IISER Pune

MTH402 Endsem Report

Hyperbolic Geometry and Knot Theory

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1 Introduction

Hyperbolic geometry is a non-Euclidean geometry, namely one which violates Euclid's parallel postulate which states that given a straight line and a point outside it, there is a unique line passing through that line parallel to the given line. In hyperbolic geometry one can find infinitely many such lines.
A *model* for hyperbolic geometry is an underlying space coupled with a metric such that the above holds. The metric gives us the *geodesics* (locally distance minimizing paths) and the *totally geodesic planes* (two dimensional subspaces that are locally distance minimizing) of the given model. A *conformal model* is one in which the angles in the underlying space are the actual angles between two curves in hyperbolic space. *Isometries* are distance-preserving homeomorphisms from the space to itself.

2 Upper half plane model

The underlying space of this model is the upper half of $\mathbb{R}^n$,

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}.$$

The metric is given by

$$ds^2 = \frac{dx_1^2 + ... + dx_n^2}{x_n^2}.$$

This is a conformal model.

2.1 Isometries of $\mathbb{H}^2$

For this section we consider $\mathbb{H}^2 = \{x + iy \in \mathbb{C} : y > 0\}$, which is equivalent to the previously defined subspace of $\mathbb{R}^2$. The boundary of this space is given by $\mathbb{R} \cup \{\infty\}$, the *circle at infinity*. The geodesics of $\mathbb{H}^2$ are vertical Euclidean lines and Euclidean semicircles centred on the real axis of $\mathbb{C}$.
Properties

1. For each pair of distinct points p and q in $\mathbb{H}^2$, there exists a unique hyperbolic line in $\mathbb{H}^2$ passing through p and q. (We say that the space is totally geodesic when this happens.)
2. Two hyperbolic lines in $\mathbb{H}^2$ are said to be parallel if they are disjoint.
3. Let L be a hyperbolic line in $\mathbb{H}^2$, and let p be a point in $\mathbb{H}^2$ not on L. Then, there exist infinitely many distinct hyperbolic lines through p that are parallel to L.
4. Given any two points p and q here exists an isometry of $\mathbb{H}^2$ (in terms of translation and dilation) that maps p to q.

5. A point and a direction uniquely determine a line.

Theorem 2.1. The following are isometries of $\mathbb{H}^2$.

1. **Dilation** $(x, y) \mapsto (ax, ay)$ for some $a > 0$
2. **Translation** $(x, y) \mapsto (x + b, y)$ for some $b \in \mathbb{R}$
3. **Reflection** $(x, y) \mapsto (\frac{cx}{x^2+y^2}, \frac{cy}{x^2+y^2})$ for $c > 0$ and $(x, y) \mapsto (d - x, y)$ for $d \in \mathbb{R}$
4. **Rotation** Rotation by angle θ about a point p. Each rotation is given by reflections in two different lines composed with each other.

Every isometry of $\mathbb{H}^2$ is some composition of one of the above.

The proof of the above uses the number of points fixed by the isometry to characterize it. Once we have determined that the isometries form the group of Möbius transforms of $\mathbb{H}^2$, we can also classify them using the trace of the matrix of the transform.

Proof. (Classification of isometries based on fixed points) Suppose $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry other than the identity map. If φ has 3 or more (noncollinear) fixed points then it must be the identity, so we are left with the following cases.
Case 1: φ has 2 fixed points. Then φ is a reflection in some line.
Suppose p and q are the fixed points. The line L joining them will be fixed by φ. Take a point p' not on L. Let m be the point on L that is closest to p'. Then the points in $\mathbb{H}^2$ which are $d(m, p')$ away from m such that m is the closest point on L are p' and $m - p'$. So each point must be mapped to itself or its reflection in L. Any two points on the same side of L must be mapped to one side, and points on the opposite sides must be mapped to opposite sides. Since φ is not the identity, it must be a reflection along L.
Case 2: φ has exactly one fixed point. Then φ is a rotation.
Let p be the fixed point. For every other point x, $d(x, p)$ is conserved. Further, lines map to lines. Hence this is a rotation about point p.
Case 3: φ has no fixed points. Then φ is a rotation or a reflection, composed with a translation and/or dilation.
Take any point $x \in \mathbb{H}^2$, and let $\varphi(x) = y$. Let f be an isometry of $\mathbb{H}^2$ that takes y to x. We have seen that such an f can always be found by composing some translation and some dilation. Then $f \circ \varphi$ has at least one fixed point, hence can be classified as one of the previous cases, either a rotation or a reflection (or the identity). We then get the nature of φ by looking at $f^{-1} \circ f \circ \varphi$. □

Matrix representation For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$, and $z = x + iy \in \mathbb{H}^2$, define the map

$$\varphi_A : \mathbb{H}^2 \to \mathbb{H}^2, z \mapsto \frac{az + b}{cz + d}.$$

This gives an isometry of $\mathbb{H}^2$. Clearly, $\varphi_A = \varphi_{-A}$. So the group of these transforms is $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\sim$, where $A \sim B$ iff $A = \pm B$.
Matrices corresponding to the previously mentioned isometries:

- $T = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ corresponds to translation by s.
- $D = \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$ corresponds to dilation by a^2.
- $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ corresponds to rotation on the pivot $(1, 0)$.

It turns out that the above matrices generate all of $PSL_2(\mathbb{R})$, which is the group of all orientation-preserving isometries of $\mathbb{H}^2$. This along with reflections gives all possible isometries.

Classification of orientation-preserving isometries based on trace
Let $A \in PSL_2(\mathbb{R})$. The characteristic polynomial of A is

$$\lambda^2 - tr(A)\lambda + 1$$

and the eigenvalues are

$$\frac{tr(A) \pm \sqrt{tr(A)^2 - 4}}{2}.$$

Case 1: $tr(A) > 2$. Then φ_A is a dilation.
A has two distinct real eigenvalues, and is similar to D above.
Case 2: $tr(A) = 2$. Then φ_A is a translation.
A has exactly one eigenvalue which is real, and is similar to T above.
Case 3: $tr(A) < 2$. Then φ_A is a rotation.
A has exactly one fixed point, as $\varphi_A(z) = z \implies cz^2 - (a-d)z - b = 0 \implies z = \frac{a-d\pm\sqrt{tr(A)^2-4}}{2c}$, and since $tr(A)^2 < 4$, the possible values of z are conjugates and only one of these is contained in $\mathbb{H}^2$. Thus by the previous classification, φ_A must be a rotation.

2.2 Isometries of $\mathbb{H}^3$

We take $\mathbb{H}^3 = \{(x + iy, t) \in \mathbb{C} \times \mathbb{R}\}$ and metric $ds^2 = \frac{dx^2+dy^3+dt^2}{t^2}$.
Here the lines will be vertical Euclidean straight lines and Euclidean semicircles orthogonal to $\partial\mathbb{H}^3 = \mathbb{C} \times \{0\} \cup \{\infty\}$, written simply as $\mathbb{C} \times 0$. The totally geodesic planes are the vertical Euclidean planes and Euclidean hemispheres centred on $\partial\mathbb{H}^3$.

Lemma 2.1. The set of orientation preserving homeomorphisms of $\overline{\mathbb{C}}$ that map circles to circles is precisely the set of Möbius transformations $PSL(2, \mathbb{C})$.

Proof. Any Möbius transform $\alpha : z \mapsto \frac{az+b}{cz+d}$ is a composition of functions $\alpha = f \circ g \circ h$, where $f : z \mapsto -z + \frac{a}{c}$, $g : z \mapsto \frac{1}{z}$ and $h : z \mapsto c^2 z + cd$. Since these are maps taking circles to circles, their composition α must also take circles to circles.
Conversely, suppose f is a homeomorphism of $\overline{\mathbb{C}}$ that takes circles to circles. We know that given any three points $p_2, p_3, p_4 \in \overline{\mathbb{C}}$, there is a unique Möbius transform Φ_p that sends p_2, p_3, p_4 to $1, 0, \infty$ respectively, given by $\Phi_p(z) = \frac{(z-p_3)(p_2-p_4)}{(z-p_4)(p_2-p_3)}$. So to take p_2, p_3, p_4 to any other given points q_2, q_3, q_4, we simply take $\Phi_q^{-1} \circ \Phi_p$.
Fix $p_1 = 0, p_2 = 1, p_3 = \infty \in \overline{\mathbb{C}}$ and let γ be the Möbius transform that takes $(f(p_1), f(p_2), f(p_3))$ to (p_1, p_2, p_3). Then $\gamma \circ f$ is a homeomorphism taking circles to circles that fixes $0, 1, \infty \in \overline{\mathbb{C}}$. We will prove that $\gamma \circ f$ is in fact the identity map, and hence $f = \gamma^{-1} \in PSL(2, \mathbb{C})$.
We now construct a dense subset of $\overline{\mathbb{C}}$ that is fixed by $\gamma \circ f$. Then $\gamma \circ f - Id_{\overline{\mathbb{C}}}$ is a continuous function that is zero on a dense subset, hence is zero everywhere. Let $U = \{z \in \mathbb{C} : im(z) > 0\}$ and $L = \{z \in \mathbb{C} : im(z) < 0\}$. Given $s \in \mathbb{R}$, let $V(s)$ be the vertical line through s and $H(s)$ be the horizontal line through is.
Since ∞ is fixed by $\gamma \circ f$, Euclidean lines of $\overline{\mathbb{C}}$ must go to Euclidean lines and Euclidean circles to Euclidean circles. In particular, $\overline{\mathbb{R}} \subset \overline{\mathbb{C}}$ is the Euclidean circle determined by $0, 1, \infty$, so $\gamma \circ f(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$. Then either $\gamma \circ f(U) = U$ or $\gamma \circ f(U) = L$. In the latter case we compose γ with the conjugation map $z \mapsto \bar{(}z)$ and take this to be γ instead, so we always have $\gamma \circ f(U) = U$.
Let A be the circle of radius $\frac{1}{2}$ centred at the point $\frac{1}{2}$ on the real axis. $\gamma \circ f$ fixes 0 and 1, so $\gamma \circ f(V(0))$ is tangent to $\gamma \circ f(A)$ at 0 and $\gamma \circ f(V(1))$ is tangent to $\gamma \circ f(A)$ at 1. Hence $V(0), V(1)$ and A are invariant (not yet fixed) under $\gamma \circ f$. Horizontal lines map to horizontal lines in the same half of the plane, and since $H(\frac{1}{2})$ and $H(-\frac{1}{2})$ are tangent to A which is invariant, $H(\pm\frac{1}{2})$ must also be invariant. The intersections of these two horizontal lines with $V(0)$ and $V(1)$ give four fixed points. The lines and circles joining these will also be invariant, so we keep getting new intersection points which must be fixed by $\gamma \circ f$. We continue this process and get a dense subset of $\overline{\mathbb{C}}$. □

Theorem 2.2. The group of orientation preserving isometries of $\mathbb{H}^3$ is $PSL(2, \mathbb{C})$. Its action on $\mathbb{H}^3$ is uniquely determined by its Möbius action on $\partial\,\mathbb{H}^3$.

Proof. It is enough to show that the orientation preserving isometries of $\mathbb{H}^3$ are in one-to-one correspondence with the homeomorphisms of $\partial\,\mathbb{H}^3$ that take circles to circles.
Given an isometry h of $\mathbb{H}^3$, we can extend it to $\partial\,\mathbb{H}^3$ using the continuity of the map h. Since isometries of $\mathbb{H}^3$ take totally geodesic planes to totally geodesic planes, the restriction $h|_{\partial\,\mathbb{H}^3} : \partial\,\mathbb{H}^3 \to \partial\,\mathbb{H}^3$ takes boundaries of planes in $\mathbb{H}^3$ to boundaries of other planes. This means that $h|_{\partial\,\mathbb{H}^3}$ is a homeomorphism of $\partial\,\mathbb{H}^3$ that takes circles to circles in $\overline{\mathbb{C}}$, where a straight Euclidean line in $\mathbb{C}$ together with the point ∞ is also a circle. Conversely, given a homeomorphism of $\partial\,\mathbb{H}^3$ that takes circles to circles, there is a unique extension to an isometry of $\mathbb{H}^3$. Given $\alpha \in PSL(2, \mathbb{C})$, define $\tilde{\alpha} : \mathbb{H}^3 \to \mathbb{H}^3$ as the isometry that takes the line with endpoints $p, q \in \partial\,\mathbb{H}^3$ to the line with endpoints $\alpha(p), \alpha(q)$. □

Theorem 2.3. (Classification of Isometries of $\mathbb{H}^3$) Let $A \in PSL(2,\mathbb{C})$ with trace t, where A is not the identity matrix. Then

1. $t = \pm 2 \implies A$ is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then A is said to be *parabolic* and corresponds to translation.

2. $t \in (-2,2) \implies A$ is conjugate to $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ for some $\theta \in \mathbb{R}$. Then A is said to be *elliptic* and corresponds to rotation about a fixed axis.

3. $t \in \mathbb{C} - [-2,2] \implies A$ is conjugate to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Then A is said to be *loxodromic* and corresponds to dilation along with rotation about an axis.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. As $A \in PSL(2,\mathbb{C})$, the characteristic polynomial of A is $\lambda^2 - t\lambda + 1$ and eigenvalues are

$$\lambda_1 = \frac{t + \sqrt{t^2-4}}{2}, \lambda_2 = \frac{t - \sqrt{t^2-4}}{2}.$$

Note that $\lambda_1\lambda_2 = 1$.
Case 1: $t = \pm 2$. If $tr(A) = -2$, then A is equivalent to the matrix $-A$ with trace 2. Hence we only need to consider $t = 2$. Then we get $\lambda_1 = \lambda_2 = 1$ and so the Jordan canonical form of A, to which A is conjugate, is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
Case 2: $t \in (-2,2)$. Then $t^2 - 4 < 0$ is a real number. So $\lambda_1 = \frac{t+i\sqrt{4-t^2}}{2}$ and $\lambda_2 = \frac{t-i\sqrt{4-t^2}}{2}$ are complex numbers written in terms of their real and imaginary parts. We know that $\lambda_2 = \frac{1}{\lambda_1}$ and $|\lambda_1| = \sqrt{\frac{t^2}{4} + \frac{4-t^2}{4}} = 1$, so we can write $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$ for some $\theta \in \mathbb{R}$. Thus the Jordan canonical form is $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ for some $\theta \in \mathbb{R}$.
Case 3: $t \in \mathbb{C} - [-2,2]$. Let $\lambda_1 = \lambda$, so $\lambda_1 = \frac{1}{\lambda}$. Then A is conjugate to the Jordan canonical form $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix}$. We have to show that $|\lambda| > 1$.
It can be seen that conjugating A with another element of $PSL(2,\mathbb{C})$ does not change its trace. Writing $\lambda = re^{i\theta}$, we get $t = \lambda + \lambda^{-1} = (r + r^{-1})cos\theta + (r - r^{-1})sin\theta$. If t is real, $t^2 > 4$ and from $\lambda_1 = \frac{t+\sqrt{t^2-4}}{2}$ we can see that $|\lambda| > 1$.
If t is not real, we must have $\theta \neq n\pi$ for any $n \in \mathbb{Z}$ and $r \neq r^{-1}$. Hence $r \neq \pm 1$. If $r > 1$ we are done. If $r < 1$ we just swap the values of λ_1 and λ_2 so that $|\lambda| > 1$. Hence A is conjugate to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. □

3 Other models of hyperbolic geometry

3.1 Hyperboloid model

Consider the symmetric bilinear form $\langle , \rangle_{(n,1)}$ on $\mathbb{R}^{n+1}$, defined as

$$\langle x, y\rangle_{(n,1)} = x_1y_1 + ... + x_ny_n - x_{n+1}y_{n+1}.$$

The underlying space for the hyperboloid model is

$$I_n = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle_{(n,1)} = -1, x_{n+1} > 0\} = \{x \in \mathbb{R}^{n+1} : x_1^2 + ... + x_n^2 - x_{n+1} = -1, x_{n+1} > 0\}.$$

By the regular value theorem, this is a smooth n-manifold. For $p \in I_n$, the tangent space at p is

$$T_pI_n = \{y \in \mathbb{R}^{n+1} : \langle p, y\rangle_{(n,1)} = 0\}.$$

Lemma 3.1. The bilinear form $\langle , \rangle_{(n,1)}$ restricted to each T_pI_n is positive definite.

Proof. Let $v \in T_pI_n$. If $\langle v, v\rangle_{(n,1)} = 0$, then

$$\langle p + v, p + v\rangle_{(n,1)} = \langle p, p\rangle_{(n,1)} + 2\langle p, v\rangle_{(n,1)} + \langle v, v\rangle_{(n,1)} = \langle p, p\rangle_{(n,1)} = -1,$$

so $p + v \in I_n$ and since

$$\langle cv, cv\rangle_{(n,1)} = c^2\langle v, v\rangle_{(n,1)} = 0$$

for any $c \in \mathbb{R}$, we have $p + cv \in I_n$. If $v \neq 0$, we can choose $c \in \mathbb{R}$ such that $p + cv$ has $(n + 1)$th coordinate $\frac{1}{2}$. But this contradicts $p + cv \in I_n$, as the $(n + 1)$th coordinate of a point in I_n is at least 1. □

Hence $\langle , \rangle_{(n,1)}$ defines a norm on each tangent space. This defines a Riemannian metric on I_n by $ds^2 = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2$, which is the hyperbolic metric on I_n. Now we shall characterise the geodesics of this space.

Proposition 3.1. If $x \in I_n$, $y \in T_xI_n$, the geodesic starting at x in the direction y is parametrized as

$$t \mapsto cosh(t)x + sinh(t)y.$$

Proof. Let W be the Euclidean plane generated by x, y in $\mathbb{R}^{n+1}$. We can see by calculation that $t \mapsto cosh(t)x + sinh(t)y$ is a parametrization of $W \cap I_n$. We now show that the maximal geodesic through x in the direction y must be $W \cap I_n$. Define

$$O(\mathbb{R}^{n+1}, \langle , \rangle_{(n,1)}) = \{A \in GL(n + 1, \mathbb{R}) : \langle Ax, Ay\rangle_{(n,1)} = \langle x, y\rangle_{(n,1)} \forall x, y \in \mathbb{R}^{n+1}\}$$

and let $O(I_n)$ be the set of elements of $O(\mathbb{R}^{n+1}, \langle , \rangle_{(n,1)})$ under which I_n is invariant. Let l be the maximal geodesic through x along y. (Existence of such a geodesic comes from defining

I_n as a Riemannian manifold and using the Hopf-Rinow theorem.) Let $\phi \in O(I_n)$ be the linear map given by $\phi(a) = a$ for $a \in W$ and $\phi(a) = -a$ for $a \in W^\perp$. Then $\phi(x) = x$ and $d\phi(x)(y) = y$ as $x, y \in W$. Hence $\phi|_l = Id$, which means that $l \subset W$. Since $l \subset I_n$ by definition, $l \subset W \cap I_n$. Since $W \cap I_n = \{cosh(t)x + sinh(t)y : t \in (-\infty, \infty)\}$, it is one-dimensional and since l is a maximal geodesic, $l = W \cap I_n$. □

Note that from the above proof, any maximal geodesic of I_n is the intersection of I_n with some plane passing through the origin. From this characterization of geodesics, it is easy to see that any two distinct points in I_n determine a unique hyperbolic line in I_n, since the two distinct points along with the origin determine a unique Euclidean plane in $\mathbb{R}^{n+1}$. The analog of totally geodesic planes in higher dimensions is a hyperbolic subspace.

Definition 3.1. We say that $N \subset I_n$ is a *hyperbolic subspace* if given any two points in N, the entire geodesic passing through them is also in N.

Proposition 3.2. $N \in I_n$ is a hyperbolic subspace iff $N = W \cap I_n$ for some linear subspace W of $\mathbb{R}^{n+1}$.

The proof of this will be similar to the one given above.

3.2 Disc model

The underlying space for this model is the unit ball

$$\mathbb{D}^n = \{(x_1, ...x_n) \in \mathbb{R}^n : x_1^2 + ... + x_n^2 < 1\}$$

and the metric is given by

$$ds^2 = \frac{4(dx_1^2 + ... + dx_n^2)}{(1 - x_1^2 - ... - x_n^2)^2}.$$

This is a conformal model. The geodesics are the intersections with $\mathbb{D}^n$ of Euclidean lines passing through the origin and Euclidean circles which are perpendicular to $\partial\mathbb{D}^n$.

3.3 Klein model

The underlying space for this model is also the unit disc of $\mathbb{R}^n$, the same as the disc model. We denote it by $\mathbb{K}^n$. The metric is given by

$$ds^2 = \frac{dx_1^2 + ... + dx_n^2}{1 - x_1^2 - ... - x_n^2} + \frac{x_1 dx_1 + ... + x_n dx_n}{(1 - x_1^2 - ... - x_n^2)^2}$$

The geodesics of this model are Euclidean straight lines intersected with the unit disc.

3.4 Isometries between the models

The models described above are all models for hyperbolic space, and they have to be isometric. Here we give explicit maps between all the models by considering all of them as subspaces of $\mathbb{R}^{n+1}$. We introduce a new model, the hemisphere model, solely for the purpose of getting maps between the other models.
Define $H, I, J, K, L \subset \mathbb{R}^{n+1}$ as follows:

- **Upper half space** $H = \{(1, x_2, ..., x_{n+1}) : x_{n+1} > 0\}$
- **Disc model** $I = \{(x_1, ..., x_n, 0) : x_1^2 + ... + x_n^2 < 1\}$
- **Hemisphere model** $J = \{(x_1, ..., x_{n+1}) : x_1^2 + ... + x_n^2 = 1, x_{n+1} > 0\}$
- **Klein disc** $K = \{(x_1, ..., x_n, 1) : x_1^2 + ... + x_n^2 < 1\}$
- **Hyperboloid** $L = \{(x_1, ..., x_{n+1}) : x_1^2 + ... + x_n^2 - x_n^2 = -1, x_{n+1} > 0\}$

Then we have isometries between each of H, I, K, L and J as follows:

$$f_1 : J \to H, (x_1, ...x_{n+1}) \mapsto (1, \frac{2x_2}{x_1 + 1}, ..., \frac{2x_{n+1}}{x_1 + 1})$$

$$f_2 : J \to I, (x_1, ...x_{n+1}) \mapsto (\frac{x_1}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}}, 0)$$

$$f_3 : K \to J, (x_1, ...x_n, 1) \mapsto (x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2})$$

$$f_4 : L \to J, (x_1, ...x_n, x_{n+1}) \mapsto (\frac{x_1}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}}, \frac{1}{x_{n+1}})$$

We can get an isometry between any two given models by simply composing the above as required.
Descriptions:

- f_1 - Given $j \in J$, $h = f_1(j)$ is the intersection of H with the line joining $(-1, 0, ..., 0)$ and j.
- f_2 - Given $j \in J$, $i = f_2(j)$ is the intersection of I with the line joining $(0, ..., 0, -1)$ and j.
- f_3 - Given $k \in K$, $j = f_3(k)$ is the projection on J of k along the x_{n+1} coordinate.
- f_4 - Given $l \in L$, $j = f_4(l)$ is the intersection of J with the line joining $(0, ..., 0, -1)$ and l.

Figure 3.1 shows where a given point $j \in J$ is mapped by the different isometries.
Using these isometries we can also define each model in terms of the hyperbolic model and pullback metrics. From this construction and the fact that geodesics of L are intersections of L Euclidean planes passing through the origin, we can see that the geodesics of the disc model and the Klein model are as described before.

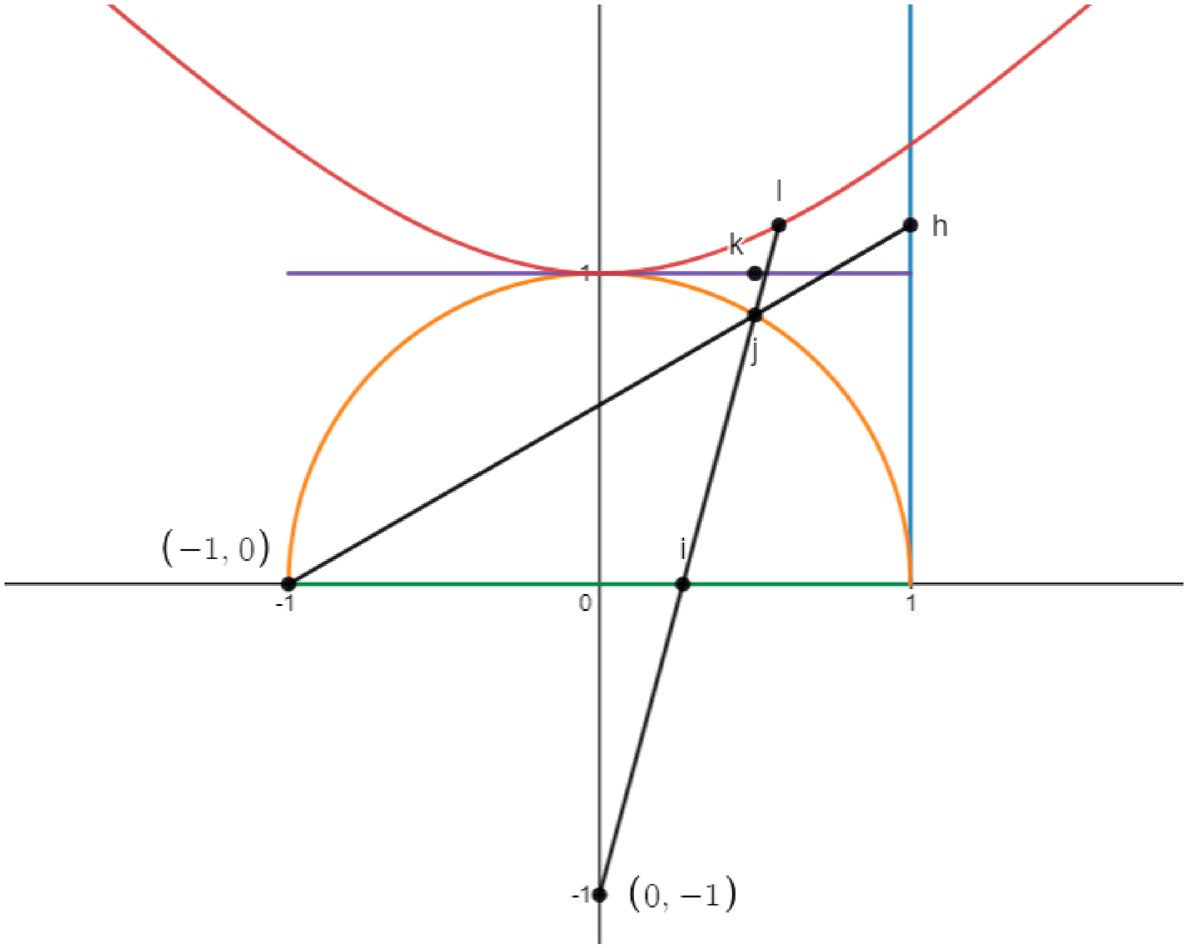

Figure 3.1: The models in one dimension, inside $\mathbb{R}^2$

4 Basics of knot theory

Definition 4.1. A *knot* K is a piecewise linear embedding of the circle S^1 in S^3, $K : S^1 \to S^3$. A *link* L is a piecewise linear embedding of disjoint copies of S^1, $L : \coprod_{i \in I} S^1_i \to S^3$.

We often also call the image $K(S^1)$ as the knot K. A knot may be thought of as a subspace of S^3 that is homeomorphic to S^1 via a piecewise linear homeomorphism. Two knots (or links) K_1 and K_2 are said to be equivalent if they are ambient isotopic, i.e. there is a homotopy $H : S^3 \times [0,1] \to S^3$ such that $H(*,t) = H_t : S^3 \to S^3$ is a homeomorphism for every $t \in [0,1]$ and $H_0(K_1) = K_1$,$H_1(K_2) = K_2$.

Definition 4.2. For a knot K, the *knot exterior* is $S - N(K)$ where $N(K)$ is a regular neighbourhood of K. The *knot* complement is $S - K$, which is homeomorphic to the interior of the knot exterior. Similar definitions work for links.

Theorem 4.1. (Gordon-Luecke theorem) If K_1 and K_2 are knots (or links) such that $S^3 - K_1$ and $S^3 - K_2$ are homeomorphic via an orientation-preserving homeomorphism, then K_1 and K_2 are equivalent.

In general, knots are not equivalent to their mirror images. We will only consider knots up to reflection, so in this case knots will be determined by their complements.
A knot diagram is a projection of a knot onto the two dimensional plane. It is a 4-regular graph which shows undercrossings and overcrossings at each vertex. The Reidemeister moves are a set of three kinds of moves that manipulate a knot diagram without changing its equivalence class.

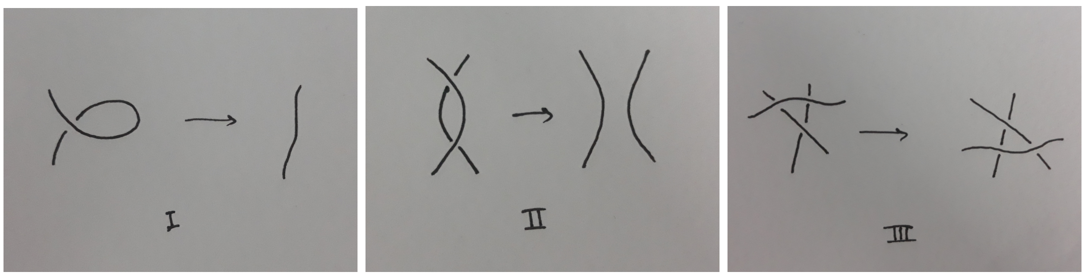

Figure 4.1: The three Reidemeister moves

Theorem 4.2. (Reidemeister-Alexander-Briggs) Two knots are equivalent if and only if their diagrams can be changed from one to the other using only Reidemeister moves and planar isotopies (isotopies of arcs of the diagram without affecting the crossings).

Definition 4.3. The *crossing number* of a knot is the minimum possible number of crossings in a diagram of the knot.

The following are some well-known open problems in knot theory:

1. Classification - Given two knots, find out whether they are equivalent.
2. Determining the geometry of the knot complement.
3. Determining the geometry for families of knots.
4. Enumerating knots by geometry rather than by crossing number as is done traditionally.
5. Finding a diagram for a given knot that encodes information about the geometry of the knot.
6. Determining geometric knot invariants.
7. Relating geometric invariants to the other well-known knot invariants.

5 Polyhedral decomposition of knot complements

The aim of this section is to present the complement $S^3 - K$ of a knot K as a gluing of two ideal polyhedra.

Definition 5.1. A *polyhedron* is a closed 3-ball with a finite graph drawn on its boundary such that there are finitely many vertices and edges, and the faces are simply connected. An *ideal polyhedron* is a polyhedron with its vertices removed.

The picture to keep in mind: The knot diagram lies flat on the XY plane, with overcrossings forming a bump above the XY plane and undercrossings below it. We imagine two closed 3-balls expanding like balloons into the diagram, one from above and one from below. They will meet at the faces cut out by the knot diagram on the plane and be glued there.

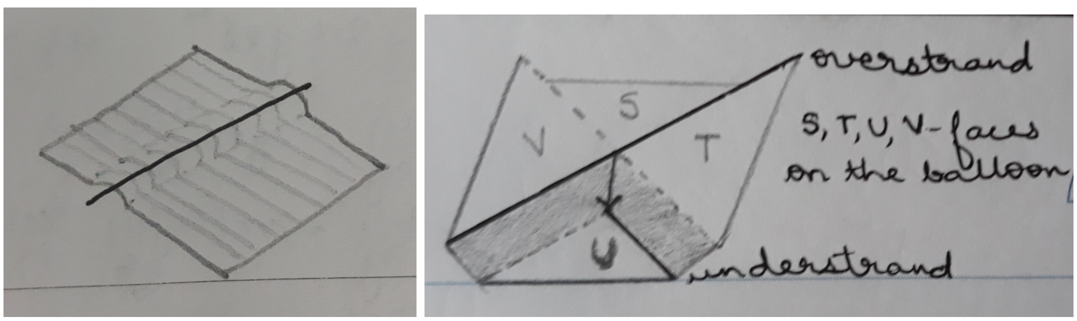

Figure 5.1: How the balloons expand onto a strand of the knot and a crossing

We work through the algorithm for the figure-8 knot.

Step 1: First, label the faces given by the knot diagram. We first construct the top polyhedron. At each crossing, draw two arrows from the overstrand to the understrand as shown in figure 5.2. These are called *crossing arcs* and will form the edges of our polyhedron. These edges must be ideal edges, i.e. their endpoints must be removed, as the endpoints are on the knot and hence not a part of $S^3 - K$.

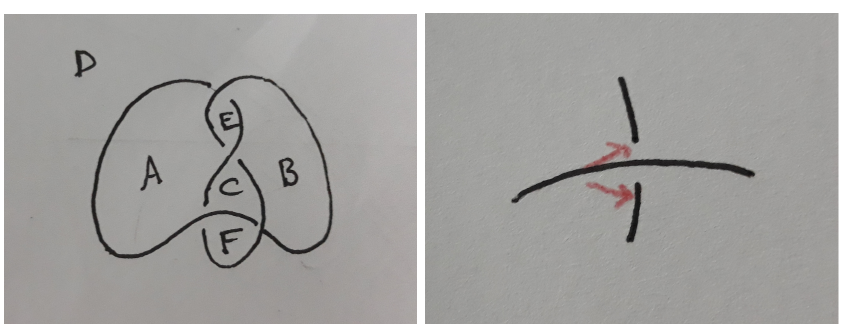

Figure 5.2: Labelled faces and crossing arcs

Step 2: Now we "cut" the knot at each undercrossing so that we get a bunch of disjoint arcs, exactly as the knot diagram would suggest. We shrink each arc to a point, and these form the vertices of the ideal polyhedron. Figure 5.3 shows the top view of the knot with crossing arcs and the graph obtained on the top polyhedron.

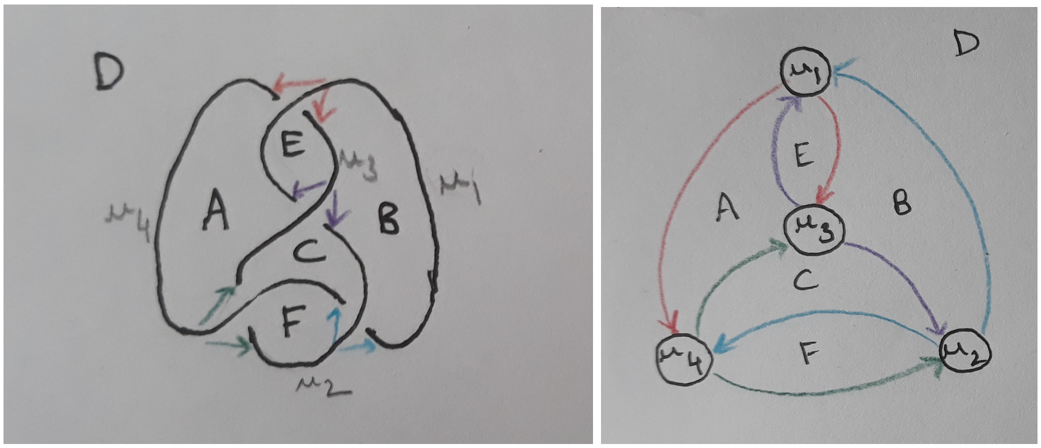

Figure 5.3: Top view of the knot and top polyhedron

Step 3: We do the same on the other side to get the bottom polyhedron, except the crossing arcs are now from understrand to overstrand as we are using the bottom view of the knot. From the top view this still looks like arrows from overstrand to understrand (figure 5.4). So if we fix the top view, there are four arrows going from overstrand to understrand at each crossing, two for the top polyhedron and two for the bottom one.

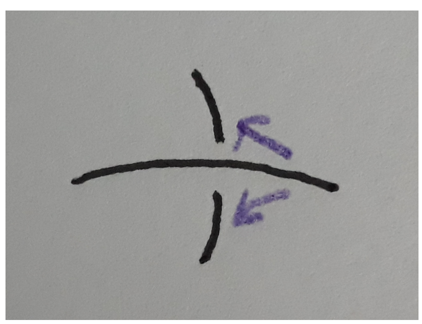

Figure 5.4: Crossing arcs for bottom polyhedron

Glue the two polyhedra face by face, and glue the corresponding edges (of the same colour) together. Then in the resultant object, glue together pairs of edges of the same colour, i.e. got from the same crossing. Now all four edges got from the same crossing have been identified.

This gluing gives us $S^3 - K$. The second gluing is to reattach the pieces of the knot that we had cut up earlier.

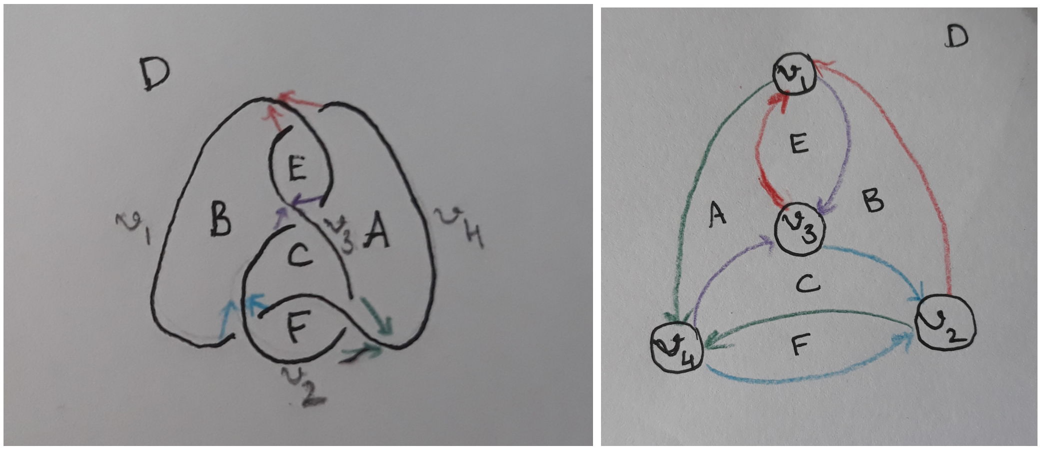

Figure 5.5: Bottom view of the knot and top view of bottom polyhedron

This procedure works for any given knot. When the knot is alternating, we can see that the polyhedra (before removing bigons) are given by labelling each 3-ball with the projection graph of the knot and removing the vertices. This is because each arc a is made of an overcrossing in between two undercrossings, so corresponding to the four arcs in contact with a there are two arrows coming into a and two going out of a, as shown in Figure 5.6. This is exactly how the knot diagram looks at a crossing. Hence for alternating knots the graph obtained by the above algorithm will be a 4-regular graph. Such a graph will admit

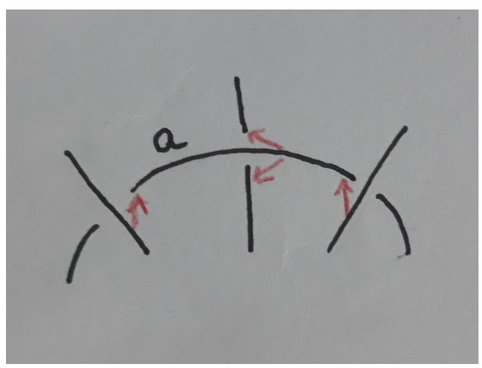

Figure 5.6: Crossing arcs for an alternating knot

a checkerboard colouring or a two-colouring. The gluing in this case will be via a "gear rotation", where the unshaded faces are rotated anti-clockwise before gluing and the shaded faces are rotated clockwise before gluing (figure 5.7). We can also see this above by comparing the top and bottom polyhedra, and seeing that the edges on the bottom are obtained by rotating the edges on the top clockwise. The reason for this is that the pair of arrows of a given colour on the top polyhedron are at an angle to the pair of the same colour on the bottom polyhedron, due to the way the arrows have been defined at a crossing. So in order to identify them we must align them by rotating, and this gives rise to the gear rotation. The

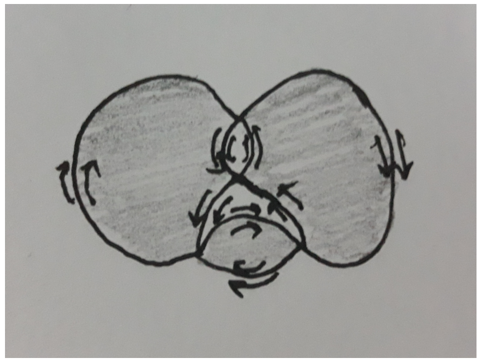

Figure 5.7: Gear rotation and checkerboard colouring for figure-8 knot

polyhedrons obtained above may contain bigons, which we may replace with a single edge that joins its two vertices, as the two edges of a bigon are isotopic since each face is simply connected. Doing this to the figure-8 knot complement gives tetrahedra. In this case we will

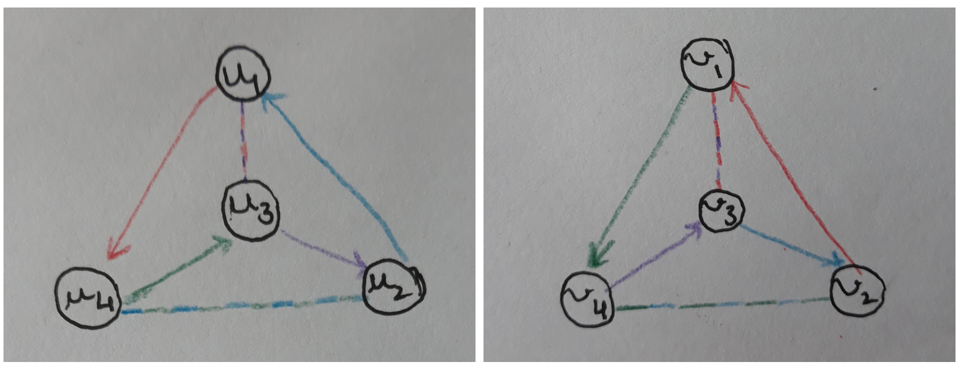

Figure 5.8: Top and bottom tetrahedra for the figure-8 knot

first glue the polyhedra together along matching faces and edges of the same colour. Some edges are coloured in two colours as they are formed by collapsing a bigon of two differently coloured edges. Such a compound edge will be labelled by both these colours and will be glued to another compound edge having the same two colours. In the second gluing step, we glue sets of three edges together, rather than pairs of edges. For instance, in the above case the red, purple and red+purple edges are identified together and the blue, green and blue+green edges are identified together.

6 Geometric structures

Definition 6.1. A *topological polygonal decomposition* of a 2-manifold M is a combinatorial way of gluing polygons (one or more of whose vertices could be ideal) by identifying faces to faces, edges to edges and vertices to vertices so that the result is homeomorphic to M.

Definition 6.2. A *geometric polygonal decomposition* of M is a topological polygonal decomposition along with a metric on each polygon such that gluing is via isometries and the result is a smooth manifold with a complete metric.

Suppose X is a manifold and G is a group acting on X. We say a manifold M has a $(G, X)-$structure if for every $x \in M$, there exists a chart (U, φ) containing x where $U \subset M$ and $\varphi(U) \subset X$ are open, $\varphi : U \to \varphi(U)$ is a homeomorphism and for any two charts (U, φ) and (V, ψ) with $U \cup V \neq \phi$, the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is the restriction to $\psi(U \cap V)$ of an element of G.
Example: Take $X = \mathbb{R}^2$ with the Euclidean metric and $G =$Isom$(\mathbb{R}^2)$. The torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ has (G, X) structure as follows. Let $p : \mathbb{R}^2 \to T^2$ be the covering map. For any $x \in T^2$, let D be an evenly covered open disc around x, and $U \subset \mathbb{R}^2$ an open disc such that $p|_U : U \to D$ is a homeomorphism. Then $(D, (p|_U)^{-1})$ is a chart around x. This gives a (Isom$(\mathbb{R}^2), \mathbb{R}^2)$ structure to T^2, called a *Euclidean structure*.

6.1 Hyperbolic Structure

Definition 6.3. An m-manifold M is said to admit *hyperbolic structure*, or be *hyperbolic*, if it admits a (Isom$(\mathbb{H}^m), \mathbb{H}^m)$ structure.

Hyperbolic polygons are polygons in $\mathbb{H}^2$ with geodesic edges and vertices in $\mathbb{H}^2$ (finite vertices) or in $\partial \mathbb{H}^2$ (ideal vertices). We can get 2-manifolds by gluing together convex hyperbolic polygons along their edges via isometries. We want to see when such a gluing will result in a hyperbolic manifold.

Lemma 6.1. Let M be a 2-manifold obtained by gluing hyperbolic polygons via isometries. M is hyperbolic with structure agreeing with that in the interior of the polygons if and only if each point in M has a neighborhood isometric to a disk in $\mathbb{H}^2$ with the isometry being an identity in the interior of the polygons.

When we say that the hyperbolic structure agrees with the structure in the interior of the polygons, we mean that for any point p in the interior of one of the polygons, the hyperbolic structure comes from a chart (U, φ) where U is a disc containing p, lying in the interior of the polygon, and φ the inclusion map from U to $U \subset \mathbb{H}^3$.
The same result holds for gluing of hyperbolic n-gons or hyperbolic polyhedra of dimension n. We will give the proof for this case.

Proof. If M is hyperbolic, every $x \in M$ has an open neighbourhood U and a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{H}^n$ such that the transition maps are isometries of $\mathbb{H}^n$. We can choose a small enough U containing x so that $\varphi(U)$ is a ball in $\mathbb{H}^n$. In the interior of polyhedron P, φ is an inclusion map, hence an isometry of $\mathbb{H}^n$. On the boundaries, this is composed with the gluing map which is also an isometry. Hence φ is an isometry of U into a ball in $\mathbb{H}^n$. Conversely, suppose each point in M has a neighbourhood isometric to a disc in $\mathbb{H}^n$ with the isometry being an identity on the interiors of the polyhedra. Then this neighbourhood and isometry give a chart around the given point. If (U, φ) and (V, ψ) are two such charts with nonempty intersection, then $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a composition of isometries and hence an isometry of $\mathbb{H}^n$. Hence these charts give M a (Isom($\mathbb{H}^n$), $\mathbb{H}^n$)-structure. Since the charts in the interior of polyhedra are identity maps, the hyperbolic structure agrees with that on the interiors of the polyhedra. □

For the 2-dimensional case, we want to find a simple condition for a given point to have a neighborhood isometric to a disk in $\mathbb{H}^2$. Let $M = \coprod_{\alpha \in A} P_\alpha / \sim$ be the gluing of the polygons P_α and $q : \coprod_{\alpha \in A} P_\alpha \to M$ be the quotient map. Take any $x \in M$.

Case (1) $\hat{x} \in q^{-1}(x)$ is in the interior of a polygon P_α. Then $q^{-1}(x) = \{\hat{x}\}$ as the interior points are not glued by q. Then we have a neighbourhood U of $\hat{x}$ inside P_α such that $q(U)$ provides a chart for x.

Case (2) $\hat{x} \in q^{-1}(x)$ lies on an edge of a polygon. Then $q^{-1}(x) = \{\hat{x}_1, \hat{x}_2\}$ as each edge is glued to one other edge. We can get "half disks" U_1, U_2 containing $\hat{x}_1, \hat{x}_2$ so that U_1, U_2 glue along the boundary to give a disk U around x that is isometric to a disk in $\mathbb{H}^2$, as shown in figure 6.1.

Case (3) $\hat{x} \in q^{-1}(x)$ is a finite vertex of a polygon. Then $q^{-1}(x)$ may have several points, which are different vertices of polygons (possibly multiple vertices of the same polygon). In this case, we can get a neighbourhood U of x isometric to a disk in $\mathbb{H}^2$ iff the sum of interior angles at each vertex in $q^{-1}(x)$ is 2π. This is proved in the following lemma.

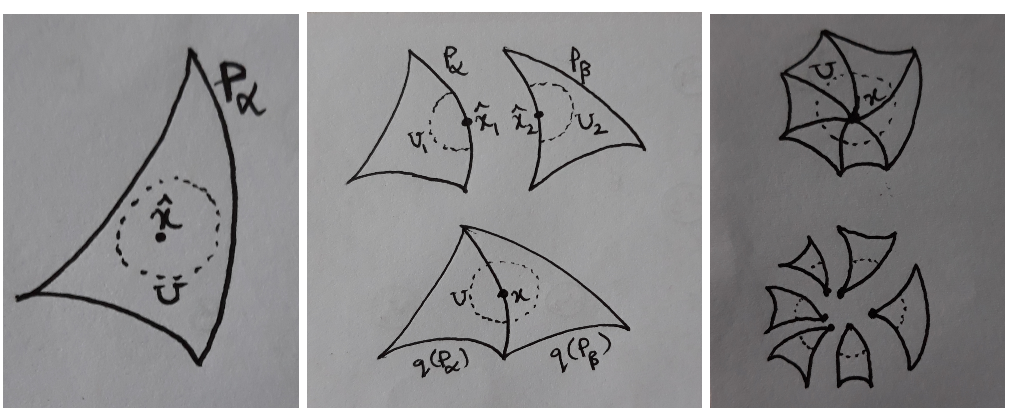

Figure 6.1: Cases 1, 2 and 3

Lemma 6.2. A gluing of hyperbolic polygons gives a 2-manifold with a hyperbolic structure iff for each finite vertex v of the polygons, the sum of interior angles at each vertex glued to v is 2π.

Proof. If the sum of angles at v is 2π, then we can find neighbourhoods in each polygon around the vertex such that the neighbourhoods glue to give a disk isometric to one in $\mathbb{H}^2$. This is because isometries are angle-preserving and the angle around each point in $\mathbb{H}^2$ is 2π. If the sum of angles is not 2π, then the resultant will have a point around which the angle is not 2π. So this cannot isometrically map into $\mathbb{H}^2$. □

6.2 Developing map and Holonomy

Given a manifold M with a (G, X)-structure such that G is a group of real analytic diffeomorphisms acting transitively on X, we construct a local homeomorphism from its universal cover $\tilde{M}$ to X, called the developing map. When M has a polygonal decomposition with polygons in X, this map contains the gluing information for attaching copies of the polygons along edges in the space X.

Let (U, φ) and (V, ψ) be two charts of the (G, X)-structure of M with $U \cap V \neq \phi$. We try to extend φ to V. When $U \cap V$ is connected, $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ determines a unique element of G (as G is a group of real analytic diffeomorphisms). For $y \in U \cap V$, define $\gamma(y)$ as the element of G determined above. We can see that this map $\gamma : U \cap V \to G$ is locally constant.

Define $\Phi : U \cup V \to X$ by

$$\Phi(x) = \begin{cases} \varphi(x) & x \in U \\ \gamma(y) \cdot \psi(x) & x \in V \end{cases}$$

for a fixed $y \in U \cap V$.

This is an extension of φ and a well-defined homeomorphism onto its image when $U \cap V$ is connected, since for $x \in U \cap V$, $\varphi(x) = \gamma(x) \cdot \psi(x) = \gamma(y) \cdot \psi(x)$. However, when $U \cap V$ is not connected and x and y are in different path components, $\gamma(x)$ need not equal $\gamma(y)$ and the above Φ is not always well defined. To get around this we define the developing map on the universal cover $\tilde{M}$ of M, which can be written as $\tilde{M} = \{[\alpha] : \alpha \text{ is a path in } M \text{ starting at } x_0\}$ for a fixed $x_0 \in M$.

Construction of developing map Let $[\alpha] \in \tilde{M}$. We choose a representative $\alpha : [0, 1] \to M$ and use this to define the developing map. Let (U_0, φ_0) be a chart containing $x_0 = \alpha(0)$. Let $(U_1, \varphi_1), ..., (U_n, \varphi_n)$ be charts such that $U_0, ..., U_n$ form a cover of the path α. Choose $0 = t_0 < t_1 < ... < t_n = 1$ such that $\alpha([t_i, t_{i+1}]) \subset U_i$ for each $i \in \{0, 1, ..., n-1\}$. Then we have $\alpha(t_i) \in U_{i-1} \cap U_i$. Write $\alpha(t_i) = x_i$ and let $\gamma_{i-1,i}(x_i)$ be the element of G determined by the transition map $\varphi_{i-1} \circ \varphi_i^{-1}$. Define $\Phi_1 : [0, t_2] \to X$ as

$$\Phi_1(t) = \begin{cases} \varphi_0(\alpha(t)) & t \in [0, t_1] \\ \gamma_{0,1}(x_1) \cdot \varphi_1(\alpha(t)) & t \in [t_1, t_2] \end{cases}$$

Since $\alpha(t)$ lies in the same connected component of $U_{i-1} \cap U_i$ as x_1, this is well-defined. Now we can inductively define $\Phi_i : [0, t_{i+1}] \to X$ as

$$\Phi_i(t) = \begin{cases} \varphi_{i-1}(\alpha(t)) & t \in [0, t_i] \\ \gamma_{0,1}(x_1) \cdot \gamma_{1,2}(x_2) \cdot ... \cdot \gamma_{i-1,i}(x_i) \cdot \varphi_i(\alpha(t)) & t \in [t_i, t_{i+1}] \end{cases}$$

This is well-defined because at t_i,

$$\begin{aligned} \Phi_{i-1}(t_i) &= \gamma_{0,1}(x_1) \cdot \gamma_{1,2}(x_2) \cdot ... \cdot \gamma_{i-2,i-1}(x_{i-1}) \cdot \varphi_{i-1}(\alpha(t_i)) \\ &= \gamma_{0,1}(x_1) \cdot \gamma_{1,2}(x_2) \cdot ... \cdot \gamma_{i-1,i}(x_i) \cdot \varphi_i(\alpha(t_i)) \end{aligned}$$

as $\varphi_{i-1}(\alpha(t_i)) = \gamma_{i-1,i}(x_i) \cdot \varphi_i(\alpha(t_i))$.
Repeating this process till $i = n - 1$, we get $\Phi_{n-1} : [0, 1] \to X$.

Definition 6.4. The *developing map* $D : \tilde{M} \to X$ is defined as $D([\alpha]) = \Phi_{n-1}(1)$.

$$D([\alpha]) = \gamma_{0,1}(x_1) \cdot \gamma_{1,2}(x_2) \cdot ... \cdot \gamma_{n-2,n-1}(x_{n-1}) \cdot \varphi_{n-1}(\alpha(1))$$

The *developing image* is the image of D in X, $D(\tilde{M}) \subset X$.

Proposition 6.1.

1. Given $x_0 \in U_0$ and chart (U_0, φ_0), D is well-defined and independent of other choices.
2. D is a local diffeomorphism.
3. If we define D' by choosing a different initial point x_0' and chart (U_0', φ_0'), we get $D = g \circ D'$ for some $g \in G$.

Proof. 1. We want to show that $D : \tilde{M} \to X$ is independent of choice of representative of $[\alpha]$, the charts (U_i, φ_i) and the points x_i chosen in the intersections $U_{i-1} \cap U_i$.
Suppose we have two sets of charts $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ covering the path α, with $(V_0, \psi_0) = (U_0, \varphi_0)$. For a fixed j, consider the developing map given by the set of charts $\{(U_i, \varphi_i)\} \cup (V_j, \psi_j)$. Let $\beta_{i-1,j}$ and $\beta_{j,i}$ be the elements of G corresponding to the transition maps $\varphi_{i-1} \circ \psi_j^{-1}$ and $\psi_j \circ \varphi_i^{-1}$ respectively. Since $x_i \in U_{i-1} \cap V_j \cap U_i$, we have $\gamma_{i-1,i} = \varphi_{i-1} \circ \varphi_i^{-1} = \varphi_{i-1} \circ \psi_j^{-1} \circ \psi_j \circ \varphi_i^{-1} = \beta_{i-1,j}\beta_{j,i}$. Thus the expression for $D([\alpha])$ remains unchanged when we insert the chart (V_j, ψ_j). We can insert charts $(V_1, \psi_1), ..., (V_n, \psi_n)$ and the developing map will remain unchanged. So the chart cover $\{(U_i, \varphi_i)\} \cup \{(V_j, \psi_j)\}$ gives the same developing map as the individual chart covers $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$,

and these two covers give the same developing map.

Now, fix a choice of charts $\{(U_i, \varphi_i)\}$. The choice of $x_i \in U_{i-1} \cap U_i$ does not affect D because any other choice $y_i \in U_{i-1} \cap U_i$ will lie in the same connected component of $U_{i-1} \cap U_i$ and hence give the same $\gamma_{i-1,i}$.

Suppose we choose a different path $\alpha_1 \in [\alpha]$. Then α_1 is path homotopic to α via a path homotopy $F : I \times I \to M$, where $F(s, 0) = F_0(s) = \alpha(s)$ and $F(s, 1) = F_1(s) = \alpha_1(s)$. We can choose a series of finitely many t_i so that the paths $F_{t_{i-1}}$ and F_{t_i} are contained in the same set of charts covering the paths. Since the choice of charts does not affect D, each path F_{t_i} determines the same developing map as the previous one. Thus all the paths chosen above give the same developing map, and in particular $\alpha = F_0$ and $\alpha_1 = F_1$ give the same developing map.

2. The covering map $p : \tilde{M} \to M$ is given by $[\alpha] \mapsto \alpha(1)$. Hence

$$D = \gamma_{0,1}(x_1) \cdot \gamma_{1,2}(x_2) \cdot ... \cdot \gamma_{n-2,n-1}(x_{n-1}) \cdot \varphi_{n-1} \circ p.$$

p is a local diffeomorphism and the rest of the maps above are diffeomorphisms. Hence their composition D is a local diffeomorphism.

3. Choose a different basepoint x'_0 and chart (U'_0, φ'_0). Define the developing map $D' : \tilde{M} \to X$ using these. M is path connected, so we can choose a path β from x_0 to x'_0. Given a path α' starting at x'_0, we have $\alpha = \beta * \alpha'$ starting at x_0 with $\alpha(1) = \alpha'(1)$. Construct developing maps D, D' starting from the points x_0, x'_0 respectively. While constructing D we choose points on α to get group elements $\gamma_{i,i+1} \in G$. Choose x'_0 as one of these points. Then we get $D = \gamma_{0,1}(x_1)...\gamma_{i,i+1}(x_{i+1})D'$, since after x'_0 the construction is the same as that of D'. Putting $\gamma_{0,1}(x_1)...\gamma_{i,i+1} = g \in G$ we get $D = g \circ D'$ as required.

□

From the above construction we can also get a map $\Phi_{[\alpha]} : V \to X$ for a small open neighbourhood V of $\alpha(1)$, defined as

$$\Phi_{[\alpha]}(x) = \gamma_{0,1}(x_1) \cdot \gamma_{1,2}(x_2) \cdot ... \cdot \gamma_{n-2,n-1}(x_{n-1}) \cdot \varphi_{n-1}(x))$$

Suppose $[\alpha]$ is a class of loops, i.e. $[\alpha] \in \pi_1(M, x_0)$. Then $(V, \Phi_{[\alpha]})$ and (U_0, φ_0) are intersecting charts, both containing x_0. Hence φ_0 and $\Phi_{[\alpha]}$ differ by an element of G, say $g_{[\alpha]}$. We have $\Phi_{[\alpha]} = g_{[\alpha]}\varphi_0$.

Lemma 6.3. The map $\rho : \pi_1(M, x_0) \to G$ defined as $\rho([\alpha]) = g_{[\alpha]}$ is a group homomorphism.

Proof. Let $T_{[\alpha]} : \tilde{M} \to \tilde{M}$ be the map $[\beta] \mapsto [\alpha] * [\beta]$. Since $\alpha(1) = \alpha(0) \in U_0$, we can choose the chart (U_{n-1}, φ_{n-1}) to be the same as (U_0, φ_0) and see that $g_{[\alpha]} = \gamma_{0,1}(x_1)\gamma_{1,2}(x_2)...\gamma_{n-2,n-1}(x_{n-1})$. For any $[\beta] \in \tilde(M)$ starting at x_0, $g_{[\alpha]} \circ D([\beta]) = D([\alpha * \beta]) = T_{[\alpha]}([\beta])$ by part 3 of the previous proposition. Hence $D \circ T_{[\alpha]} = g_{[\alpha]} \circ D$. Then for any $[\beta_1], [\beta_2] \in \tilde{M}$, $\rho([\beta_1] * [\beta_2]) = g_{[\beta_1 * \beta_2]}$ and

$$g_{[\beta_1*\beta_2]} \circ D = D \circ T_{[\beta_1*\beta_2]} = D \circ T_{[\beta_1]} \circ T_{[\beta_2]} = g_{[\beta_1]} \circ (D \circ T_{[\beta_2]}) = g_{[\beta_1]} \circ g_{[\beta_2]} \circ D.$$

This means that $g_{[\beta_1*\beta_2]}$ and $g_{[\beta_1]} \circ g_{[\beta_2]}$ agree on the image of D which is open in X as D is a local diffeomorphism. Since they are real analytic maps, they are determined uniquely by restriction to an open set, hence $g_{[\beta_1*\beta_2]} = g_{[\beta_1]} g_{[\beta_2]}$, or $\rho([\beta_1] * [\beta_2]) = \rho([\beta_1])\rho([\beta_2])$. □

Definition 6.5. The *holonomy element* of $[\alpha]$ is $g_{[\alpha]}$, the *holonomy* of M is the map ρ and its image $\rho(\pi_1(M, x_0))$ is the *holonomy group* of M.

6.3 Completeness of polygonal gluings

Now we want to see when a given gluing of hyperbolic polygons that gives a hyperbolic 2-manifold M, also gives a geometric polygonal decompositional of M. That is, we want to see if the metric on M that comes from the hyperbolic metric on the polygons makes M a complete metric space. We know that the angle around each finite vertex must be 2π for M to be a hyperbolic manifold. Completeness depends on what happens near ideal vertices.

Definition 6.6. A *horocycle* centred at $p \in \partial \mathbb{H}^2$ is a curve in $\mathbb{H}^2$ perpendicular to every geodesic through p in $\mathbb{H}^2$. A *horoball* is the interior of a horocycle.

If $p = \infty$, the horocycles at p are the Euclidean horizontal lines in $\mathbb{H}^2$. For any other p, the horocycles are Euclidean circles in $\mathbb{H}^2$ tangent to $\partial \mathbb{H}^2$ at p.
Let v be an ideal vertex of M, got by gluing the ideal vertices v_0 of P_0, v_1 of P_1 and so on until v_{n-1} of P_{n-1}, where the P_i are hyperbolic polygons. Let h_0 be the segment in P_0 of a horocycle centred at v_0. h_0 meets an edge of P_0 at a point that is glued to a point x on the edge of P_1. Let h_1 be the horocycle segment in P_1 about v_1 determined by this point x. Continue extending the horocycle this way to obtain $h_2, ..., h_{n-1}$. Then extend it back into P_0 as h_n. Now, h_0 and

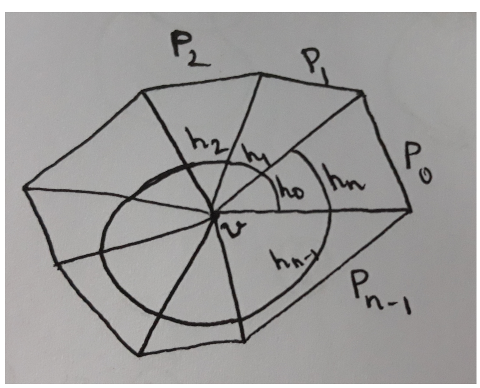

Figure 6.2: Image of the polygons and h_i inside M

h_n are horocycles in the same polygon in $\mathbb{H}^2$, centred around the same point v_0. Hence they are equidistant. Define $d(v)$ to be the fixed hyperbolic distance between h_0 and h_n, with the sign convention that $d(v)$ is positive if h_n is closer to v_0 than h_0, and negative otherwise.

Lemma 6.4. $d(v)$ is well-defined, i.e. independent of the choice of h_0 and P_0.

Proof. Note that if we extend two horocycles g_0, h_0 in P_0 to g_1, h_1 in P_1, then the distance between g_0 and h_0 is the same as that between g_1 and h_1 because they meet at an edge. Extending further, we see that the distance between g_i and h_i is the same as that between g_j and h_j for any $i, j \in \mathbb{N}$.
Suppose we start at P_0 and extend two different horocycles g_0 and h_0 at a distance c from each other. Then the distance between g_n and h_n will also be c, i.e. shifting the initial horocycle by c also shifts the final horocycle by c. Hence $d(v)$ is independent of the choice of initial horocycle.
Now, given $h_0, ..., h_n$ constructed starting from P_0, we see what happens when we start from a different polygon P_k. Since initial horocycle does not matter, we start with h_k and extend it to h_{n+k} in P_k. Then distance between h_k and h_{n+k} is the same as the distance between h_0 and h_n by the same argument as before. Hence $d(v)$ is independent of initial polygon. □

Proposition 6.2. Let S be a 2-manifold with hyperbolic structure obtained by gluing hyperbolic polygons along their edges via isometries. Then the metric on S is complete if and only if $d(v) = 0$ for each ideal vertex v.

Proof. Suppose $d(v) \neq 0$ for some ideal vertex v. Then we construct a Cauchy sequence that does not converge in S as follows. Let $P_0, ..., P_{n-1}$ be the polygons around v. Construct horocycle segments $h_0, h_1, ..., h_n, ...$ as before, going till infinity. Let a_i be the intersection of h_i with an edge meeting v as shown in figure 6.3. To show that the sequence (a_n) is Cauchy, we

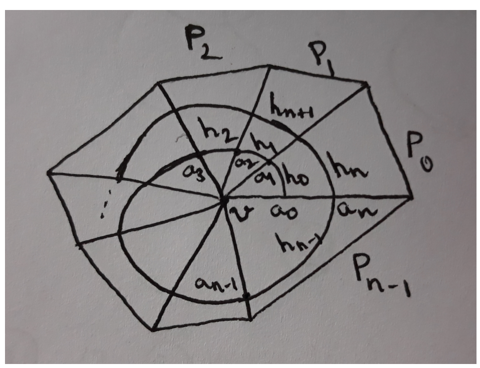

Figure 6.3: Infinite horocycle around v

take isometric copies of $P_0, ..., P_{n-1}$ such that the ideal vertex v lies at infinity and the edges to be glued are vertical geodesics, all glued in $\mathbb{H}^2$ except one edge each of P_0 and P_n. This is shown in figure 6.4. The horocycle segments h_i will be horizontal Euclidean line segments in this setup.

Let the x-coordinate of the leftmost edge of P_0 be a, and that of the rightmost edge of P_{n-1} be b. Let the y-coordinates of the horizontal lines be y_0, y_1, etc. The hyperbolic distance between the horocycles at y_i and y_{i+1} is $log(\frac{y_{i+1}}{y_i}) = d(v)$ for every $i \in \mathbb{N}$. Hence the ratio $\frac{y_{i+1}}{y_i}$ is the same for any i. Let $\frac{y_{i+1}}{y_i} = r$. Then $y_1 = ry_0, y_2 = ry_1 = r^2y_0$ and so on, $y_k = r^ky_0$. The

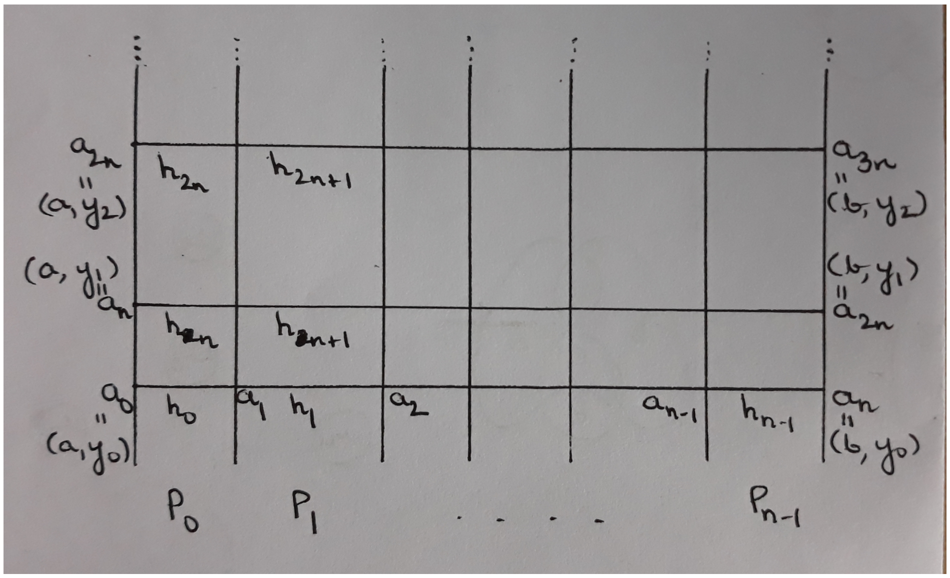

Figure 6.4: Image of the polygons in $\mathbb{H}^2$ with v at infinity

distance between any two points a_k and a_l is less than the length of the horocycle segments joining them. Now, the hyperbolic length of the horizontal Euclidean line segment joining (a, y_k) and (b, y_k) is $\frac{b-a}{y_k}$. Suppose y_j is the y-coordinate of the lower of the two points a_k and a_l. Then

$$d_H(a_k, a_l) < \sum_{i=j}^{\infty} \frac{b-a}{y_i} = (b-a)\sum_{i=j}^{\infty} \frac{1}{r^i y_0} = \frac{b-a}{y_0}\frac{r}{r-1}r^{-j}.$$

Thus given any $\varepsilon > 0$, we can take large enough j such that all the points in (a_n) above y_j are at distance less than ε from each other. Hence (a_n) is Cauchy.

For any point $x \in S$, we can find an open neighbourhood around x which does not intersect at least one edge in the gluing. Each edge contains infinitely many points of the sequence (a_n), so x cannot be the limit of (a_n). Hence (a_n) is not convergent.

Conversely, suppose $d(v) = 0$ for each ideal vertex v of S. Then the horocycles close up around each v and we can remove the interior horoball around each ideal vertex to get a compact manifold with boundary. For $t > 0$, let S_t be the compact manifold obtained by removing interiors of horocycles at a distance t from our original choice of horocycles. Then each point of S is in S_t for large enough t, i.e. $S = \bigcup_{t \in \mathbb{R}^+} S_t$. So any Cauchy sequence in S will be contained in S_t for large enough t. Since S_t is compact, this Cauchy sequence must converge. Thus S is complete. □

Examples

1. Consider the torus T^2 with Euclidean structure, i.e. $(\text{Isom}(\mathbb{R}^2), \mathbb{R}^2)$-structure. Take any point $x \in T^2$ and closed curve γ at x. There is a chart mapping x to $\mathbb{R}^2$ that also maps a segment of the curve γ to $\mathbb{R}^2$. For simplicity, take the chart to be an open square containing x, whose boundaries can be glued to give T^2. Then as γ passes over a meridian or longitude in T^2, in the developing image we glue a new square to the appropriate side of the square we just left. The tiling of $\mathbb{R}^2$ by such squares gives the developing image of T^2. This is shown in figure 6.5.

   If we take T^2 to be the affine torus, i.e. having $(G, \mathbb{R}^2)$-stucture where G is the group of

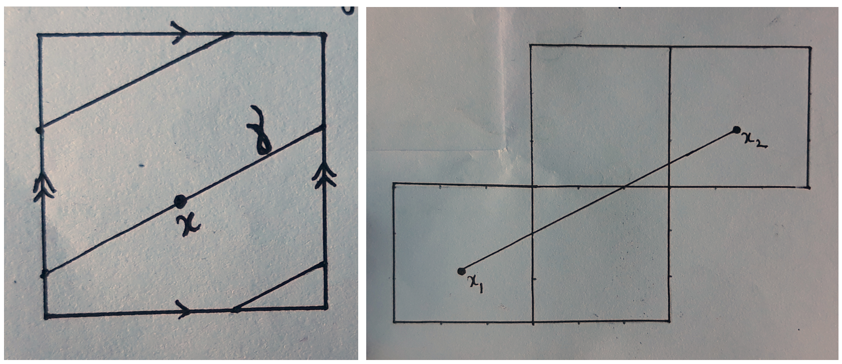

Figure 6.5: Closed curve at a point in the torus, developing image in $\mathbb{R}^2$

   invertible affine transformations of $\mathbb{R}^2$, then we can do something similar to the above process using a more general quadrilateral. Then the above process can be done by choosing a suitable chart, and the developing image will be copies of the quadrilateral in $\mathbb{R}^2$, glued together after resizing and rotating suitably. Such a gluing will not always be a tiling and unless the quadrilateral is a parallelogram, it will miss one point of $\mathbb{R}^2$ in the developing image.

2. Let us give a complete structure on a 3-punctured sphere S, which is the 2-sphere with three points removed. A topological polygonal decomposition of S is given in figure 6.6. The geometric polygonal decomposition can be constructed by looking at the developing image in $\mathbb{H}^2$ around the ideal vertex at infinity. Put the first copy of triangle A in $\mathbb{H}^2$ as the ideal triangle with vertices at $(0,0), (1,0)$ and ∞. Glue a copy of B to its right, with vertices $(1,0), \infty$ and $(x,0)$ for some $x > 1$. The next copy of A will have vertices $\infty, (x,0)$ and $(y,0)$ for some $y > x$. This determines an isometry $g_{[\alpha]}$ of $\mathbb{H}^2$, where α is a closed curve in S going once around the vertex at infinity. The image of the first copy of B under this isometry is where the next copy of B must be glued.

   We can now find the entire developing image around ∞ by successively applying $g_{[\alpha]}$ and $g_{[\alpha]}^{-1}$ to the triangles obtained before. This gives a hyperbolic structure on S.

   For this structure to be complete we want horocycles to close up around the ideal

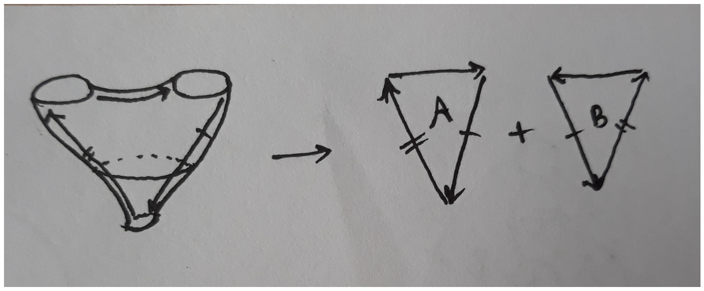

Figure 6.6: Topological polygonal decomposition of S

vertices. Let h, h_0, h_1 and h_x be horocycles about $\infty, (0,0), (1,0)$ and $(x,0)$ in the above construction, as shown in figure 6.7.

Label the distance between h and h_0 as l_1. The isometry $g_{[\alpha]}$ takes h_0 to h_x and pre-

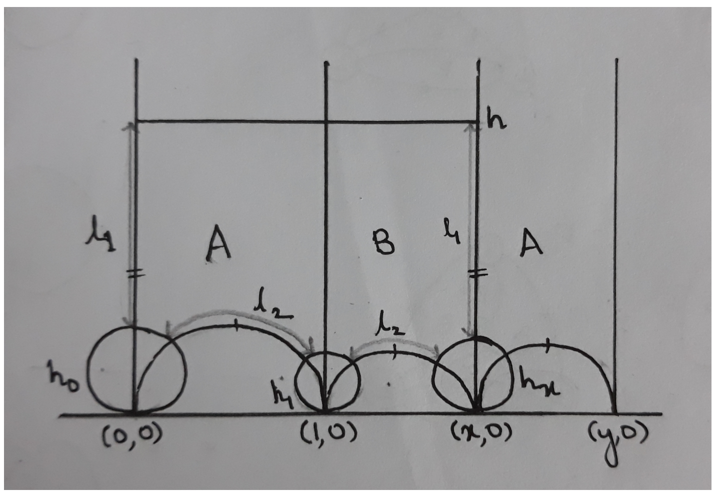

Figure 6.7: Developing image of S

serves distances, hence the distance between h_x and h is also l_1. For the structure to be complete, h must be invariant under the action of $g_{[\alpha]}$. Hence the Euclidean diameter of h_0 and h_x must be the same. Now, let l_2 be the distance between h_0 and h_1. For a closed curve β in S encircling the vertex mapped to $(1,0)$, $g_{[\beta]}$ is an isometry taking the geodesic joining $(0,0)$ and $(1,0)$ to the geodesic joining $(x,0)$ and $(1,0)$. The horocycle h_1 remains invariant under $g_{[\beta]}$ because of completeness. The horocycle h_0 is again mapped to h_x. Thus the distance between h_x and h_1 must be the same as the distance between h_0 and h_1, which is l_2. We know that h_0 and h_x have the same diameter, hence this is only possible if we have symmetry about the vertical geodesic at $(1,0)$, so we must have $x = 2$.

Now, α and β generate the fundamental group of S, so $g_{[\alpha]}$ and $g_{[\beta]}$ generate the holonomy group pf S. Hence this determines a complete structure on S. This complete

structure is unique up to isometry, since there is always an isometry taking a given ideal triangle to another, so starting with a copy of A at any other ideal triangle will yield a structure which is isometric to the one obtained above.
In the above setup, if we take x to be any value other than 2, we get an incomplete structure on S. If we try gluing copies of A and B as before, the triangles will approach a vertical line l which is not in the developing image of S. This is because the intersection of a horocycle at infinity with successive vertical edges will give a Cauchy sequence which is not complete. The space S with this structure can be made into a complete metric space by adding the image of l to S, as this contains the limits of all the nonconverging Cauchy sequences in S.

6.4 A more general criterion for completeness

Lemma 6.5. Let X be a manifold and Y a Hausdorff space. Let $p : Y \to X$ be a local homeomorphism such that every path in X lifts to a unique path in Y with the given initial point in Y.

Proof. Take any $x_0 \in X$ and let $p^{-1}(x) = \{y_j\}_{j \in J}$. Let U be an open neighbourhood of x_0 homeomorphic to a ball and let $f : U \to X$ be the inclusion map. For every $y_j \in p^{-1}(x)$, we will construct a lift $\hat{f}_j : U \to Y$ of f such that $\hat{f}_j(x_0) = y_j$.
For any $x \in U$, let $\alpha : I \to U$ be a curve from x_0 to x in U. Then $\beta = f \circ \alpha$ is a curve in X from x_0 to x. Let $\hat{\beta} : I \to Y$ be the lift of β starting at y_j. Define $\hat{f}_j(x) = \hat{\beta}(1)$. This is independent of the choice of curve α as any other curve from x_0 to x in U will be homotopic to α as U is simply connected. This gives a path homotopic to β in X, so its lift is path homotopic to $\hat{\beta}$, as the homotopy will also lift. We can see that $\hat{f}_j(x_0) = y_j$ by definition, and $p \circ \hat{f}_j(x) = p(\hat{\beta}(1)) = x = f(x)$, hence $p \circ \hat{f}_j(x) = f$.
$\hat{f}_j$ is continuous: Let $x \in U$ and $y = \hat{f}_j(x)$. Let $V \subset Y$ be an open neighbourhood of y. Since p is a local homeomorphism, we can choose V such that $p|_V : V \to p(V)$ is a homeomorphism. Let $q : p(V) \to V$ be the inverse map of $p|_V$. Since f is continuous and U is locally path connected, we can find a path connected neighbourhood W of x such that $f(W) \subset p(V)$. Then $\hat{f}_j(W) \subset V$. We have found for each $x \in U$ a neighbourhood W such that $\hat{f}_j(W) \subset V$ a neighbourhood of y. Hence $\hat{f}_j$ is continuous.
Let $V_j = \hat{f}_j(U)$. Then $p^{-1}(U) = \coprod_{j \in J} V_j$, and $p|_{V_j} : V_j \to U$ is a homeomorphism. Hence U is an evenly covered neighbourhood of x_0, and p is a covering map. □

Theorem 6.1. Let M be an n-manifold with (G, X)-structure, where G acts transitively on X and the metric on X is G-invariant and complete. Then the metric on M inherited from X via the (G, X) charts is complete iff the developing map $D : \tilde{M} \to X$ is a covering map.

Proof. Let $p : \tilde{M} \to M$ be the covering map and give $\tilde{M}$ the metric obtained by pulling back the metric on M using p, known as the lift of the metric on M. Then p is a local isometry.

Note that D is a local isometry by construction.
Suppose D is a covering map. Let (x_n) be a Cauchy sequence in M. This lifts to a Cauchy sequence $(\tilde{x}_n)$ in $\tilde{M}$ as p is a local isometry. D is a local isometry, so $(D(\tilde{x}_n))$ is a Cauchy sequence in X. Since X is complete, $(D(\tilde{x}_n))$ converges in X, say to y. Let U be an open neighbourhood of y that is evenly covered by D. Let $\tilde{U} \subset \tilde{M}$ an open set homeomorphic to U via D, such that $\tilde{U}$ contains infinitely many points of the sequence $(\tilde{x}_n)$. Then the lift of y in $\tilde{U}$, $\tilde{y}$ is the limit of $(\tilde{x}_n)$. Hence $p(\tilde{y})$ is the limit of (x_n) in M. This process can be used to find the limit of any Cauchy sequence in M, hence M is complete.
Now suppose M is complete. Then $\tilde{M}$ is complete with respect to the lift of the metric on M. Let $\alpha : [0,1] \to X$ be a path in X. We will construct its lift $\tilde{\alpha}$ in $\tilde{M}$ with respect to the map $D : \tilde{M} \to X$. Since D is a local homeomorphism, we can lift the portion of α from $t \in [0, t_0)$ for some $t_0 > 0$, to get $\tilde{\alpha} : [0, t_0) \to \tilde{M}$. Since $\tilde{M}$ is complete, this can be extended to $[0, t_0]$. Again using the fact that D is a local homeomorphism, we can choose an evenly covered neighbourhood of $\alpha(t_0)$ to extend $\tilde{\alpha}$ to $[0, t_0 + \varepsilon)$ for some $\varepsilon > 0$. We continue in this fashion until $\tilde{\alpha}$ is defined on all of $[0,1]$.
Thus any path in X lifts to a path in $\tilde{M}$ with respect to D. Also, D is a local homeomorphism. Hence by the previous lemma, D is a covering map. □

Corollary. If X is simply connected and M is a manifold with (G, X)- structure as before, then M is complete iff D is an isometry of X.

Proof. Since X and $\tilde{M}$ are both simply connected, D is a covering map iff it is a covering isomorphism. This is the same as saying that D is an isometry, since D is a local isometry. From the previous theorem, the two statements above are equivalent to saying that M is complete. □

References

1. Jessica S. Purcell. (2019). *Hyperbolic Knot Theory.*
   Retrieved from https://arxiv.org/pdf/2002.12652.pdf.

2. R. Benedetti, C. Petronio. (1991). *Lectures on Hyperbolic Geometry.* New York: Springer-Verlag.

3. James W. Anderson. (2000). *Hyperbolic Geometry.* 2nd ed. London: Springer-Verlag.

4. Jennifer Schultens. (2014). *Introduction to 3-Manifolds.* AMS Graduate studies in mathematics; v. 151.

5. John Stillwell. (1992). *Geometry of Surfaces.* 1st ed. New York: Springer-Verlag.

6. Albert Marden. (2007). *Outer Circles - An Introduction to Hyperbolic 3-Manifolds.* New York: Cambridge University Press.

7. Kathy Snyder. (2009). *Isometries of the Hyperbolic Plane*. Retrieved from http://www.math.uchicago.edu/ may/VIGRE/VIGRE2009/REUPapers/Snyder.pdf.

8. Otto Foster. (1991). *Lectures on Riemann Surfaces*. New York: Springer-Verlag.