\partial_n : C_n \to C_{n-1}$ is then constructed, which generally captures a natural way of going one dimension down. This map satisfies $\partial_{n-1}\partial_n = 0$, or simply ∂^2. The resulting sequence

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

is called a **chain complex**, owing to the condition $\partial^2 = 0$. This condition is equivalent to saying that $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$, and so we now consider the groups $H_n := \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$ (called the homology group of the chain complex in dimension n). These homology groups then turn out to be homotopy invariant, and the broad goal of homology theory is to study these groups and the information they contain about the space X. This construction is natural and covariant, in the sense that a continuous map $f : X \to Y$ between topological spaces induces group homomorphisms from the homology groups of X to those of Y in each dimension.

The cohomology groups are constructed in a similar way, except that the analogues of the boundary maps go up in dimension (from dimension n to $n + 1$). This construction is also natural, although it is contravariant — a map $f : X \to Y$ induces group homomorphism from the cohomology groups of Y to those of X in each dimension.

2 Homology

The most general form of homology, called singular homology, attempts to make precise the idea of capturing 'holes' of various 'shapes' in a topological space X. This is done using a procedure which effectively boils down to considering equivalence classes of maps from spaces called Δ-complexes to X, obtained by gluing simplices together along their faces. The equivalence relation in question is loosely described by saying that two such maps are considered equivalent if the 'gap' between them can be 'filled in' with a Δ-complex of one dimension higher.

Let Δ^n be the **standard n-simplex**, viewed as the convex hull of the basis $v_0, \ldots, v_n \in \mathbb{R}^{n+1}$ obtained by scaling the standard basis by a factor of $\frac{1}{\sqrt{2}}$. The reason for this scaling is so that $v_i - v_j = \delta_{ij}$, i.e. the standard simplices have side-length 1. *Throughout this report, we will use v_i to denote the vertices of the standard simplices.* A **k-face** of Δ^n, for $0 \leq k \leq n$ is an affine linear map $\Delta^k \to \Delta^n$ which takes the vertices of $v_0, \ldots, v_k$ of Δ^k injectively to vertices of Δ^n. Hence, we may denote a k-face of Δ^n by $[v_{i_0}, \ldots, v_{i_k}]$, where v_j is mapped to v_{i_j} by the face. More generally, if Y is a convex space and $a_0, \ldots, a_n \in Y$, the map $[a_0, \ldots, a_n] : \Delta^n \to Y$ is defined to be the affine-linear map taking v_i to a_i.

A **singular n-simplex** σ in X is a continuous map $\sigma : \Delta^n \to X$ and a **singular n-chain** is a formal $\mathbb{Z}$-linear combination of singular n-simplices. Denote by $C_n(X)$ the free abelian group of singular n-chains. For $n \geq 1$, given a singular n-simplex $\sigma : \Delta^n \to X$ we define its boundary $\partial_n \sigma$ as

$$\partial_n \sigma = \sum_i (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

where $\sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ is defined to be the $(n-1)$-singular simplex $\sigma \circ [v_0, \ldots, \hat{v}_i, \ldots, v_n] : \Delta^{n-1} \to X$. For $n = 0$, we simply define $\partial_0 \sigma = 0$. If the choice of X is not clear, we may write ∂_n^X to denote the boundary operator for X.

Before continuing the construction of singular homology, we make a digression to provide the motivation behind the signs in the above definition, which have to do with the orientation induced on $\partial \Delta^n$. Orient $\operatorname{int} \Delta^n$ with the ordered basis $(v_1 - v_0, \ldots, v_n - v_0)$ of the tangent space at every point and fix $0 < i \leq n$ (a slight modification of the following argument will work for $i = 0$). The basis $(v_1 - v_0, \ldots, v_n - v_0)$ is oriented the same as

$$(-1)^i (v_0 - v_i, v_1 - v_0, \ldots, \widehat{v_i - v_0}, v_n - v_0)$$

which in turn is oriented the same as

$$(-1)^i \left(n(v_0 - v_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n} (v_j - v_0), v_1 - v_0, \ldots, \widehat{v_i - v_0}, v_n - v_0 \right)$$

Now, at a point in $\operatorname{int}\operatorname{conv}(v_0, \ldots, \hat{v}_i, \ldots, v_n)$, the vector

$$\left(\frac{1}{n}\sum_{\substack{j \\ j\neq i}} v_j\right) - v_i = \frac{1}{n}\sum_{\substack{j \\ j\neq i}} (v_j - v_i)$$

points outward. Hence, the orientation on $\operatorname{int}\operatorname{conv}(v_0, \ldots, \hat{v}_i, \ldots, v_n)$ induced by $\operatorname{int}\Delta^n$ is the same as

$$(-1)^i(v_1 - v_0, \ldots, \widehat{v_i - v_0}, \ldots, v_n - v_0)$$

Finally, we observe that $(v_1 - v_0, \ldots, \widehat{v_i - v_0}, \ldots, v_n - v_0)$ gives the orientation on $\operatorname{int}\operatorname{conv}(v_0, \ldots, \hat{v}_i, \ldots, v_n)$ induced by $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$. Hence, a factor of $(-1)^i$ accounts for the discrepancy in these two orientations on $\operatorname{int}\operatorname{conv}(v_0, \ldots, \hat{v}_i, \ldots, v_n)$.

We now return to the construction of singular homology. By extending ∂_n $\mathbb{Z}$-linearly, we obtain the following diagram.

$$\ldots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

It follows from a straightforward calculation that $\partial_{n-1} \circ \partial_n = 0$ for $n \geq 1$, so the above is a chain complex. The n-th singular homology group $H_n(X)$ is defined to be the n-th **homology group** of this chain complex.

$$H_n(X) := \frac{\ker \partial_n}{\operatorname{im}\partial_{n+1}}$$

Given a cycle $\sigma \in \ker \partial_n$, its class in $H_n(X)$ is called its **homology class** and will be denoted by $[\sigma]$. If the choice of X is unclear, we will write $[\sigma]_X$ instead. A continuous function $f : X \to Y$ induces a homomorphism

$$f_\# : C_n(X) \to C_n(Y); \sum_i n_i\sigma_i \mapsto \sum_i n_i(f \circ \sigma_i)$$

It is clear that $f_\#\partial = \partial f_\#$, i.e. the following diagram commutes.

$$\begin{array}{ccccccc}
\ldots & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \ldots \\
 & & \downarrow f_\# & & \downarrow f_\# & & \\
\ldots & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \ldots
\end{array}$$

Such a map $f_\#$ is called a **chain morphism**, and it is easy to see that it naturally induces a homomorphism

$$f_* : H_n(X) \to H_n(Y); [\sigma] \mapsto [f_\#\sigma]$$

The following follows easily from the definitions.

Proposition 2.1. *For continuous maps $f : X \to Y$ and $g : Y \to Z$, we have*

- $(fg)_\# = f_\# g_\#$ *and* $(fg)_* = f_* g_*$.
- $(\mathrm{id}_X)_\# = \mathrm{id}_{C_n(X)}$ *and* $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.
- *If f is a homeomorphism, then $f_\#$ and f_* are isomorphisms.*

We now return to the initial idea of studying maps to X from spaces obtained by gluing simplices in order to study holes in X, and how that relates to the above construction of the homology groups. Given a cycle $\xi \in C_n(X)$, we can write it as $\sum_i \epsilon_i \sigma_i$ where $\epsilon_i = \pm 1$ and the σ_i's are singular simplices. Make a copy Δ_i^n of Δ^n corresponding to each σ_i, with Δ_i^n oriented in the usual way if $\epsilon_i = 1$ and opposite to the usual way if $\epsilon_i = -1$. Hence we have a map $\bigsqcup_i \sigma_i : \bigsqcup_i \Delta_i^n \to X$. Since $\partial \xi = 0$, this map factors through a quotient K_ξ of $\bigsqcup_i \Delta_i^n$ which is obtained by gluing $(n-1)$-faces of the Δ_i^n's to each-other in pairs so that each $(n-1)$-face is glued to precisely one other face (and these two faces cancel in the expression for $\partial \xi$).

Note that for a given ξ, K_ξ may not be unique. Note also that K_ξ is locally homeomorphic to $\mathbb{R}^n$ at points in the interior of n- and $(n-1)$-simplices of K_ξ, although this may not be the case at points in interiors of simplices of smaller dimension. The orientations of adjacent n-simplices transition consistently through the glued faces, which can be seen using the preceding discussion about the signs appearing in the definition of the boundary map. Hence, removing all k-simplices of K_ξ for $k \le n-2$ yields an orientable manifold.

2.1 Δ-complexes and simplicial homology

The preceding construction of K_ξ motivates the following definition. For a topological space X, a **Δ-complex structure** on X is a collection of maps (called **characteristic maps**) $\Sigma = \{\sigma_\alpha : \Delta^{n_\alpha} \to X \mid \alpha\}$ satisfying the following.

1. $\sigma_\alpha | \operatorname{int} \Delta^{n_\alpha}$ is injective and the images of these restrictions form a disjoint cover of X. Here, we take $\operatorname{int} \Delta^0 = \Delta^0$.
2. $\sigma_\alpha | [v_0, \ldots, \hat{v}_i, \ldots, v_n] \in \Sigma$.
3. The topology on X is the finest topology which makes all σ_α's continuous.

X is said to be an n-dimensional Δ-complex if it is non-empty and is covered by its n-simplices. Hence, an n-dimensional Δ-complex has no simplices of dimension $> n$ and every simplex of dimension $< n$ is a face of an n-simplex.

It is now clear that K_ξ as constructed previously is an n-dimensional Δ-complex with the obvious Δ-complex structure. For Δ-complexes, there is a homology theory similar to but simpler than singular homology, called simplicial homology. For X a Δ-complex with a Δ-complex structure as above, let $\Delta_n(X)$ to be the free abelian group generated by those σ_α for which $n_\alpha = n$. Define the boundary map $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ on the basis elements in the same way as in singular homology and extend $\mathbb{Z}$-linearly.

$$\partial_n \sigma_\alpha := \sum_{i=0}^{n} (-1)^n \sigma_\alpha|[v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

By the second condition in the definition of Δ-complex structure, the image of ∂_n is indeed contained in its codomain. Once again we observe that $\partial_{n-1}\partial_n = 0$, so we obtain a chain complex:

$$\ldots \xrightarrow{\partial} \Delta_n(X) \xrightarrow{\partial} \Delta_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Delta_0(X) \xrightarrow{\partial} 0$$

The homology groups $H_n^\Delta(X)$ of this chain complex are called the **simplicial homology groups** of X. The homology class of a cycle $\sigma \in \Delta_n(X)$ is denoted by $[\sigma]^\Delta$, or $[\sigma]_X^\Delta$ if the choice of X is not clear. For a finite Δ-complex, it is clear that the simplicial homology groups are finitely generated (since $\mathbb{Z}$ is a PID), but this cannot be said as easily for the singular homology groups. However, it is not obvious whether two different Δ-complex structures on X will always give rise to isomorphic simplicial homology groups (i.e. whether simplicial homology is a topological invariant), whereas we were easily able to show that the singular homology groups of homeomorphic spaces are isomorphic.

We have $\Delta_n(X) \subset C_n(X)$ and the simplicial boundary map is just a restriction of the singular boundary map, so if two simplicial n-cycles yield the same simplicial homology class then they also yield the same singular homology class. Hence, we have a canonical homomorphism

$$H_n^\Delta(X) \to H_n(X); [\sigma]^\Delta \mapsto [\sigma]$$

2.2 Homotopy invariance of singular homology

We previously showed that singular homology groups are invariant under homeomorphism. We in fact have a stronger (and expected) result.

Theorem 2.2. *If $f, g : X \to Y$ are homotopic maps, then $f_* = g_*$.*

Corollary 2.3. *If $f : X \to Y$ is a homotopy equivalence then f_* is an isomorphism.*

Proof. Let $g : Y \to X$ be the homotopy inverse of f. Hence, by **theorem 2.2** and **proposition 2.1** we see that f_* and g_* are inverses of each other. □

To prove **theorem 2.2**, we first motivate the geometry behind the argument. Let $\sigma : \Delta^n \to X$ be a singular simplex. If $H : X \times [0,1] \to Y$ is a homotopy from $H(\cdot, 0) = f$ to $H(\cdot, 1) = g$, then we obtain a map

$$\tilde{\sigma} := H \circ (\sigma \times \mathrm{id}_{[0,1]}) : \Delta^n \times [0,1] \to Y$$

which equals $f_\# \sigma$ on $\Delta^n \times \{0\}$ and $g_\# \sigma$ on $\Delta^n \times \{1\}$. Observe that

$$\partial(\Delta^n \times [0,1]) = \Delta^n \times \{0\} \cup \Delta^n \times \{1\} \cup \partial\Delta^n \times [0,1]$$

with $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$ having opposite induced orientations. Suppose we construct a method of subdividing Δ^n which respects this decomposition, i.e. we construct a singular $(n+1)$-chain $\xi \in C_{n+1}(\Delta^n \times [0,1])$ satisfying the condition that $\partial\xi$ can be written (loosely speaking) as

$$(\Delta^n \times \{1\}) - (\Delta^n \times \{1\}) \pm (\text{subdivision of faces of } \partial\Delta^n \times [0,1]) \tag{1}$$

Hence, left-composing $\tilde{\sigma}$ yields

$$\partial(\tilde{\sigma}\xi) = g_\#\sigma - f_\#\sigma \pm (\text{subdivision of } \tilde{\sigma}|\,\partial\Delta^n \times [0,1])$$

Hence, if $\lambda \in C_n(X)$ is a cycle then applying the above idea to λ term-wise yields

$$\operatorname{im}(\partial^X_{n+1}) \ni g_\#\lambda - f_\#\lambda$$

since $\partial\lambda = 0$. It would then follow that $g_*[\lambda] = f_*[\lambda]$, as desired. The following formal proof makes the above precise.

Proof of ***theorem 2.2***. Let H be as above. Let $u_i = (v_i, 0)$ and $w_i = (v_i, 1)$ be vertices of $\Delta^n \times [0,1]$. Define $P : C_n(X) \to C_{n+1}(Y)$, the prism operator, on a singular simplex σ as follows and extend $\mathbb{Z}$-linearly to $C_n(X)$.

$$P\sigma := \sum_i (-1)^i \tilde{\sigma}|[\ldots, u_i, w_i, \ldots, w_n]$$

This definition is motivated by an inductive viewpoint, since for $n = 0, 1$ it is easy to make the preceding idea of subdivision precise. Careful but mundane computation shows that P satisfies the following identity, reminiscent of (1).

$$\partial P = g_\# - f_\# - P\partial$$

Hence, if $\lambda \in C_n(X)$ is a cycle then we see that $g_\#\lambda - f_\#\lambda$ is a boundary, and hence $g_*[\lambda] = f_*[\lambda]$ as desired. □

More generally, given any two chain complexes A_n and B_n and chain morphisms $\phi, \psi : A_n \to B_n$ between them, a map $\Phi : A_n \to B_{n+1}$ is called a **chain homotopy** between ϕ and ψ if

$$\partial\Phi + \Phi\partial = \psi - \phi$$

ϕ and ψ are said to be chain homotopic if there is a chain homotopy between them, and chain homotopic maps produce the same homomorphisms on the homology groups.

2.3 Barycentric subdivision and the Mayer-Vietoris sequence

The goal of this subsection is to prove the following theorem, which is often a useful computational tool.

Theorem 2.4 (Mayer-Vietoris sequence)**.** *Let* $A, B \subset X$ *so that* $\operatorname{int} A \cup \operatorname{int} B = X$*. Then we have an exact sequence*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & H_n(A\cap B) & \xrightarrow{\phi} & H_n(A)\oplus H_n(B) & \xrightarrow{\psi} & H_n(X) & \xrightarrow{\partial} & H_{n-1}(A\cap B) \\ & & & & & & & & \downarrow \\ & & & & 0 & \longleftarrow & H_0(X) & \longleftarrow & \dots \end{array}$$

For convenience, set $C = A \cap B$. The map ∂ will be defined later. ϕ and ψ are defined as follows.

$$\begin{aligned} \phi: & \quad [\mu]_C & \mapsto [\mu]_A \oplus [-\mu]_B \\ \psi: & \quad [\mu]_A \oplus [\tau]_B & \mapsto [\mu+\tau]_X \end{aligned}$$

It is immediate that these maps are well-defined. The key idea in the proof of the theorem is to subdivide a cycle (by subdividing each singular simplex) in a way which does not change its homology class, and then use the Lebesgue number lemma. To do this, we first establish some notation. Let Y be a convex space and denote by $LC_n(Y)$ the free abelian group generated by affine linear singular n-simplices in Y. Hence, every singular simplex in $LC_n(Y)$ is given by $[a_0, \dots, a_n]$ for some $a_0, \dots, a_n \in Y$.

Points $b \in Y$ may (uniquely) be identified with a homomorphism

$$b : LC_n(Y) \to LC_{n+1}(Y); [a_0, \dots, a_n] \mapsto [b, a_0, \dots, a_n]$$

Observe that

$$\partial b = \mathrm{id} - b\partial \tag{2}$$

so b is a chain homotopy between $\mathrm{id}_{LC_n(Y)}$ and the 0 chain map. For $\lambda = [a_0, \dots, a_n] \in LC_n(Y)$, let $b_\lambda \in Y$ be the barycenter of λ, i.e.

$$b_\lambda := \frac{1}{n+1}\sum_{i=0}^{n} a_i$$

We now define the **barycentric subdivision operator** $S : LC_n(Y) \to LC_n(Y)$. For $n = 0$, let $S = \mathrm{id}_{LC_0(Y)}$. For $n \geq 1$, we define S recursively. For $\lambda \in LC_n(Y)$ a singular simplex we set

$$S\lambda := b_\lambda \, S \, \partial\lambda$$

and extend $\mathbb{Z}$-linearly to $LC_n(Y)$. Hence, (2) yields

$$\partial \, S\lambda = S\,\partial\lambda - b_\lambda \,\partial\, S\,\partial\lambda$$

Now, observe that $\partial S \partial = 0$ for $n = 0$ (since $\partial_0 = 0$). Hence, if

$$\partial \lambda = \sum_{i=0}^{n} (-1)^i \lambda_i$$

for $\lambda_i \in LC_{n-1}(Y)$, then inducting on the dimension n we obtain

$$\begin{aligned}
\partial S \lambda &= S\partial\lambda - b_\lambda(\partial S)\partial\lambda \\
&= S\partial\lambda - \underbrace{b_\lambda(S\partial)\partial\lambda}_{0,\text{ since } \partial^2 = 0} + b_\lambda \sum_i (-1)^i \underbrace{b_{\lambda_i}(\partial S \partial)\lambda_i}_{\substack{0,\text{ since } \partial S\partial = 0 \\ \text{in dimension} \\ n-1}} \\
&= S\partial\lambda
\end{aligned} \tag{3}$$

Hence, we see that S is a chain morphism. Geometrically, S decomposes Δ^n into several smaller simplices and then applies λ to each 'piece' in the decomposition. For every permutation $i_0, \ldots, i_n$ of $\{0, \ldots, n\}$, a unique simplex in the decomposition is spanned by the barycentres of the decreasing sequence of simplices starting at $[v_0, \ldots, v_n]$ and removing $v_{i_0}, \ldots, v_{i_{j-1}}$ to obtain the j-th element of the sequence ($0 \le j \le n$). Conversely, every simplex in the decomposition yields a unique permutation of $\{0, \ldots, n\}$. In other words, every permutation yields a unique simplex in the decomposition with vertices

$$\frac{1}{n+1-j}\left(\sum_{i=0}^{n} v_i - \sum_{k=0}^{j-1} v_{i_k}\right),\ j = 0, \ldots, n$$

and conversely. These simplices then inherit orientation from Δ^n. It can be verified that this geometric description for the algebraic definition of S is accurate by induction on the dimension. A more formal (but weaker) statement is that $S\lambda$ *is a* $\mathbb{Z}$*-linear combination of singular simplices given by restrictions of* λ *to simplices obtained from permutations as above* (upto ordering of vertices).

By endowing Y with the metric of the ambient Euclidean space, we claim that the diameter of the image of every singular simplex appearing in the expression for $S\lambda$ is at most

$$\frac{n}{n+1}\operatorname{diam}(\lambda)$$

To see this, we use induction on n (for $n = 0$ the claim is trivial). If the claim is true for $n-1$ then it suffices to show that the distance of b_λ from any point in $\partial\lambda$[1] is at most

$$\frac{n}{n+1}\operatorname{diam}(\lambda)$$

since any singular simplex in the expression for $S\lambda$ is obtained by applying b_λ to a singular simplex in the expression for $\partial\lambda$. Hence it suffices to show that

$$\|b_\lambda - \lambda(x)\| \le \frac{n}{n+1}\operatorname{diam}(\lambda)\ \forall\, x \in \Delta^n$$

[1] Here we are slightly abusing notation by using $\partial\lambda$ to refer to the union of the images of all singular simplices in the expression for $\partial\lambda$. This may or may not be the same as the boundary of the image of λ.

Letting $\lambda = [u_0, \ldots, u_n]$ and $x = \sum_{j=0}^n x_j v_j$ we have

$$\begin{aligned}
\|b_\lambda - \lambda(x)\|^2 &= \left\| \sum_{j=0}^n \left(\frac{1}{n+1} - x_j \right) u_j \right\|^2 \\
&= 2 \sum_{0 \le j < k \le n} \left(\frac{1}{n+1} - x_j \right) \left(\frac{1}{n+1} - x_k \right) \langle u_j, u_k \rangle + \sum_{j=0}^n \left(\frac{1}{n+1} - x_j \right)^2 \|u_j\|^2
\end{aligned}$$

This is a subharmonic function, so by the strong maximum principle it attains its maximum for x a vertex of Δ^n. Hence we see that

$$\begin{aligned}
\|b_\lambda - \lambda(x)\| &\le \max_{0 \le i \le n} \|b_\lambda - u_i\| \\
&= \max_{0 \le i \le n} \left\| \frac{1}{n+1} \sum_{j=0}^n (u_j - u_i) \right\| \\
&\le \frac{n}{n+1} \operatorname{diam}(\lambda)
\end{aligned}$$

as desired. Now clearly

$$\lim_{k \to \infty} \left(\frac{n}{n+1} \right)^k = 0$$

so we obtain the following.

Lemma 2.5. *For $\epsilon > 0$, let $LC_n^\epsilon(Y)$ be the free abelian group on affine linear singular simplices whose images have diameter less than ϵ. Then for every $\epsilon > 0$ and singular simplex $\lambda \in LC_n(Y)$ there exists $k \in \mathbb{N}$ such that $S^k \lambda \in LC_n^\epsilon(Y)$.*

Given a singular simplex $\sigma \in C_n(X)$ we define $S\sigma$ as $\sigma_*(S[v_0, \ldots, v_n])$, where we take $Y = \Delta^n$ in the above discussion with $\sigma : Y \to X$ a continuous function. S is then extended $\mathbb{Z}$-linearly to $C_n(X)$. By (3), it follows that $S : C_n(X) \to C_n(X)$ is a chain morphism.

Lemma 2.6. *$S : C_n(X) \to C_n(X)$ is chain homotopic to $\mathrm{id}_{C_n(X)}$. In particular, α and $S\alpha$ have the same homology class for every cycle $\alpha \in C_n(X)$.*

To prove the above, we wish to construct a chain homotopy $T : C_n(X) \to C_{n+1}(X)$ satisfying

$$\partial T + T\partial = S - \mathrm{id} \tag{4}$$

For this, we must construct a way to subdivide $\Delta^n \times [0,1]$ into simplices so that the 'bottom' n-face $\Delta^n \times \{0\}$ is not subdivided whereas the 'top' n-face $\Delta^n \times \{1\}$ is subdivided as per the barycentric procedure. Furthermore, for any two $(n-1)$-faces λ and λ' of Δ^n the subdivision should be, in some sense, 'the same' on $\lambda \times [0,1]$ and $\lambda' \times [0,1]$. This can be done inductively as follows. One piece in the subdivision will be $b_{\Delta^n \times \{1\}}(\Delta \times \{0\})$, and the remaining region is subdivided by applying $b_{\Delta^n \times \{1\}}$ to the subdivision of $\partial \Delta^n \times [0,1]$. To then define $T\sigma$ for a singular simplex $\sigma \in C_n(X)$, we push forward the subdivision of $\Delta^n \times [0,1]$ via $\sigma \times \mathrm{id}_{[0,1]} : \Delta^n \times [0,1] \to X$.

Proof of ***lemma 2.6***. To make the above precise, we first define $T : LC_n(Y) \to LC_{n+1}(Y)$ satisfying (4) for a convex space Y. For $n = 0$ define $T[p] = [p, p]$ for every $p \in Y$ and extend $\mathbb{Z}$-linearly, so both sides of (4) are 0. For $n > 0$ and $\lambda \in LC_n(Y)$ a singular simplex, define

$$T\lambda := b_\lambda(-T\partial\lambda - \lambda)$$

and extend $\mathbb{Z}$-linearly. To see that T satisfies (4), we use induction on n.

$$\begin{aligned}
\partial T\lambda &= \partial b_\lambda(-T\partial\lambda - \lambda) \\
&= b_\lambda(\partial T(\partial\lambda) + \partial\lambda) - T\partial\lambda - \lambda \\
&= b_\lambda[\underbrace{-T\partial(\partial\lambda)}_{0,\text{ since } \partial^2 = 0} + S\partial\lambda - \partial\lambda] + b_\lambda\partial\lambda - T\partial\lambda - \lambda \text{ (by induction on } n) \\
&= b_\lambda S\partial\lambda - T\partial\lambda - \lambda \\
&= S\lambda - T\partial\lambda - \lambda \text{ (by recurrence for } S)
\end{aligned}$$

Now, given a singular simplex $\sigma \in C_n(X)$ we take $Y = \Delta^n$ in the above and define

$$T\sigma := \sigma_* T[v_0, \dots, v_n]$$

and extend $\mathbb{Z}$-linearly. □

For a singular simplex $\sigma \in C_n(X)$, the sets $\sigma^{-1}(\operatorname{int} A)$ and $\sigma^{-1}(\operatorname{int} B)$ form an open cover of the compact set Δ^n. By the Lebesgue number lemma together with **lemma 2.5**, we have the following, where $C_n(A + B) := C_n(A) + C_n(B)$.

Theorem 2.7. *For any n-chain $\mu \in C_n(X)$ there exists $k \in \mathbb{N}$ such that $S^k\mu \in C_n(A + B)$.*

We will now build up to the definition of ∂ in **theorem 2.4**. For an n-chain $\mu \in C_n(X)$, let $k(\mu)$ be the smallest $k \in \mathbb{N}$ given by **theorem 2.7** (note that $k(\mu) \geq k(\partial\mu)$ but equality may not always hold). Let $p : C_n(X) \to C_n(A)$ be the projection which fixes singular simplices in $C_n(A)$ and sends all other singular simplices to 0. Let $\mu_A = p \circ S^{k(\mu)}\mu$ and $\mu_B = S^{k(\mu)}\mu - \mu_A$ (note that $\partial\mu_A$ and $(\partial\mu)_A$ are not the same in general). Hence for $k \geq k(\mu)$ we see that $(S^k\mu)_A = p(S^k\mu)$. In particular, if $\mu' \in C_n(X)$ and $k \geq k(\mu), k(\mu')$ then

$$(S^k(\mu + \mu'))_A = (S^k\mu)_A + (S^k\mu')_A \tag{5}$$

Define $\delta : \ker(\partial_n^X) \to H_{n-1}(C)$ on n-cycles by

$$\delta\mu := [\partial\mu_A]_C$$

We claim that δ factors through a map $H_n(X) \to H_{n-1}(C)$, which we define to be ∂. First we check that δ is a well-defined homomorphism. Observe that $\partial\mu_A + \partial\mu_B = \partial S^{k(\mu)}\mu = 0$, so

$$C_{n-1}(A) \ni \partial\mu_A = -\partial\mu_B \in C_{n-1}(B)$$

This yields

$$\partial\mu_A \in C_{n-1}(A) \cap C_{n-1}(B) = C_{n-1}(C)$$

Hence $\partial\mu_A \in C_{n-1}(C)$. Also, $\partial\mu_A$ is a cycle since $\partial^2 = 0$. To prove that δ is a homomorphism, we require a lemma.

Lemma 2.8. *$[\partial(S^k\mu)_A]_C = \delta(\mu)$ for all $k \geq k(\mu)$.*

Proof. Let $r = k - k(\mu)$. We have

$$\begin{aligned}(S^k\mu)_A - S^r\mu_A &= (S^r(S^{k(\mu)}\mu))_A - S^r\mu_A \\ &= (S^r(\mu_A + \mu_B))_A - S^r\mu_A \\ &= (S^r\mu_B)_A\end{aligned}$$

Since $S^r\mu_B \in C_n(B)$, we have

$$(S^k\mu)_A - S^r\mu_A = (S^r\mu_B)_A \in C_n(A) \cap C_n(B) = C_n(C)$$

This yields

$$\begin{aligned}[\partial(S^k\mu)_A]_C &= [\partial S^r\mu_A]_C \\ &= [S^r\partial\mu_A]_C \text{ (since } S \text{ is a chain morphism)} \\ &= [\partial\mu_A]_C \text{ (by } \textbf{lemma 2.6}) \\ &= \delta(\mu)\end{aligned}$$

□

Now, for cycles $\mu, \tau \in C_n(X)$ let $k = \max\{k(\mu), k(\tau), k(\mu+\tau)\}$ so that

$$\begin{aligned}\delta(\mu+\tau) &= [\partial(\mu+\tau)_A]_C \\ &= [\partial(S^k(\mu+\tau))_A]_C \text{ (by } \textbf{lemma 2.8}) \\ &= [\partial(S^k\mu)_A + \partial(S^k\tau)_A]_C \text{ (by (5))} \\ &= [\partial(S^k\mu)_A]_C + [\partial(S^k\tau)_A]_C \\ &= \delta(\mu) + \delta(\tau) \text{ (by } \textbf{lemma 2.8})\end{aligned}$$

Hence δ is a homomorphism. Lastly, to see that δ factors through $H_n(X)$ let $\xi \in C_{n+1}(X)$ and we will show that $\delta(\partial\xi) = 0$. Let $k = k(\xi) \geq k(\partial\xi)$. Hence, by **lemma 2.8** we have

$$\begin{aligned}\delta(\partial\xi) &= [\partial(S^k\partial\xi)_A]_C \\ &= [\partial(\partial S^k\xi)_A]_C\end{aligned}$$

Now, observe that

$$\begin{aligned}(\partial S^k\xi)_A &= (\partial(S^k\xi)_A)_A + (\partial(S^k\xi)_B)_A \text{ (by (5))} \\ &= \partial(S^k\xi)_A + (\partial(S^k\xi)_B)_A \\ &\in \partial(S^k\xi)_A + C_n(C)\end{aligned}$$

Hence it follows by the preceding expression for $\delta(\partial\xi)$ that $\delta(\partial\xi) = 0$ as desired.

To prove **theorem 2.4**, it now only remains to show that the sequence is exact. From definitions, it is at least clear that the sequence is a chain complex; we now prove the remaining inclusions.

- $\ker \psi \subset \operatorname{im} \phi$.
  Let $\mu \in C_n(A)$ and $\tau \in C_n(B)$ be cycles with $\psi([\mu]_A \oplus [\tau]_B)$. Hence we obtain $\xi \in C_{n+1}(X)$ with $\partial \xi = \mu + \tau$. Let $k = k(\xi)$. Hence, we have

$$\begin{aligned} \partial \xi_A + \partial \xi_B &= S^k \mu + S^k \tau \\ \implies S^k \mu - \partial \xi_A &= -(S^k \tau - \partial \xi_B) \in C_n(C) \end{aligned}$$

  Hence $\sigma = S^k \mu - \partial \xi_A$ is a cycle in C with $[\sigma]_A = [S^k \mu]_A = [\mu]_A$ and $[\sigma]_B = [-S^k \tau]_B = -[\tau]_B$, i.e.

$$\phi([\sigma]_C) = [\mu]_A \oplus [\tau]_B$$

- $\ker \partial \subset \operatorname{im} \psi$.
  Let $\sigma \in C_n(X)$ be a cycle with $\partial([\sigma]_X) = 0$. Hence, $\partial \sigma_A = \partial \xi$ for some $\xi \in C_n(C)$. Let $\mu = \sigma_A - \xi \in C_n(A)$ and $\tau = \sigma_B + \xi \in C_n(B)$. Hence, μ and τ are cycles with

$$\psi([\mu]_A \oplus [\tau]_B) = [\sigma_A + \sigma_B]_X = [\sigma]_X$$

- $\ker \phi \subset \operatorname{im} \partial$.
  Let $\sigma \in C_n(C)$ be a cycle with $\phi([\sigma]_C) = 0$. Hence, $\sigma = \partial \xi = \partial \eta$ for some $\xi \in C_{n+1}(A)$ and $\eta \in C_{n+1}(B)$. Hence, $\mu := \xi - \eta \in C_{n+1}(X)$ is a cycle. Furthermore, $k(\mu) = 1$ and $\mu_A - \xi \in C_{n+1}(C)$, so

$$\partial([\mu]_X) = [\partial \mu_A]_C = [\partial \xi]_C = [\sigma]_C$$

The procedure followed above also gives a Mayer-Vietoris sequence for the simplicial homology groups, except no subdivision is required.

Theorem 2.9. *Let X be a Δ-complex and $A, B \subset X$ be subcomplexes with $A \cup B = X$. Then we have an exact sequence*

$$\begin{array}{ccccccccc} \dots \longrightarrow H_n^\Delta(A \cap B) & \xrightarrow{\phi} & H_n^\Delta(A) \oplus H_n^\Delta(B) & \xrightarrow{\psi} & H_n^\Delta(X) & \xrightarrow{\partial} & H_{n-1}^\Delta(A \cap B) \\ & & & & & & \downarrow \\ & & 0 & \longleftarrow & H_0^\Delta(X) & \longleftarrow & \dots \end{array}$$

All maps are defined in the same way as before. However, observe that no subdivision is required to define ∂ since $\Delta_n(X) = \Delta_n(A) + \Delta_n(B)$. This essentially means that the proof of **theorem 2.4** carries forward directly to a proof of **theorem 2.9** by ignoring all occurrences of S, or alternatively by defining S to be the identity.

2.3.1 Singular homology for simplicial complexes

For X a Δ-complex, we have a natural map $H_n^\Delta(X) \to H_n(X)$ induced by inclusion of $\Delta_n(X)$ in $C_n(X)$. In this section, we will use **theorem 2.4** to show that this map is an isomorphism.

Theorem 2.10. *Let X be a Δ-complex. The inclusion $\Delta_n(X) \hookrightarrow C_n(X)$ of chain complexes induces an isomorphism $H_n^\Delta(X) \xrightarrow{\sim} H_n(X)$.*

The following version of **theorem 2.4** (which follows easily from **theorem 2.4**) will be the main topological tool.

Theorem 2.11. *Let X be a topological space and $A, B \subset X$ such that $A \cup B = X$. Suppose also that U and V are open neighbourhoods of A and B respectively such that U deforms onto A, V deforms onto B and $U \cap V$ deforms onto $A \cap B$. Then we have an exact sequence*

$$\cdots \longrightarrow H_n(A\cap B) \xrightarrow{\phi} H_n(A)\oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \longrightarrow \cdots \longrightarrow H_0(X) \longrightarrow 0$$

Here ϕ and ψ are defined as before. We define $\partial([\mu]_X)$ for a cycle $\mu \in C_n(X)$ as follows. Define μ_U using subdivisions as before ($S^k\mu \in C_n(U) + C_n(V)$ for sufficiently large $k \in \mathbb{N}$). Let $r : U \cap V \to A \cap B$ be the known retraction. Then $\partial([\mu]_X) := [r_*\partial\mu_U]_{A\cap B}$.

The following algebraic lemma will also be useful to prove **theorem 2.10**.

Lemma 2.12 (Five lemma)**.** *Consider the following commutative diagram in the category of abelian groups.*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\epsilon}\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

If $\alpha, \beta, \delta, \epsilon$ are isomorphisms and the rows are exact, then γ is also an isomorphism.

*Proof of **theorem 2.10**.* We will prove the case when X is a finite Δ-complex using strong induction on $|X|$, the number of simplices in X. This suffices to prove the general statement since the image of a singular simplex is compact. Note that the claim is trivially true for the case of $|X| = 1$, i.e. the case of X being a point.

Let $A \subset X$ be a simplex of top dimension, $B = X - \operatorname{int} A = \overline{X - A}$ and $C = A \cap B = \partial A$. Hence $|B|, |C| < |X|$ and so the induction hypothesis applies on B and C. We have the following diagram with exact rows given by **theorem 2.9** and **theorem 2.11** respectively, and vertical maps induced by the inclusion $\Delta_n(X) \hookrightarrow C_n(X)$.

$$\begin{array}{ccccccccc} H_n^\Delta(C) & \longrightarrow & H_n^\Delta(A)\oplus H_n^\Delta(B) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_{n-1}^\Delta(C) & \longrightarrow & H_{n-1}^\Delta(A)\oplus H_{n-1}^\Delta(B) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\epsilon}\downarrow \\ H_n(C) & \longrightarrow & H_n(A)\oplus H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_{n-1}(C) & \longrightarrow & H_{n-1}(A)\oplus H_{n-1}(B) \end{array}$$

The commutativity of the squares can be verified easily using the definitions of all the maps involved. By the induction hypothesis we see that $\alpha, \beta, \delta, \epsilon$ are isomorphisms, so the five lemma (**lemma 2.12**) completes the proof. $\square$

2.4 Relative homology and excision

Let C be a chain complex with maps ∂ and let A be a subcomplex. Since ∂ maps A_n to A_{n-1}, we obtain a well-defined map $\partial : C_n/A_n \to C_{n-1}/A_{n-1}$. This yields a chain complex C/A — the **relative chain complex**.

$$\ldots \longrightarrow C_n/A_n \xrightarrow{\partial} C_{n-1}/A_{n-1} \longrightarrow \ldots$$

This corresponds with a **short exact sequence of chain complexes**.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & C_n/A_n & \xrightarrow{\partial} & C_{n-1}/A_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

It is clear that the squares are commutative — the inclusion $A \hookrightarrow C$ and the surjection $C \twoheadrightarrow C/A$ are chain morphisms. Such a short exact sequence of chain complexes yields a long exact sequence of homology groups.

Theorem 2.13. *Let A, B, C be chain complexes. A short exact sequence*

$$0 \longrightarrow A_n \xrightarrow{i} C_n \xrightarrow{p} B_n \longrightarrow 0$$

of chain complexes induces the following long exact sequence of homology groups.

$$\ldots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(C) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \ldots$$

Here the map $\partial : H_n(B) \to H_{n-1}(A)$ (called the connecting map) is defined as follows. Let $\beta \in B_n$ be a cycle. Since p is surjective, β has a preimage $\gamma \in C_n$ with $p\gamma = \beta$. This yields

$$\begin{aligned}
0 &= \partial\beta \\
&= \partial\, p\gamma \\
&= p\,\partial\gamma \\
\implies \quad & \partial\gamma \in \ker(p) = \operatorname{im} i.
\end{aligned}$$

By the injectivity of i, there exists a (unique) cycle $\alpha \in A_{n-1}$ with $i(\alpha) = \partial\gamma$. We define

$$\partial([\beta]) := [\alpha]$$

This definition is independent of the choices made when picking the representative β of $[\beta]$ and the preimage $\gamma \in p^{-1}(\beta)$, but we omit the proof here.

One consequence of **theorem 2.13** is an alternate proof of **theorem 2.4**, the Mayer-Vietoris sequence, which we sketch below. For A, B, C, X as in **theorem 2.4**, we have the following short exact sequence of chain complexes.

$$0 \longrightarrow C_n(C) \xrightarrow{\tilde{\phi}} C_n(A) \oplus C_n(B) \xrightarrow{\tilde{\psi}} C_n(A+B) \longrightarrow 0 \tag{6}$$

The maps $\tilde{\phi}$ and $\tilde{\psi}$ are defined as follows.

$$\tilde{\phi} : \mu \mapsto \mu \oplus (-\mu)$$
$$\tilde{\psi} : \mu \oplus \tau \mapsto \mu + \tau$$

Exactness of (6) is immediate. Since $\tilde{\phi}_* = \phi$, **theorem 2.13** yields the following long exact sequence.

$$\ldots \longrightarrow H_n(C) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\tilde{\psi}_*} H_n(A+B) \xrightarrow{\partial^\wedge} H_{n-1}(C) \longrightarrow \ldots$$

It can now be shown using **theorem 2.7**, **lemma 2.6** and **lemma 2.8** that the inclusion $C_n(A+B) \hookrightarrow C_n(X)$ induces an isomorphism on the homology groups (in fact this inclusion is a chain homotopy equivalence), so the above exact sequence becomes

$$\ldots \longrightarrow H_n(C) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial^{\wedge\wedge}} H_{n-1}(C) \longrightarrow \ldots$$

Here $\partial^{\wedge\wedge}$ is the composition of $\partial^\wedge$ with the isomorphism $H_n(A+B) \xrightarrow{\sim} H_n(X)$ induced by $C_n(A+B) \hookrightarrow C_n(X)$. From the definition of $\partial^\wedge$, it can then be seen that $\partial^{\wedge\wedge}$ is precisely the map ∂ from **theorem 2.4**, completing the proof. This technique of obtaining the Mayer-Vietoris sequence also works for proving **theorem 2.9** and makes more direct use of the fact that $\Delta_n(X) = \Delta_n(A+B)$.

For an arbitrary topological space X, a subspace $A \subset X$ gives a natural choice of a subcomplex of $C_n(X)$, namely $C_n(A)$. Denoting the relative chain complex $C_n(X)/C_n(A)$ by $C_n(X, A)$ (and the corresponding homology groups by $H_n(X, A)$), we have a short exact sequence of chain complexes

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

which yields a long exact sequence of homology groups, called the **long exact sequence of the pair** (X, A).

$$\ldots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \ldots$$

For $A = \emptyset$, this yields

$$H_n(X, \emptyset) \approx H_n(X)$$

For $A = \{x_0\}$ a point, $H_n(\{x_0\}) = 0$ when $n > 0$ and $H_0(A) = \mathbb{Z}$. Hence, we see that

$$H_n(X, \{x_0\}) \approx H_n(X) \text{ for } n > 0$$

For $n = 0$, we have a short exact sequence describing $H_0(X, \{x_0\})$ coming from the long exact sequence of the pair $(X, \{x_0\})$.

$$0 \longrightarrow H_0(\{x_0\}) \longrightarrow H_0(X) \longrightarrow H_0(X, \{x_0\}) \longrightarrow 0$$

Hence, in a loose sense, $H_0(X, \{x_0\}) \approx H_0(X)/\mathbb{Z}$. The groups $H_n(X, \{x_0\})$ are called the **reduced homology groups** of X. For $n > 0$, the basepoint independence of these groups is clear from the above discussion. For $n = 0$ and $x_1 \in X$, there is an isomorphism $H_0(X) \to H_0(X)$ which takes the generator corresponding to the path component of x_0 to the path-component of x_1. Hence, the above short exact sequence gives a corresponding isomorphism between $H_0(X, \{x_0\})$ and $H_0(X, \{x_1\})$.

A basepoint-independent approach can also be taken to define the reduced homology groups. This is done by appending the chain complex $C_n(X)$ with $\mathbb{Z}$ in the -1-th dimension. The 0-th boundary map $\varepsilon : C_0(X) \to \mathbb{Z}$ takes all singular 0-simplices to 1. The homology group in dimension n of this extended chain complex, denoted by $\tilde{H}_n(X)$, is the n-th reduced homology group.

$$\ldots \longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \ldots \longrightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Cycles in the chain complex $C_n(X, A)$ are those chains whose boundary lies in $C_n(A)$.[2] This suggests that $H_n(X, A) \approx H_n(X/A, A/A)$, since $H_n(X, A)$ essentially ignores the 'internal structure' of A. If this were true, we would have

$$H_n(X, A) \approx \tilde{H}_n(X/A)$$

Indeed, this turns out to be true when the way A 'sits inside' X is sufficiently nice. Say that (X, A) is a **good pair** if there is an open neighbourhood $U \subset X$ of A with A as a deformation retract.

Theorem 2.14. *If (X, A) is a good pair, then the map of pairs $(X, A) \to (X/A, A/A)$ induces an isomorphism $H_n(X, A) \xrightarrow{\sim} H_n(X/A, A/A)$. In particular,*

$$H_n(X, A) \approx H_n(X/A, A/A) \approx \tilde{H}_n(X/A)$$

To prove **theorem 2.14** we require a versatile tool called the excision theorem, which makes precise another aspect of how $H_n(X, A)$ ignores the internal structure of A — $H_n(X, A)$ remains unchanged if some (sufficiently nice) subspace of A is removed from both X and A.

[2]Technically, cycles in $C_n(X, A)$ are equivalence classes in the quotient $C_n(X)/C_n(A)$ which are *represented* by a chain in $C_n(X)$ whose boundary lies in $C_n(A)$.

Theorem 2.15 (Excision theorem)**.** *Let $Z \subset A \subset X$ with $\overline{Z} \subset \operatorname{int} A$. The inclusion of pairs $(X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism*

$$H_n(X - Z, A - Z) \xrightarrow{\sim} H_n(X, A)$$

Equivalently, if $A, B \subset X$ such that the interiors of A and B cover X, then the inclusion of pairs $(B, B \cap A) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(B, B \cap A) \xrightarrow{\sim} H_n(X, A)$$

*Proof of **theorem 2.14**.* Since A is closed, the quotient map $X \to X/A$ restricts to homeomorphisms $X - A \to (X/A) - (A/A)$ and $U - A \to (U/A) - (U/A)$. By treating these as identifications, we obtain the following diagram.

$$\begin{array}{ccc} H_n(X - A, U - A) & \xrightarrow{\approx} & H_n(X/A, U/A) \\ {\scriptstyle\approx}\big\downarrow & \nearrow & \\ H_n(X, U) & & \end{array} \tag{7}$$

The isomorphisms come from **theorem 2.15** and are induced by inclusions of pairs. The diagonal map is induced by the quotient map. Hence the commutativity of the diagram is immediate — the diagonal map is an isomorphism. Since U deforms onto A, the inclusion $A \hookrightarrow U$ induces an isomorphism on homology groups. The long exact sequences of the pairs (X, U) and (X, A), along with the five lemma, now show that the inclusion $(X, A) \hookrightarrow (X, U)$ also induces an isomorphism on homology groups. Likewise for the inclusion $(X/A, A/A) \hookrightarrow (X/A, U/A)$. These isomorphisms together with (7) yield the following diagram.

$$\begin{array}{ccccc} H_n(X, U) & \xrightarrow{\approx} & H_n(X/A, U/A) & \xleftarrow{\approx} & H_n(X/A, A/A) \\ & {\scriptstyle\approx}\nwarrow & & \nearrow & \\ & & H_n(X, A) & & \end{array}$$

Commutativity is immediate from the definitions of all the maps, and so the claim follows. $\square$

*Proof of **theorem 2.15**.* We will prove the second statement. Let $i : (B, A \cap B) \hookrightarrow (X, A)$ be the inclusion. Injectivity of i_* is immediate since

$$C_n(A \cap B) + \operatorname{im}(\partial_n^B) \subset C_n(A) + \partial_n(X)$$

To see surjectivity, let $\mu \in C_n(X)$ with $\partial\mu \in C_n(A)$. Recalling the chain homotopy T between S and id from the proof of **lemma 2.6**, we have

$$\begin{aligned} S\mu - \mu &= T\partial\mu + \partial T\mu \\ S^2\mu - S\mu &= T\partial S\mu + \partial T S\mu \\ &\cdots \\ S^k\mu - S^{k-1}\mu &= T\partial S^{k-1}\mu + \partial T S^{k-1}\mu \end{aligned}$$

Adding all these equations and using the fact that $\partial S^i \mu = S^i \partial \mu \in C_n(A)$ (since $\partial \mu \in C_n(A)$), we obtain

$$S^k \mu \equiv \mu \mod C_n(A) + \operatorname{im}\left(\partial_n^X\right)$$

Using notation from §2.3, setting $k = k(\mu)$ yields

$$\mu_B \equiv \mu \mod C_n(A) + \operatorname{im}(\partial_n)$$

In particular, we have

$$[\mu]_{(X,A)} = i_* \left([\mu_B]_{(B,A\cap B)}\right) \in \operatorname{im}(i_*) \quad \square$$

A useful invariant of the local topology of a space X at a point $x \in X$ is the local homology group $H_n(X, X - \{x\})$. For any open neighbourhood $U \subset X$ of x, excision yields $H_n(X, X - \{x\}) \approx H_n(U, U\{x\})$ (assuming that $\{x\}$ is closed in X). Hence for any other space Y and a point $y \in Y$ (with $\{y\}$ closed in Y), a local homeomorphism from x to y must induce an isomorphism on local homology groups. The following is an important consequence of this idea.

Proposition 2.16 (Invariance of dimension)**.** *If nonempty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$.*

Proof. A homeomorphism $f : U \to V$ induces an isomorphism of local homology groups $H_k(U, U - \{x\})$ and $H_k(V, V - \{f(x)\})$. By excision, we have

$$H_k(U, U - \{x\}) \approx H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\})$$

Also, the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ yields

$$H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \approx \tilde{H}_{k-1}(\mathbb{R}^n - \{x\}) \approx \tilde{H}_{k-1}(S^{n-1})$$

Hence, the local homology groups $H_k(U, U - \{x\})$ and $H_k(V, V - \{f(x)\})$ are 0 if $k \neq n$ and $\mathbb{Z}$ otherwise. Likewise, by using excision on $H_k(V, V - \{f(x)\})$, we see that the above groups are 0 if $k \neq m$ and $\mathbb{Z}$ otherwise. Hence $n = m$. $\square$

2.4.1 Relative homology for Δ-complexes

For X a Δ-complex and A a subcomplex, the homology groups of the relative chain complex $\Delta_n(X, A) := \Delta_n(X)/\Delta_n(A)$ are the relative simplicial homology groups $H_n^\Delta(X, A)$, and by **theorem 2.13** we obtain sequences of pairs and triples as in the case of singular homology. Using this framework of relative homology, we obtain an alternative proof of the fact that simplicial and singular homology are equivalent for Δ-complexes (namely **theorem 2.10**).

*Alternate proof of **theorem 2.10**.* We will prove the claim for all finite-dimensional skeletons X^k of X by induction on k (hence proving the claim for the case when X is finite-dimensional), and once again attribute the general statement to the compactness of the image of singular simplices.

For $k = 0$ the claim is trivial, so let $k > 0$. We now make some observations regarding the various homology groups of the pair (X^k, X^{k-1}).

1. $\Delta_n(X^k, X^{k-1})$ is 0 if $n \neq k$, and for $n = k$ it is the free abelian group on the k-simplices of X. In particular, we also have $\Delta_n(X^k, X^{k-1}) = H_n^\Delta(X^k, X^{k-1})$ for all n.
2. By **theorem 2.14**, the quotient map
$$(X^k, X^{k-1}) \to (X^k/X^{k-1}, X^{k-1}/X^{k-1})$$
induces an isomorphism
$$H_n(X^k, X^{k-1}) \xrightarrow{\sim} H_n(X^k/X^{k-1}, X^{k-1}/X^{k-1}) = \tilde{H}_n(X^k/X^{k-1})$$
3. X^k/X^{k-1} is a wedge sum of k-spheres, with each k-sphere the image of a unique k-simplex of X. Hence, $\tilde{H}_n(X^k/X^{k-1})$ is 0 if $n \neq k$ and it is the free abelian group with basis given by the homology classes of (the characteristic maps of) the k-simplices of X.

Putting these observations together, we see that the inclusion $\Delta_n(X^k, X^{k-1}) \hookrightarrow C_n(X^k, X^{k-1})$ induces an isomorphism on the homology groups. Hence, we have the following diagram with exact rows coming from the long exact sequences of the pair (X^k, X^{k-1}) and vertical maps induced by the inclusions of the respective simplicial chain complexes in the respective singular chain complexes.

$$\begin{array}{ccccccccc}
H_{n+1}^\Delta(X^k, X^{k-1}) & \longrightarrow & H_n^\Delta(X^{k-1}) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X^k, X^{k-1}) & \longrightarrow & H_{n-1}^\Delta(X^{k-1}) \\
\alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\
H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

The commutativity of the diagram is immediate from the definitions of all the maps. We know that β and ϵ are isomorphisms by the induction hypothesis. Also, α and δ are isomorphisms based on the preceding discussion. Hence, the claim follows by the five lemma. □

It is natural to expect a relative version of **theorem 2.10** as well, and indeed it follows immediately from the long exact sequence of pairs and the five-lemma.

Corollary 2.17. *Let X be a Δ-complex and $A \subset X$ be a subcomplex. The inclusion $\Delta_n(X, A) \hookrightarrow C_n(X, A)$ induces an isomorphism $H_n^\Delta(X, A) \xrightarrow{\sim} H_n(X, A)$.*

Proof. Consider the following diagram with exact rows coming from the (simplicial and singular) long exact sequences of the pair (X, A) and vertical maps induced by the inclusions of the respective simplicial chain complexes in the respective singular chain complexes.

$$\begin{array}{ccccccccc}
H_n^\Delta(A) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X, A) & \longrightarrow & H_{n-1}^\Delta(A) & \longrightarrow & H_{n-1}^\Delta(X) \\
\alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\
H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X)
\end{array}$$

The commutativity of the diagram is immediate from the definitions of all the maps. We know that α, β, δ and ϵ are isomorphisms by **theorem 2.10**, so the claim follows by the five lemma. □

2.5 Degree of maps $S^n \to S^n$

For a map $f : S^n \to S^n$ $(n > 1)$, the **degree** $\deg f$ is the integer d such that $f_* : H_n(S^n) \to H_n(S^n)$ is given by multiplication by d. Such d exists and is unique since $H_n(S^n) \approx \mathbb{Z}$. By homotopy invariance of homology, it is clear that homotopic maps have the same degree. The converse is also true and essentially boils down to the fact that $\pi_n(S^n) \approx \mathbb{Z}$, with generator given by the homotopy class of id_{S^n}.

Proposition 2.18. $\deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$.

Proof. Identify S^n with the space

$$\left\{ \sum_{i=0}^{n} x_i v_i \,\middle|\, \sum_{i=0}^{n} |x_i| = 1 \right\}$$

which is canonically a simplicial complex with 2^{n+1} simplices, each of the form $[\pm v_0, \ldots, \pm v_n]$. For $s = (s_0, \ldots, s_n) \in \{-1, 1\}^{n+1}$, let $s\Delta^n = [s_0 v_0, \ldots, s_n v_n]$. Observe that for any $s \in \{-1, 1\}^{n+1}$, the $(n-1)$-simplex

$$[s_0 v_0, \ldots, \widehat{s_i v_i}, \ldots, s_n v_n]$$

has the same sign in the expressions for

$$\partial[s_0 v_0, \ldots, s_i v_i, \ldots, s_n v_n]$$

and

$$\partial[s_0 v_0, \ldots, -s_i v_i, \ldots, s_n v_n]$$

Hence, the n-chain

$$\sigma := \sum_{s \in \{-1,1\}} (s_0 \cdot \ldots \cdot s_n)\, s\Delta^n$$

is a cycle in $\Delta_n(S^n)$. It can now be seen that σ is a generator of the group of cycles in $\Delta_n(S^n)$,[3] which is the same as $H_n^{\Delta}(S^n)$. Consequently, $[\sigma]$ is a generator of $H_n(S^n)$ by **theorem 2.10**. Finally, observe that

$$(-\mathrm{id}_{S_n})_{\#}\sigma = (-1)^{n+1}\sigma \quad \square$$

Proposition 2.19. *If $f : S^n \to S^n$ has no fixed points then $f \cong -\mathrm{id}_{S^n}$ and $\deg f = (-1)^{n+1}$.*

Proof. By homotopy invariance of degree, it suffices to show that $f \cong -\mathrm{id}_{S^n}$. Since $\mathbb{R}^{n+1} - \{0\}$ retracts onto S^n, it suffices to show that f and $-\mathrm{id}_{S^n}$ are homotopic as maps from S^n to $\mathbb{R}^{n+1} - \{0\}$. Indeed, we have the following homotopy from f to $-\mathrm{id}_{S^n}$.

$$H : S^n \times [0,1] \to \mathbb{R}^{n+1} - \{0\}; (x,t) \mapsto (1-t)f(x) - tx$$

Since $\|f(x)\| = \|x\| = 1$ and $f(x) \neq x$ for all $x \in S^n$, the image of H is indeed contained in $\mathbb{R}^{n+1} - \{0\}$. $\square$

[3] This follows from the fact that every $(n-1)$-simplex in S^n is in the boundary of exactly two n-simplices.

Corollary 2.20. *The only non-trivial group which acts freely on S^n for n even is $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Suppose G acts freely on S^n and $g \in G$ is not the identity. Hence, the homeomorphism $x \mapsto gx$ from S^n to S^n has no fixed points. By **proposition 2.19**, this map has degree -1. Since this holds for all non-trivial $g \in G$, the group homomorphism $G \to \{1, -1\}$ given by $g \mapsto \deg(x \mapsto gx)$ has trivial kernel. $\square$

A notion related to degree is that of local degree. Let $f : S^n \to S^n, x_0 \in S^n$ and $U \subset S^n$ be an open neighbourhood of x_0 such that $f(y) \neq f(x_0)$ for $y \in U - \{x_0\}$. Hence, we have an induced homomorphism

$$f_* : H_n(U, U - \{x_0\}) \to H_n(S^n, S^n - \{f(x_0)\})$$

The domain and codomain of this homomorphism are both $\mathbb{Z}$, so it seems natural to define the local degree of f at x_0 in the same way we defined $\deg f$. However, this requires natural isomorphisms $\mathbb{Z} \xrightarrow{\sim} H_n(U, U - \{x_0\})$ and $H_n(S^n, S^n - \{f(x_0)\}) \xrightarrow{\sim} \mathbb{Z}$ to be produced, since otherwise there will be an ambiguity of sign in the definition of local degree. For this, we consider the following diagram.

$$\begin{array}{ccccc} H_n(S^n, S^n - \{x_0\}) & \xleftarrow{\approx} & H_n(U, U - \{x_0\}) & \xrightarrow{f_*} & H_n(S^n, S^n - \{f(x_0)\}) \\ \uparrow \approx & & & & \uparrow \approx \\ H_n(S^n) & & \dashrightarrow & & H_n(S^n) \end{array} \tag{8}$$

The isomorphisms are the obvious ones obtained via excision and the long exact sequence of pairs, and the dashed arrow is chosen to make the diagram commute. This dashed arrow is given by multiplication by a unique integer called the **local degree** of f at x_0, written $\deg f \mid x_0$. Observe that if $U' \subset U$ is a smaller neighbourhood of x_0, then we have the following diagram.

$$\begin{array}{ccccc} & \swarrow^{\approx} & H_n(U', U' - \{x_0\}) & \searrow^{f_*} & \\ & & \downarrow \approx & & \\ H_n(S^n, S^n - \{x_0\}) & \xleftarrow[\approx]{} & H_n(U, U - \{x_0\}) & \xrightarrow{f_*} & H_n(S^n, S^n - \{f(x_0)\}) \end{array}$$

The isomorphisms all come from excision and so the diagram commutes. This shows that the dashed arrow in (8) does not change when U is replaced by U' — the local degree of f is indeed a local quantity. The next result gives an alternate definition of local degree.

Proposition 2.21. *Let $f : S^n \to S^n$ and $x_0 \in S^n$ such that $\deg f \mid x_0$ is defined. If $g : S^n \to S^n$ is a homeomorphism with $gf(x_0) = x_0$ and $g \cong \mathrm{id}_{S^n}$, then*

$$\deg f \mid x_0 = \deg gf \mid x_0$$

Since $gf(x_0) = x_0$, the above yields that the dashed arrow in the following diagram, defined to make the diagram commute, is simply mulitplication by $\deg f \mid x_0$. A particularly useful case of this, for the purpose of computation, is when g is of the form $x \mapsto \frac{Ax}{\|Ax\|}$ for some $A \in GL(n+1)$ with positive determinant.

$$\begin{array}{ccc} H_n(U, U - \{x_0\}) & & \\ \approx \big\downarrow & \searrow^{(gf)_*} & \\ H_n(S^n, S^n - \{x_0\}) & \dashrightarrow & H_n(S^n, S^n - \{x_0\}) \end{array}$$

Proof of ***proposition 2.21****.* Since g is a homeomorphism, it is clear that $\deg gf \mid x_0$ is defined. Since $g \cong \mathrm{id}_{S^n}$, the map $g_* : H_n(S^n) \to H_n(S^n)$ is the identity. Consider the following diagram.

$$\begin{array}{ccc} H_n(S^n, S^n - \{f(x_0)\}) & \xrightarrow{g_*} & H_n(S^n, S^n - \{x_0\}) \\ \approx \big\uparrow & & \big\uparrow \approx \\ H_n(S^n) & \xrightarrow{g_* = \mathrm{id}} & H_n(S^n) \end{array}$$

The isomorphisms come from the long exact sequences of pairs, so the diagram commutes. Appending this diagram to (8) on the right and using $g_* f_* = (gf)_*$ now proves the claim. □

Now that we have a tool to aid in computation of local degrees, we develop a tool which allows for calculating the degree of a map in terms of local degrees in most situations that arise in practice.

Theorem 2.22. *If $f : S^n \to S^n$ such that for some $y \in S^n$ the preimage $f^{-1}(y)$ is finite, then*

$$\deg f = \sum_{x \in f^{-1}(y)} \deg f \mid x$$

Proof. Let $x_1, \ldots, x_m \in S^n$ be all the preimages of y, and let $U_i \subset S^n$ be disjoint neighbourhoods of the x_i's. Let $U = U_1 \cup \ldots \cup U_m$. Fix a generator $\alpha \in H_n(S^n)$. Let $\beta_i \in H_n(S^n, S^n - \{x_i\})$ and $\gamma_i \in H_n(U_i, U_i - \{x_i\})$ be images of α under the the isomorphisms given by the left-upper corner of (8):

$$\begin{array}{ccc} H_n(S^n, S^n - \{x_i\}) & \xleftarrow{\approx} & H_n(U_i, U_i - \{x_i\}) \\ \approx \big\uparrow & & \\ H_n(S^n) & & \end{array}$$

Hence, the γ_i's generate

$$\bigoplus_{i=1}^{m} H_n(U_i, U_i - \{x_i\}) \approx H_n(U, U - f^{-1}(y)) \approx H_n(S^n, S^n - f^{-1}(y)) \tag{9}$$

Now we have a diagram

$$\begin{array}{ccc} H_n(S^n, S^n - \{x_i\}) & \xleftarrow{\approx} & H_n(U_i, U_i - \{x_i\}) \\ \approx \big\uparrow & \nwarrow^{p_i} & \big\downarrow \\ H_n(S^n) & \xrightarrow{j} & H_n(S^n, S^n - f^{-1}(y)) \end{array}$$

All maps are the obvious ones, so it can be checked that the triangles commute. Hence, we see that $p_i \circ j(\alpha) = \beta_i$. By (9), it follows that $j(\alpha) = \sum_i \beta_i$ under appropriate identification. Finally, we have

$$\begin{array}{ccc}
 & H_n(U, U - f^{-1}(y)) & \\
 \swarrow^{\approx} & & \searrow^{f_*} \\
H_n(S^n, S^n - f^{-1}(y)) & \xrightarrow{\quad f_* \quad} & H_n(S^n, S^n - \{y\}) \\
{\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \approx} \\
H_n(S^n) & \xrightarrow{\quad f_* \quad} & H_n(S^n)
\end{array}$$

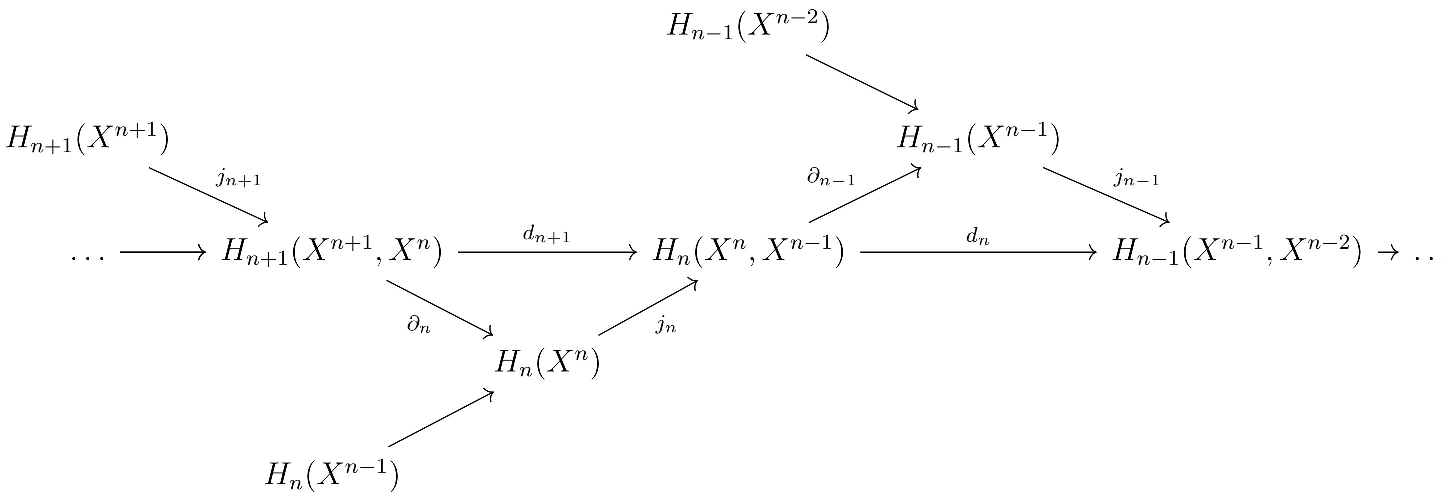
Commutativity of the triangle and square are immediate. Let $\delta \in H_n(S^n, S^n - \{y\})$ be the image of α under the isomorphism on the right. By definition of local degree, the topmost f_* map takes β_i to $\deg f \mid x_i \cdot \delta$. Since $j(\alpha) = \sum_i \beta_i$ and the lowermost f_* map takes α to $\deg f \cdot \alpha$, the claim follows from commutativity. $\square$

2.6 Cellular homology

In this section, we will define a version of homology, called cellular homology, for CW complexes which comes directly from their CW complex structure. The cellular homology groups are isomorphic to the singular homology groups, and hence cellular homology is a useful computational tool analogous to simplicial homology. Let $j_n : H_n(X^n) \to H_n(X^n, X^{n-1})$ and $\partial_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ be the respective maps from the long exact sequence of the pair (X^n, X^{n-1}), and define the cellular boundary map

$$d_n := j_{n-1} \circ \partial_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

Here, we take $X^{-1} = \emptyset$ and $d_0 = 0$. Hence we obtain a sequence involving the d_n's, which is the horizontal sequence below.



(10)

The diagonal sequences are exact, so in particular $d_n d_{n+1} = 0$ since $\partial_{n-1} j_n = 0$. Hence the horizontal sequence is a chain complex, called the **cellular chain complex**. Its homology groups are the cellular homology groups, temporarily denoted by $H_n^{CW}(X)$. For $n > 0$, observe that

$$H_i(X^n, X^{n-1}) \approx \tilde{H}_i(X^n/X^{n-1}) \; \forall \, i$$

Furthermore X^n/X^{n-1} is a wedge sum of n-spheres, each of the form $e^n_\alpha/\partial e^n_\alpha$ for an n-cell e^n_α of X. Hence $H_n(X^n, X^{n-1})$ is free abelian with basis corresponding to the n-cells of X. For $n = 0$, $H_0(X^0, \emptyset) = H_0(X^0)$ is also free abelian with generators corresponding to the 0-cells of X.

Theorem 2.23. $H_n^{CW}(X) \approx H_n(X)$.

We omit the proof of this isomorphism here. In brief, it follows using the following lemma together with diagram-chasing.

Lemma 2.24. *$H_i(X^n) = 0$ for $i > n$. Furthermore, the inclusion $X^n \hookrightarrow X$ induces isomorphisms $H_i(X^n) \xrightarrow{\sim} H_i(X)$ for $i < n$.*

The proof of this lemma in turn requires another lemma.

Lemma 2.25. *Let Y be a topological space and $n > 0$. Let Z be obtained by gluing $(n+1)$-cells $\{e^{n+1}_\alpha \mid \alpha\}$ to Y via gluing maps $\partial e^{n+1}_\alpha \to Y$. Then the inclusion $Y \hookrightarrow Z$ induces an isomorphism on the i-th homology group for $i < n$.*

Proof. Since $n > 0$, the claim is trivially true for $i = 0$. Hence, suppose $0 < i < n$. Let U be the union of the interiors of all the $(n+1)$-cells and let x_α be the centre of e_α^{n+1}. Let $V = Z - \{x_\alpha \mid \alpha\}$, so $Z = U \cup V$. Now consider the following section of the corresponding the Mayer-Vietoris sequence.

$$\tilde{H}_i(U \cap V) \longrightarrow \tilde{H}_i(U) \oplus \tilde{H}_i(V) \longrightarrow \tilde{H}_i(Z) \longrightarrow \tilde{H}_{i-1}(U \cap V)$$

Observe that $U \cap V$ deforms onto a disjoint union of n-spheres, so since $0 < i < n$ we see that the first and last groups in this sequence are zero. Furthermore, U is contractible so we see that $V \hookrightarrow Z$ induces an isomorphism $\tilde{H}_i(V) \xrightarrow{\sim} \tilde{H}_i(Z)$. Finally, $Y \hookrightarrow V$ is a homotopy equivalence since V deforms onto Y. Hence $Y \hookrightarrow Z$ induces an isomorphism $\tilde{H}_i(Y) \xrightarrow{\sim} \tilde{H}_i(Z)$. Since $i > 0$, reduced homology groups can be replaced by the absolute homology groups. □

Proof of ***lemma 2.24****.* The proof of the first claim uses induction on n, with the claim being trivial for $n = 0$. The induction step follows from our observations regarding $H_i(X^n, X^{n-1})$ together with the long exact sequence of the pair (X^n, X^{n-1}).

Lemma 2.25 proves the second statement in the case when X is finite dimensional. The general result follows from the fact that the image of a singular simplex is compact and hence meets only finitely many cells of X. □

The next result, which describes the cellular boundary maps d_n, together with **theorem 2.22** completes the computation tool box that is cellular homology. For this, we make precise some required identifications. View each n-cell e_α^n of X as a copy of D^n (the closed unit ball centred at the origin in $\mathbb{R}^n$). Hence, ∂e_α^n is correspondingly identified with S^{n-1}. Identify $e_\alpha^n/\partial e_\alpha^n$ with S^n via the following quotient map.

$$q_n : D^n \to S^n; x \mapsto \begin{cases} \left(1 - 2\|x\|,\, x \cdot \frac{2\sqrt{1-\|x\|}}{\sqrt{\|x\|}}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Remark. Note that the choice of q_n only matters insofar as how it transfers the orientation of D^n to an orientation of S^n.

Next, we make explicit the basis element of $H_n(X^n, X^{n-1})$ which a given n-cell e_α^n corresponds to. Fix a homeomorphism $h_n : \Delta^n \to D^n$ which is an orientation-preserving diffeomorphism on the interior. Composing h_n with the characteristic map $D^n \to X$ of e_α^n we obtain a singular simplex in $C_n(X)$, which we will denote by σ_α^n. Note that $\partial\sigma_\alpha^n$ is an $(n-1)$-chain in $\partial e_\alpha^n \subset X^{n-1}$, so σ_α^n is a cycle in $C_n(X^n, X^{n-1})$. The homology class $[\sigma_\alpha^n] \in H_n(X^n, X^{n-1})$ is the basis element corresponding to e_α^n. Let $r_\alpha^n : X^n/X^{n-1} \to e_\alpha^n/\partial e_\alpha^n$ be the retraction.

Theorem 2.26 (Cellular boundary formula)**.** *For $n > 1$, we have*

$$d_n[\sigma_\alpha^n] = \sum_\beta d_{\alpha\beta}[\sigma_\beta^{n-1}]$$

where $d_{\alpha\beta}$ is the degree of the retracted gluing map

$$\partial e_\alpha^n \longrightarrow X^{n-1}/X^{n-2} \xrightarrow{r_\beta^{n-1}} e_\beta^{n-1}/\partial e_\beta^{n-1}$$

when viewed as a map $S^{n-1} \to S^{n-1}$ with the identifications mentioned previously.

Proof. By identifying $H_{n-1}(X^{n-1}, X^{n-2})$ with $\tilde{H}_{n-1}(X^{n-1}/X^{n-2})$, we have

$$d_n[\sigma_\alpha^n] = \sum_\beta (r_\beta^{n-1})_*[\partial\sigma_\alpha^n]$$

This is a consequence of the fact that X^{n-1}/X^{n-2} is a wedge sum of the spheres $e_\beta^{n-1}/\partial e_\beta^{n-1}$. Now, observe that ∂h_n (viewing h_n as a singular simplex in D^n) and σ_β^{n-1} (viewed as a singular simplex in $e_\beta^{n-1}/\partial e_\beta^{n-1}$) give the same generator for $H^{n-1}(S^{n-1})$ under the previously mentioned identifications. Hence the claim now follows by recalling the definitions of σ_α^n and σ_β^{n-1}. □

2.6.1 Euler characteristic

For a finite CW complex X, its Euler characteristic is defined as

$$\chi(X) := \sum_n (-1)^n |X|_n$$

where $|X|_n$ is the number of n-cells in X. Hence, a priori the definition depends on the CW complex structure on X. However, the following shows that $\chi(X)$ depends only on the topology of X.

Theorem 2.27. $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$.

The proof uses the following lemma and some straightforward rearrangement of terms by realising $H_n(X)$ as the n-th cellular homology group, so we omit it here.

Lemma 2.28 (Left as exercise in [1])**.** *If A, B, C are abelian groups with finite ranks and we have a short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

then $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$*.*

Proof. Recall that $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module (more generally, the field of fractions of an integral domain R is a flat R-module). Hence, we may tensor the given short exact sequence by $\mathbb{Q}$ to obtain a new short exact sequence.

$$0 \longrightarrow A \otimes \mathbb{Q} \longrightarrow B \otimes \mathbb{Q} \longrightarrow C \otimes \mathbb{Q} \longrightarrow 0$$

The action of $\mathbb{Z}$ on A naturally extends to an action of $\mathbb{Q}$ on $A \otimes \mathbb{Q}$, making it a vector space over $\mathbb{Q}$. Consequently the above is a short exact sequence of vector spaces over $\mathbb{Q}$, and hence we have

$$\dim_{\mathbb{Q}}(B \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}) + \dim_{\mathbb{Q}}(C \otimes \mathbb{Q})$$

Lastly, we observe that $\operatorname{rank}(A) = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$. □

2.7 Homology with coefficients

Instead of viewing $C_n(X)$ as a formal $\mathbb{Z}$-linear combination of singular simplices, we can also view it as a direct sum of copies of $\mathbb{Z}$ indexed by singular simplices. This yields a natural generalisation by replacing $\mathbb{Z}$ by an arbitrary (additively written) abelian group G. Hence, we define $C_n(X; G)$ to be finite linear combinations of the form

$$\sum_i g_i \sigma_i$$

for $g_i \in G$ and singular simplices σ_i. The boundary maps are then defined in the obvious way.

$$\partial \sum_i g_i \sigma_i := \sum_i \sum_j (-1)^j g_i \sigma_i \mid [v_0, \ldots, \hat{v_j}, \ldots, v_n]$$

As before, we see that $\partial^2 = 0$ and so a chain complex is formed.

$$\ldots \longrightarrow C_n(X; G) \xrightarrow{\partial} C_{n-1}(X; G) \longrightarrow \ldots$$

The homology groups of this chain complex, written $H_n(X; G)$ are the **homology groups of** X **with coefficients in** G. The relative homology groups $H_n(X, A; G)$ with G coefficients are then defined to be the homology groups of the chain complex

$$C_n(X, A; G) := C_n(X; G)/C_n(A; G)$$

The long exact sequence of pairs is obtained immediately from **theorem 2.13**. In fact, for H a subgroup of G we have another naturally obtained short exact sequence of chain complexes.

$$0 \longrightarrow C_n(X; H) \longrightarrow C_n(X; G) \longrightarrow C_n(X; G/H) \longrightarrow 0$$

The maps are the obvious ones. This yields the following long exact sequence.

$$\ldots \longrightarrow H_n(X; H) \longrightarrow H_n(X; G) \longrightarrow H_n(X; G/H) \xrightarrow{\partial} H_{n-1}(X; H) \longrightarrow \ldots$$

By examining the map ∂, it is easy to see that if the short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

splits then $\partial = 0$ and we have a split short exact sequence

$$0 \longrightarrow H_n(X; H) \longrightarrow H_n(X; G) \longrightarrow H_n(X; G/H) \longrightarrow 0$$

Next, we observe that the subdivision constructions from §2.3 carry forward to the setup with coefficients as well, so the Mayer-Vietoris sequence and the excision theorem hold for homology with coefficients as well. Simplicial and cellular homology also carry forward to homology with coefficients as expected.

3 Cohomology

We start the discussion with an abstract algebraic set up — let C_n be a chain complex and fix an abelian group G. Let $C_n^* := \text{Hom}(C_n, G)$ be the dual group of C_n. The boundary operator $\partial_{n+1} : C_{n+1} \to C_n$ yields a dual map

$$\delta_n : C_n^* \to C_{n+1}^*; \phi \mapsto \phi \circ \partial_{n+1}$$

Since $\partial^2 = 0$, we have $\delta^2 = 0$. Hence we obtain a chain complex

$$\ldots \longrightarrow C_n^* \xrightarrow{\delta_n} C_{n+1}^* \longrightarrow \ldots$$

The homology groups $H^n(C; G) := \ker(\delta_n)/\text{im}(\delta_{n-1})$ of this chain complex are the **cohomology groups** of the chain complex C_n. A chain morphism $f : C_n \to C'_n$ induces a dual chain morphism $f^* : C'^*_n \to C_n^*$, and hence yields a homomorphism at the cohomology level.

$$f^* : H^n(C'; G) \to H^n(C; G)$$

To understand how the cohomology groups relate to the homology groups algebraically, we start by understanding $\ker(\delta_n)$. Let $B_n = \text{im}(\partial_{n+1})$ be the n-boundaries and $Z_n = \ker(\partial_n)$ be the n-cycles. For $\phi \in C_n^*$, we have

$$\begin{aligned} \phi \in \ker(\delta_n) &\iff \phi\, \partial_{n+1} = 0 \\ &\iff B_n \subset \ker(\phi) \\ &\iff \phi \text{ uniquely factors through } C_n/B_n \end{aligned}$$

Hence $\ker(\delta_n) = (C_n/B_n)^*$, the subgroup of C_n^* consisting of maps $C_n \to G$ which vanish on B_n. Since $H_n(C) = Z_n/B_n \subset C_n/B_n$, restriction to $H_n(C)$ gives a map

$$\tilde{h} : \ker(\delta_n) \to (Z_n/B_n)^*$$

Remark. Observe that $(Z_n/B_n)^*$ is simply $\text{Hom}(H_n(C), G)$. However, we will continue referring to it as $(Z_n/B_n)^*$ since that is more natural in the present setting.

In fact, we can say more about this map. Since all groups in the following short exact sequence are free abelian, the sequence splits.

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0 \tag{11}$$

Remark. Note that there is a subtle difference between the map labelled ∂_n here and the usual boundary map, namely its codomain is taken to be B_{n-1} instead of C_{n-1}. This distinction becomes important upon dualising.

Hence, the following is split exact as well.

$$0 \longrightarrow Z_n/B_n \longrightarrow C_n/B_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0$$

The dual of a split exact sequence is split exact, so the following is split exact sequence.

$$0 \longleftarrow (Z_n/B_n)^* \xleftarrow{\tilde{h}} \ker(\delta_n) \xleftarrow{\partial_n^*} B_{n-1}^* \longleftarrow 0 \tag{12}$$

In particular, $\tilde{h}$ is surjective. Furthermore if $\phi \in \operatorname{im}(\delta_{n-1})$, i.e. $\phi = \psi\partial_n$ for some $\psi \in C_{n-1}^*$, then $Z_n \subset \ker(\phi)$. Consequently $\tilde{h}\phi = 0$. In other words, $\tilde{h}$ factors through a surjective map

$$h : H^n(C;G) \to (Z_n/B_n)^*$$

Hence we have obtained a map from the cohomology group to the dual of the homology group. We will now use (12) to obtain a split exact sequence involving h. Consider the following diagram, with the top row coming from (12) and the bottom row coming from the dual of (11) (with $n-1$ in place of n).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_{n-1}^* & \xrightarrow{\partial_n^*} & \ker(\delta_n) & \xrightarrow{\tilde{h}} & (Z_n/B_n)^* & \longrightarrow & 0 \\
 & & {\scriptstyle i_{n-1}^*}\uparrow & \nwarrow & \uparrow{\scriptstyle \delta_{n-1}} & & & & \\
0 & \longleftarrow & Z_{n-1}^* & \longleftarrow & C_{n-1}^* & \xleftarrow{\partial_{n-1}^*} & B_{n-2}^* & \longleftarrow & 0
\end{array}$$

Here $i_{n-1} : B_{n-1} \to Z_{n-1}$ is the inclusion. The diagonal and lower edge of the square are also duals of inclusions. We know that both rows are split exact, and commutativity of the triangles is immediate. In particular, this means that ∂_n^* maps $\operatorname{im}(i_{n-1}^*)$ isomorphically onto $\operatorname{im}(\delta_{n-1})$. Quotienting out these images, we obtain a split exact sequence involving h.

$$0 \longrightarrow B_{n-1}^*/\operatorname{im}(i_{n-1}^*) \xrightarrow{\partial_n^*} H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C),G) \longrightarrow 0 \tag{13}$$

Hence the group $B_{n-1}^*/\operatorname{im}(i_{n-1}^*)$ measures how much the cohomology group $H^n(C;G)$ 'deviates' from being isomorphic to the dual of the homology group $H^n(C)$. Observe that $\operatorname{im}(i_{n-1}^*)$ simply consists of those maps $B_{n-1} \to G$ which have an extension $Z_{n-1} \to G$, so this deviation is also a measure of the extent to which one can find maps $B_{n-1} \to G$ which cannot be extended to Z_{n-1}. We consider some special cases when this question of extendability is easy to answer.

Proposition 3.1. *h is an isomorphism when $G = \mathbb{Q}$.*

Proof. We have a canonical embedding $B_{n-1} \hookrightarrow B_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q}; x \mapsto x \otimes 1$ and every map $\phi : B_{n-1} \to \mathbb{Q}$ extends uniquely to a map $\phi' : B_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$. This in turn extends to a map $\tilde{\phi}' : Z_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$. Now $Z_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q}$ canonically contains Z_{n-1}, so restricting $\tilde{\phi}'$ to Z_{n-1} gives an extension $\tilde{\phi} : Z_{n-1} \to \mathbb{Q}$ of ϕ. □

Proposition 3.2. *If $H_n(C)$ is free abelian for some n, then $h : H^n(C;G) \to \operatorname{Hom}(H_n(C),G)$ is an isomorphism.*

Proof. We have the following short exact sequence with all groups free abelian.

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \longrightarrow 0$$

Hence this sequence splits and its dual is also short exact. In particular, i^*_{n-1} is surjective. □

3.1 The universal coefficient theorem

We will now describe how to compute $B^*_{n-1}/\mathrm{im}(i^*_{n-1})$ in general. For an abelian group H, a **free resolution** of H is a short exact sequence

$$0 \longrightarrow B \xrightarrow{i} Z \longrightarrow H \longrightarrow 0$$

where B and Z are both free abelian. It is easy to see that a free resolution always exists since every abelian group is a quotient of a free abelian group. The dual of a free resolution yields the following exact sequence.

$$0 \longleftarrow B^*/\mathrm{im}(i^*) \longleftarrow B^* \xleftarrow{i^*} Z^* \longleftarrow H^* \longleftarrow 0$$

The following lemma, which we state without proof, shows that the group $B^*/\mathrm{im}(i^*)$ depends only on H (and G), and this dependence is natural on H.

Lemma 3.3. *A homomorphism $f : H \to H'$ can be extended to free resolutions of H and H' to obtain a commutative diagram as follows.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i} & Z & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & B' & \xrightarrow{i'} & Z' & \longrightarrow & H' & \longrightarrow & 0 \end{array}$$

Furthermore, the induced map $B'^*/\mathrm{im}(i'^*) \to B^*/\mathrm{im}(i^*)$ is unique.

Corollary 3.4. *In the above lemma, if f is an isomorphism then the induced map $B'^*/\mathrm{im}(i'^*) \to B^*/\mathrm{im}(i^*)$ is also an isomorphism.*

The unique group $B/\mathrm{im}(i^*)$ is written $\mathrm{Ext}(H, G)$ to emphasise that it depends only on H and G, and not on the choice of free resolution of H. However, to talk about maps to and from $\mathrm{Ext}(H, G)$ it is necessary to fix a realisation of $\mathrm{Ext}(H, G)$ by fixing a free resolution of H. Returning to the context of the cohomology groups of a chain complex C_n, we see that if the C_n's are free abelian then the group $B^*_{n-1}/\mathrm{im}(i^*_{n-1})$ appearing in (13) is $\mathrm{Ext}(H_{n-1}(C), G)$. Hence, our discussion so far are summarised as follows.

Theorem 3.5 (Universal coefficient theorem)**.** *The cohomology groups of a chain complex C_n of free abelian groups admit a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \longrightarrow 0$$

where $\mathrm{Ext}(H_{n-1}(C), G)$ is realised as $B^*_{n-1}/\mathrm{im}(i^*_{n-1})$ and its inclusion in $H^n(C; G)$ is given by ∂^*_n.

$\operatorname{Ext}(H, G)$ satisfies the following straightforward and useful properties.

- $\operatorname{Ext}(H \oplus H', G) \approx \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G)$.
- $\operatorname{Ext}(H, G) = 0$ if H is free (cf. **proposition 3.2**).
- $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \approx G/nG$.

The short exact sequence provided by the universal coefficient theorem is natural in the following sense — the proof relies only on **lemma 3.3**.

Theorem 3.6. *A chain morphism $f : C_n \to C'_n$ yields a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}(H_{n-1}(C'), G) & \longrightarrow & H^n(C'; G) & \longrightarrow & \operatorname{Hom}(H_n(C'), G) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (f_*)^*} & & \\
0 & \longrightarrow & \operatorname{Ext}(H_{n-1}(C), G) & \longrightarrow & H^n(C; G) & \xrightarrow{h} & \operatorname{Hom}(H_n(C), G) & \longrightarrow & 0
\end{array}$$

where the f^* map on the left is induced by f upon realising $\operatorname{Ext}(H_{n-1}(C), G)$ *as in **theorem 3.5**.*

Finally **corollary 3.4**, **theorem 3.6** and **lemma 2.12** yield the following.

Corollary 3.7. *If a chain morphism $f : C_n \to C'_n$ induces isomorphisms on the homology groups, then it also induces isomorphisms on the cohomology groups.*

3.2 Cohomology for spaces

The cochain complex of a topological space X with coefficients in G is $C^n(X; G) := C_n(X)^*$, and its homology groups $H^n(X; G)$ are the cohomology groups of X. The relative cochain complex $C^n(X, A; G)$ is likewise defined to be the dual of $C_n(X, A)$, and its homology groups are the relative cohomology groups of the pair (X, A). We will now discuss the cohomology analogues of the various results obtained so far for homology — they mostly follow from the universal coefficient theorem and **theorem 3.6**.

3.2.1 Induced homomorphisms

A map $f : X \to Y$ induces a chain morphism $f^\# : C^n(Y; G) \to C^n(X; G)$ and maps $f^* : H^n(Y; G) \to H^n(X; G)$ at the cohomology level. By **theorem 3.6**, these maps also yield a commutative diagram involving the split exact sequences given by the universal coefficient theorem.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}(H_{n-1}(Y), G) & \longrightarrow & H^n(Y; G) & \longrightarrow & \operatorname{Hom}(H_n(Y), G) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (f_\#)^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (f_*)^*} & & \\
0 & \longrightarrow & \operatorname{Ext}(H_{n-1}(X), G) & \longrightarrow & H^n(X; G) & \xrightarrow{h} & \operatorname{Hom}(H_n(X), G) & \longrightarrow & 0
\end{array}$$

3.2.2 Homotopy invariance

Homotopic maps $f, g : X \to Y$ induces chain homotopic chain morphisms $f_\#, g_\# : C_n(X) \to C_n(Y)$. The dual of a chain homotopy between $f_\#$ and $g_\#$ is itself a chain homotopy between the dual morphisms $f^\#$ and $g^\#$, so we see that $f^\#$ and $g^\#$ are also chain homotopic. In particular, the induced maps f^* and g^* at the cohomology level are equal.

3.2.3 Relative cohomology

Recall the short exact sequence of chain complexes corresponding to a pair (X, A):

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

Since this a short exact sequence of free abelian groups, its dual is also short exact.

$$0 \longleftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \longleftarrow 0$$

Consequently, we obtain a corresponding long exact sequence of cohomology groups.

$$\ldots \longleftarrow H^n(A; G) \xleftarrow{i^*} H^n(X; G) \xleftarrow{j^*} H^n(X, A; G) \xleftarrow{\delta} H^{n-1}(A; G) \longleftarrow \ldots$$

Furthermore the connecting map δ is naturally dual to the connecting map ∂ which occurs in long exact sequence of (X, A) for homology, in the sense that the following diagram commutes.

$$\begin{array}{ccc} H^n(A; G) & \xrightarrow{\delta} & H^{n+1}(X, A; G) \\ \downarrow h & & \downarrow h \\ \mathrm{Hom}(H_n(A), G) & \xrightarrow{\partial^*} & \mathrm{Hom}(H_{n+1}(X, A)) \end{array} \tag{14}$$

Commutativity can be verified easily by spelling out the definitions all maps involved.

3.2.4 The Mayer-Vietoris sequence

As above, the short exact sequence (6) of chain complexes can be dualised to obtain the following a long exact sequence for cohomology involving the cohomology groups of $A, B, A+B$ and $C := A \cap B$. Then **corollary 3.7** may be used to replace $A + B$ by X, finally yielding the following long exact sequence.

$$\ldots \longleftarrow H^n(C; G) \xleftarrow{\phi^*} H^n(A; G) \oplus H^n(B; G) \xleftarrow{\psi^*} H^n(X; G) \xleftarrow{\delta} H^{n-1}(C; G) \longleftarrow \ldots$$

3.2.5 Excision

The statement of excision for cohomology is that for $Z \subset A \subset X$ with $\overline{Z} \subset \operatorname{int} A$, the inclusion of pairs

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces isomorphisms at the cohomology level.

$$H^n(X, A; G) \xrightarrow{\sim} H^n(X - Z, A - Z; G)$$

This follows at once from the excision theorem for homology and **corollary 3.7**.

3.2.6 Simplicial cohomology

For a simplicial pair (X, A), the chain morphism $\Delta_n(X, A) \hookrightarrow C_n(X, A)$ induces isomorphism on the homology groups, and hence the same holds at the cohomology level by **corollary 3.7**.

3.2.7 Cellular cohomology

For a CW complex X, the homology groups $H_i(X^n, X^{n-1})$ are all free abelian and so

$$h : H^n(X^n, X^{n-1}; G) \to \operatorname{Hom}(H_n(X^n, X^{n-1}), G)$$

is an isomorphism by the universal coefficient theorem. Hence the dual of the cellular chain complex, with appropriate identifications, takes the following form.

$$\dots \longrightarrow H^n(X^n, X^{n-1}; G) \xrightarrow{d_n^*} H^{n+1}(X^{n+1}, X^n; G) \longrightarrow \dots$$

In fact, this chain complex is the same as that obtained by using the long exact sequences of pairs for cohomology (analogous to how the cellular chain complex was constructed) as is done in [1] — this can be seen using the commutativity of (14). As expected, the cellular cohomology groups (the homology groups of the above chain complex) are isomorphic to the singular cohomology groups of X. However, this does not simply follow from **corollary 3.7** since the isomorphism between the cellular and singular homology groups was not induced by a chain morphism. We omit the proof here.

3.3 Cup product and the cohomology ring

The cohomology groups of a space can be endowed with an additional algebraic structure, called the cup product, when the coefficient group comes from a ring. The construction, as usual, starts at the level of chain complexes. For R a ring, the cup product $\smile : C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R)$ for cochains is constructed as follows. For $\phi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$ and $\sigma : \Delta^{k+\ell} \to X$ a singular simplex, we define

$$(\phi \smile \psi)(\sigma) := \phi(\sigma \mid [v_0, \ldots, v_k]) \cdot \psi(\sigma \mid [v_k, \ldots, v_{k+\ell}])$$

Here, the product in the right side is evaluated in R. It is easy to see that the cup product is R-bilinear (under the natural action of R on cochains). A straightforward computation yields the following formula.

$$\delta(\phi \smile \psi) = (\delta\phi) \smile \psi + (-1)^k \phi \smile (\delta\psi) \tag{15}$$

In particular we have the following two crucial observations.

- If ϕ and ψ are both cocylces then so is $\phi \smile \psi$.
- If ϕ is a coboundary (say $\phi = \delta\phi'$) and ψ is a cocyle, then $\phi \smile \psi$ is a coboundary:

$$\delta(\phi' \smile \psi) = \phi \smile \psi$$

  Likewise if ϕ is a cocylce and ψ is a coboundary.

Hence, the cup product factors through as an R-bilinear product at the level of cohomology groups.

$$\smile : H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R)$$

This makes

$$\bigoplus_{n \geq 0} H^n(X;R)$$

into a graded R-algebra, written $H^*(X;R)$. To study the commutativity (or lack thereof) of this R-algebra, the following lemma is key.

Lemma 3.8. *Define the reversal map $\rho : C_n(X) \to C_n(X)$ on a singular simplex $\sigma : \Delta^n \to X$ as*

$$\rho(\sigma) := \varepsilon_n \sigma \mid [v_n, v_{n-1}, \ldots, v_0]$$

and extend $\mathbb{Z}$-linearly to $C_n(X)$, where $\varepsilon_n := (-1)^{\frac{n(n+1)}{2}}$. Then ρ is a chain morphism and is chain homotopic to $\mathrm{id}_{C_n(X)}$.

It is easy to see that ρ is a chain morphism. That it is chain homotopic to the identity can be seen using a prism construction similar to that used to prove **theorem** 2.2. We omit the details of the argument here.

Corollary 3.9. *For a cocycle $\phi \in C^n(X;G)$, the cohomology classes $[\phi]$ and $[\phi \circ \rho]$ are equal.*

Theorem 3.10. *For cocylces $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, we have*

$$[\phi] \smile [\psi] = (-1)^{k\ell}[\psi] \smile [\phi]$$

Proof. The claim follows at once from **corollary 3.9**. For a singular simplex $\sigma \in C^{k+\ell}(X)$, observe that

$$\begin{aligned}
(\phi \smile \psi) \circ \rho(\sigma) &= \varepsilon_{k+\ell}\, \phi(\sigma \mid [v_{k+\ell}, \dots, v_\ell]) \cdot \psi(\sigma \mid [v_\ell, \dots, v_0]) \\
&= \epsilon_{k+\ell}\, \varepsilon_k\, \varepsilon_\ell\, ((\phi \circ \rho) \smile (\psi \circ \rho))(\sigma) \\
\implies (\phi \smile \psi) \circ \rho &= \epsilon_{k+\ell}\, \varepsilon_k\, \varepsilon_\ell\, (\phi \circ \rho) \smile (\psi \circ \rho)
\end{aligned}$$

Now we have $\epsilon_{k+\ell}\, \varepsilon_k\, \varepsilon_\ell = (-1)^{k\ell}$ so the claim follows by **corollary 3.9**. □

Corollary 3.11. *Let $\alpha \in H^k(X;R)$ and $\beta \in H^\ell(X;R)$ be cohomology classes. If at least one of k and ℓ is even, then*

$$\alpha \smile \beta = \beta \smile \alpha$$

If both k and ℓ are odd, then

$$\alpha \smile \beta = -\beta \smile \alpha$$

4 Some exercises

This section contains brief sketches of the solutions to some exercises from [1].

Section 2.1

Exercise 14

Suppose an abelian group fits in a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^m} \longrightarrow A \longrightarrow \mathbb{Z}_{p^n} \longrightarrow 0$$

for some prime p. Hence the size of A is $|A| = p^{n+m}$. Consequently for some $k \geq 1$ and positive integers $n_1, \ldots, n_k$ with $n_1 + \ldots + n_k = n + m$ we can write

$$A \approx \bigoplus_{i=1}^{k} \mathbb{Z}_{p^{n_i}}$$

Since $\mathbb{Z}_{p^m}$ is a subgroup of A, we must have $n_\alpha \geq m$ for some α. Likewise, since $\mathbb{Z}_{p^n}$ is a quotient of A we must have $n_\beta \geq n$ for some β. From here it follows that either $k = \alpha = \beta = 1$ and $n_1 = n+m$ or $k = 2$ and $n_\alpha = m, n_\beta = n$. Hence $A \approx \mathbb{Z}_{p^{n+m}}$ or $A \approx \mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^m}$.

Next, suppose A fits in a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

Let $a \in A$ be the image of $1 \in \mathbb{Z}$ and $b \in A$ be a premiage of generator of $1 + n\mathbb{Z} \in \mathbb{Z}_n$. Hence a and b generate A and n is the smallest positive integer i such that $ib \in \langle a \rangle$. Let $m \in \mathbb{Z}$ be the unique integer such that $nb = ma$.

Let $g = (n, m)$ be the GCD of n and m, and let $n' = \frac{n}{g}$ and $m' = \frac{m}{g}$. Let $p, q \in \mathbb{Z}$ such that $pn' + qm' = 1$ (Bezout's lemma). In the case of $g = 1$, we claim that $pa + qb$ generates A. Indeed, we have

$$\begin{aligned} n(pa + qb) &= (1 - qm)a + q(nb) \\ &= (1 - qm)a + qma \text{ (since } nb = ma) \\ &= a \end{aligned}$$

and likewise $m(pa + qb) = b$. Hence we see that $A \approx \mathbb{Z}$ when $g = 1$. Now consider the general case. Let $s := m'a - n'b$ (an element of order g) and $t := pa + qb$. We have

$$\begin{aligned} qs + n't &= a \\ -ps + m't &= b \end{aligned}$$

Hence s and t generate A. Since s is of order g, we see that t is of infinite order and hence $A \approx \mathbb{Z}_g \oplus \mathbb{Z}$. The case of $g = 1$ is also covered here since $\mathbb{Z}_1 = 0$. Finally, we must verify that the map $A \to \mathbb{Z}_n$ taking a to 0 and b to $1 + n\mathbb{Z}$ is well-defined (i.e. it does not violate any relations satisfied by a and b) in order to conclude that $A \approx \mathbb{Z}_g \oplus \mathbb{Z}$ is not only possible but also achieved. For this, observe that this map sends s to $-n' + n\mathbb{Z}$, which has order g. Since s also has order g, this map is well-defined.

Exercise 18

The boundary map $\partial : H_1(\mathbb{R}, \mathbb{Q}) \to \tilde{H}_0(\mathbb{Q})$ from the sequence of the pair $(\mathbb{R}, \mathbb{Q})$ is an isomorphism since $\tilde{H}_1(\mathbb{R})$ and $\tilde{H}_0(\mathbb{R})$ are trivial. A basis for the free abelian group $\tilde{H}_0(\mathbb{Q})$ is given by

$$\{[c_q - c_0]_{\mathbb{Q}} \mid q \in \mathbb{Q} - \{0\}\}$$

where c_q is the 0-simplex at q. The 1-simplex

$$\sigma_q : \Delta^1 \to \mathbb{R}; t \mapsto tq$$

in $\mathbb{R}$ has boundary $c_q - c_0$, so a basis of $H_1(\mathbb{R}, \mathbb{Q})$ is given by

$$\{[\sigma_q]_{(\mathbb{R},\mathbb{Q})} \mid q \in \mathbb{Q} - \{0\}\}$$

Exercise 19

Let X be the given space and let A be the subspace of X obtained by removing all irrational points from the top edge. Since A has deformation retract I, we see that $\tilde{H}_n(A)$ is trivial for all n. Hence the canonical map $\tilde{H}_n(X) \to H_n(X, A)$ is an isomorphism for all n. Now, let Z be the lower half of X, so $\overline{Z} \subset \operatorname{int} A$. By excision it can now be seen that $H_n(X, A) \approx H_n(\mathbb{R}, \mathbb{Q})$ (which is free abelian for $n = 1$ and trivial otherwise).

Exercise 21

The 'obvious' choice can be justified using **theorem 2.14** on the pair (CX, A), where A is the base of the cone CX.

Exercise 26

It can be seen that $H_1(X, A)$ is a countably generated free abelian group using the idea used for exercise 18. However, X/A is the Hawaiian earring space and so $\tilde{H}_1(X/A)$ is not countably generated.

Exercise 31

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\approx} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\approx} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\approx} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Section 2.2

Exercise 1

Let $N : D^n \to S^n$ be the embedding of D^n as the northern hemisphere and $S : D^n \to S^n$ be the embedding as the southern hemisphere. Define

$$F : S^n \to S^n; x \mapsto \begin{cases} SfN^{-1}(x) & x \text{ is in the Northern hemisphere} \\ SfS^{-1}(x) & x \text{ is in the Southern hemisphere} \end{cases}$$

Hence F is not surjective and so $\deg F = 0$. If F had no fixed points then by **proposition 2.19** we would have $\deg F = \pm 1$, so we conclude that F has a fixed point $x_0 \in S^n$. Hence x_0 lies in the southern hemisphere and $S^{-1}(x_0)$ is a fixed point of f.

Exercise 2

If f has no fixed points then by **proposition 2.19** we see that $\deg f = -1$. Likewise if $-f$ has no fixed points then $\deg(-f) = -1$. However, we also have $\deg(-f) = -\deg f$ by **proposition 2.18** so we see that if $-f$ has no fixed points then $\deg f = 1$. Hence at least one of f and $-f$ must have a fixed point.

Exercise 3

If at least one of f and $-f$ has no fixed points then by **proposition 2.19**, $\deg f = \pm 1$. Hence when $\deg f = 0$ we see that both f and $-f$ have fixed points.

Normalising the vector field F yields a map $\tilde{F} : D^n \to S^{n-1}$. Restricting to the boundary,

$$\tilde{F}|_{S^{n-1}} : S^{n-1} \to S^{n-1}$$

is a null-homotopic map (in particular it has degree 0), so the preceding result applies.

Exercise 14

For n even we have $\deg f = \deg(f \circ (-\mathrm{id}_{S^n})) = -\deg(f)$, where the first equality comes from the fact that f is even and the second comes from **proposition 2.18**. Hence $\deg f = 0$.

Next we will show that when n is odd, $\deg f$ is even. We will outline a method different from the one given as a hint in [1]. First, observe that $-\mathrm{id}_{S^n} \cong \mathrm{id}_{S^n}$ (since n is odd). Hence if there exists $y \in S^n$ such that $f^{-1}(y)$ is finite, then $\deg f \mid x$ is defined for each $x \in f^{-1}(y)$ and also $\deg f \mid x = \deg f \circ (-\mathrm{id}_{S^n}) \mid x = \deg f \mid (-x)$, where the first equality follows from an argument similar to the proof of **proposition 2.21**. Hence in the formula given by **theorem 2.22**, the terms corresponding to x and $-x$ pair up so that $\deg f$ turns out to be even.

It now suffices to show that f is homotopic to a map $g : S^n \to S^n$ such that $g^{-1}(y)$ is finite for some $y \in S^n$. Without loss of generality assume $\deg f \neq 0$, since otherwise clearly we have that $\deg f$ is even. For this, it suffices to show that f is homotopic to a smooth map $g : S^n \to S^n$. Indeed, if such g exists then it must be surjective (since $\deg g = \deg f \neq 0$) so by Sard's theorem $g^{-1}(y)$ is a 0-dimensional manifold for some $y \in S^n$. In particular, $g^{-1}(y)$ is discrete for some $y \in S^n$. Hence $g^{-1}(y)$ is a closed (by continuity) and discrete subset of the compact space S^n, i.e. $g^{-1}(y)$ is finite.

To show that f is homotopic to a smooth map, let $U, V \subset S^n$ be open discs which cover S^n and let $\phi : B^n \to U$ and $\psi : B^n \to V$ be diffeomorphisms, where B^n is the open n-ball. By Weierstrass' approximation theorem, we can uniformly approximate the continuous maps $f\phi : B^n \to \mathbb{R}^{n+1}$ and $f\psi : B^n \to \mathbb{R}^{n+1}$ by polynomial (in particular, smooth) maps $P, Q : B^n \to \mathbb{R}^{n+1}$ with error bounded by ϵ. Let $\Phi, \Psi : S^n \to [0, 1]$ be a smooth partition of unity subordinate to the cover $\{U, V\}$, with Φ subordinate to U and Ψ to V. We now define $g_\epsilon : S^n \to \mathbb{R}^{n+1}$ as follows.

$$g_\epsilon := \Phi \cdot P\phi^{-1} + \Psi \cdot Q\psi^{-1}$$

It is easy to see that g_ϵ uniformly approximates f with error bounded by ϵ. In particular, for $\epsilon = 1$ we see that g_1 has no roots (since $\|g_1(x)\| > \|f(x)\| - 1 = 0$ for all $x \in S^n$) and also the line segment joining $g_1(x)$ and $f(x)$ does not pass through the origin for all $x \in S^n$. Hence, we have a homotopy between g_1 and f given by

$$[0, 1] \times S^n \to \mathbb{R}^{n+1} - \{0\}; (t, x) \mapsto (1 - t)g_1(x) + tf(x)$$

Hence, letting $g = \frac{g_1}{\|g_1\|}$ works.

Exercises 20 and 21

These are immediate consequences of the definition of the Euler characteristic using the number of cells in each dimension.

Exercise 25

It is easy to see that $\phi(X \vee Y) = \phi(X) + \phi(Y)$. Hence, if X is d-dimensional with c_d many d-cells then

$$\begin{aligned}\phi(X) &= \phi(X^{d-1}) + \phi(X/X^{d-1}) \\ &= \phi(X^{d-1}) + c_d\,\phi(S^d) \qquad (16)\end{aligned}$$

In particular, if $X = S^d$ with cell-structure given containing exactly two d-cells, then $X^{d-1} = S^{d-1}$ and so

$$\begin{aligned}&\phi(S^d) = \phi(S^{d-1}) + 2\phi(S^d) \\ \implies &\phi(S^d) = -\phi(S^{d-1})\end{aligned}$$

Hence we see that $\phi(S^d) = (-1)^d n$ and

$$\phi(X) = \phi(X^{d-1}) + (-1)^d n$$

by (16). Now it follows that $\phi(X) = n(\chi(X) - 1)$. Also, under this definition we see that all conditions are satisfied and ϕ is also homotopy invariant.

Exercise 27

The projection $C_n(X) \to C_n(A)$ and inclusion $C_n(X, A) \to C_n(X)$ need not always be chain morphisms, and hence need not induce maps at the homology level.

Exercise 33

The result follows by induction on n by viewing X as the union of the two open sets A_n and $A_1 \cup \ldots \cup A_{n-1}$ and using the Mayer-Vietoris sequence. To see that this is the best possible result, we will inductively define a decomposition of S^n as a union of $n + 2$ open sets with the desired intersection property.

For S^0, this is clear — we let A_1^0 and A_2^0 be two singletons. Now if $n > 0$ and a desired decomposition $A_1^{n-1}, \ldots, A_{n+1}^{n-1}$ for S^{n-1} is known, we construct one for S^n by viewing S^n as the suspension

$$S^{n-1} \times [0, 1]/ \sim$$

of S^{n-1}. For $i \leq n + 1$, let A_i^n be the image of $A_i^{n-1} \times [0, \frac{2}{3})$ in the suspension. Let A_{n+2}^n be the image of $S^{n-1} \times (\frac{1}{3}, 1]$. It is easy to check that the desired intersection property is satisfied by this decomposition.

References

[1] Hatcher, A. (2002). Algebraic Topology. Cambridge University Press.