

Grushko's Theorem on free products, a Topological Proof

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This is to certify that this minor thesis entitled “Grushko’s Theorem on free products, a Topological Proof” submitted towards the partial fulfillment of the Mathematics PhD degree program at the Indian Institute of Science Education and Research Pune, represents work carried out by Makarand Sarnobat under the supervision of Dr. Tejas Kalelkar.

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1 Introduction

This is a paper written by John R. Stallings titled “**A topological proof of Grushko’s Theorem on free products**” which was published in Math. Zeitschr in the year 1965. It uses no more than basic topology and combinatorics to give a proof of an important theorem in Group Theory. One of the main advantages of this paper, other than the fact that it is a simple proof, is that the proof is algorithmic in nature. The main result proved in this paper is the following:

Grushko’s Theorem: Let $\phi : \Gamma \rightarrow *_{\alpha}\Pi_{\alpha}$ be a homomorphism of the free group Γ onto the free product of groups $\{\Pi_{\alpha}\}$. Then Γ is itself a free product, $\Gamma = *_{\alpha}\Gamma_{\alpha}$, such that $\phi(\Gamma_{\alpha}) \subseteq \Pi_{\alpha}$.

Grushko’s Theorem has some interesting implications. Here are a few of them.

1. **Kneser’s Theorem:** Every compact orientable 3 manifold M is a connected sum of prime manifolds i.e. it can be written as

$$M = P_1 \# P_2 \# \dots \# P_n,$$

such that each P_i is a prime manifold.

2. For any groups A and B of finite rank, we have

$$\text{rank}(A * B) = \text{rank}(A) + \text{rank}(B).$$

Proof: The fact that

$$\text{rank}(A) + \text{rank}(B) \geq \text{rank}(A * B)$$

is obvious as the union of generating sets of A and B will be a generating set for $A * B$.

Let $\{x_1, x_2, \dots, x_n\}$ be a minimal generating set for $A * B$. This is finite as we have an upper bound on the rank of $A * B$. Let Γ be the free group on n generators $\{y_1, y_2, \dots, y_n\}$. We then define a map ϕ from Γ to $A * B$ sending y_i to x_i to obtain a surjective homomorphism from Γ to $A * B$. By Grushko’s Theorem, $\Gamma = \Gamma_1 * \Gamma_2$ such that $\phi(\Gamma_1) = A$ and $\phi(\Gamma_2) = B$. We then have

$$\text{rank}(A) \leq \text{rank}(\Gamma_1) \text{ and } \text{rank}(B) \leq \text{rank}(\Gamma_2).$$

So that,

$$\text{rank}(A) + \text{rank}(B) \leq \text{rank}(\Gamma_1) + \text{rank}(\Gamma_2) \leq \text{rank}(\Gamma) = \text{rank}(A * B).$$

Hence, we have the result.

2 Notations and Definitions

We start by listing some definitions and assumptions.

Note: The set of path classes in X is endowed with a natural multiplication given by concatenation provided the right end point of a path class is the same as the left end point of the other path class. Under this natural operation the set of path classes forms a groupoid.

Let J be an indexing set. We shall now give some definitions.

1. All spaces we shall consider will be CW -complexes. A **path** in a space X will mean a map $P : [0, 1] \rightarrow X$, such that $P(0)$ and $P(1)$ are 0-cells of X , which will be called left and right end point, respectively. Two paths P and Q will be called **homotopic** if there is a homotopy between them relative to the endpoints. Homotopy classes of paths in X are called path-classes. Each path class will have unique left and right end points. We will denote the fundamental group of the space X by $\pi_1(X)$.
2. **J-ad:** A J -ad is a complex X along with a set of sub-complexes $\{A_\alpha\}$, indexed by J , such that $X = \bigcup_{\alpha \in J} A_\alpha$ and whenever $\alpha \neq \beta$, we have $A_\alpha \cap A_\beta = \bigcap_{\gamma \in J} A_\gamma$. We choose the base point of X as a point in $\bigcap_{\gamma \in J} A_\gamma$. We shall denote such a J -ad by $(X; \{A_\alpha\})$. The figure below gives an example of a J -ad $(X; \{A_\alpha\})$.
3. Let $(X; \{A_\alpha\})$ and $(Y; \{B_\alpha\})$ be J -ads. A map $f : (X; \{A_\alpha\}) \rightarrow (Y; \{B_\alpha\})$ is called a **map of J -ads** if $f : X \rightarrow Y$ is a map of CW complexes, i.e. it sends the n skeleton of X into the n skeleton of Y , it sends base point to base point and for all $\alpha \in J$, maps A_α to B_α .
4. In a J -ad $(X; \{A_\alpha\})$, a **loop** in X is a path whose end points coincide and lie in $\bigcap_{\gamma \in J} A_\gamma$. The loop L in the figure is a loop in the $(X; \{A_\alpha\})$.
5. A path in a J -ad $(X; \{A_\alpha\})$ is called a **tie** if the end points of the path lie in different components of $\bigcap_{\gamma \in J} A_\gamma$. The path T in the figure is an example of a tie.
6. We call the elements of J as colors and we say a path P , is **monochromatic** if there exists an $\alpha \in J$ such that $P[0, 1] \subseteq A_\alpha$. The tie T in the figure is monochromatic whereas the loop L is not monochromatic.

7. Let $f : (X; \{A_\alpha\}) \rightarrow (Y; \{B_\alpha\})$ be a map of J -ads. A path P is called a **binding tie** if $P[0, 1] \subseteq A_\alpha$ for some $\alpha \in J$, and fP is homotopic in B_α to a path in $\bigcap_{\gamma \in J} B_\gamma$.

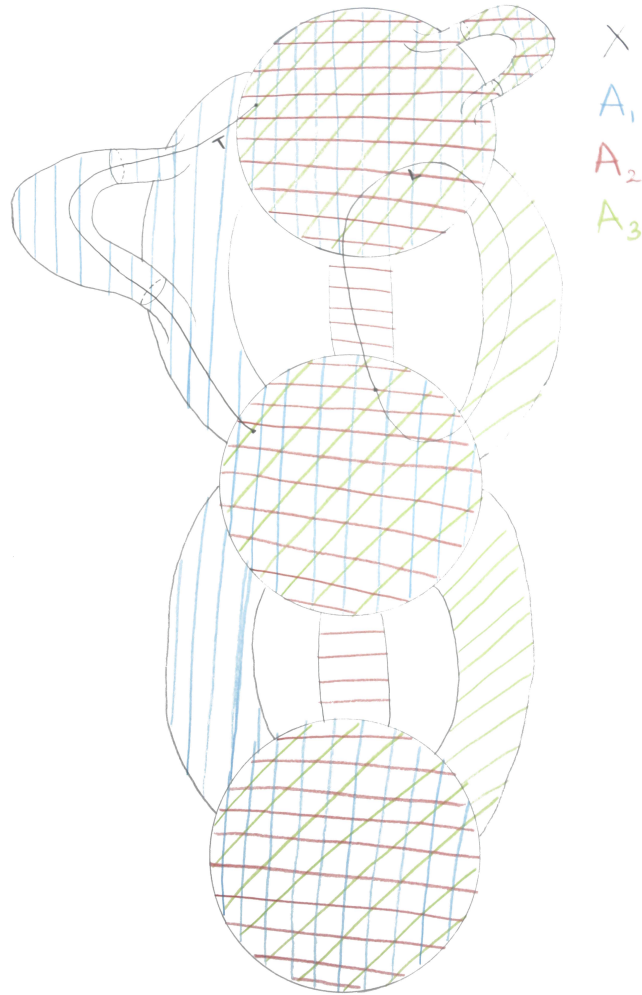


Figure 1: The J -ad X

3 Example

To get a feel for the favor of the proof of the theorem, we work out an example that demonstrates the effectiveness of the method of proof given by Stallings in this paper.

In this example, we consider the free product of two groups $\Gamma_1 * \Gamma_2$ where Γ_1 is a cyclic group of order two with presentation $\langle a | a^2 \rangle$ and Γ_2 is cyclic of order three with presentation $\langle b | b^3 \rangle$. Let F_2 be a free group on two generators x and y . Consider the map

$$\begin{aligned} \phi : F_2 &\rightarrow \Gamma_1 * \Gamma_2 \\ \phi(x) &\mapsto ab^2 \\ \phi(y) &\mapsto aba. \end{aligned}$$

For each $i = 1, 2$, construct a two-dimensional *CW* complex B_i with a single vertex v_i that is determined by the presentation of Γ_i . We then have $\pi_1(B_i) = \Gamma_i$. Let Y be obtained from the disjoint union of B_1 and B_2 by identifying v_1 with v_2 . Then

$$\pi_1(Y) = \Gamma_1 * \Gamma_2.$$

Now, we construct the space X corresponding to F_2 . Construct a space X composed of two 1-circles, one for each generator of F_2 , identified at a base point, as follows. Let $g \in \{x, y\}$ be a generator of F_2 , and consider the representation of $\phi(g) \in \Gamma_1 * \Gamma_2$ as a reduced word in $\{a, b\}$. Let S_g denote the 1-circle corresponding to g . Divide S_g into n segments by n vertices, one for each letter in $\phi(g)$.

Now, define a map

$$f : S_g \rightarrow Y$$

so that the restriction of f to each segment is a path in one of the B_i representing the generator of Γ_i . In this fashion, f defines a labeling scheme for X so that reading around S_g from the base point is exactly $\phi(g) \in \Gamma_1 * \Gamma_2$ as shown in figure 1. It is helpful to think of each segment in X as having a color corresponding to the B_i into which it is mapped. Say for example, the segments mapped into Γ_1 are red and those mapped into Γ_2 are blue. We color each vertex both red and blue.

Now $\pi_1(X) = F_2$ and the union of the maps $f : S_g \rightarrow Y$ defines a unique map $f : X \rightarrow Y$ such that the induced homomorphism

$$f_* : \pi_1(X) \rightarrow \pi_1(Y)$$

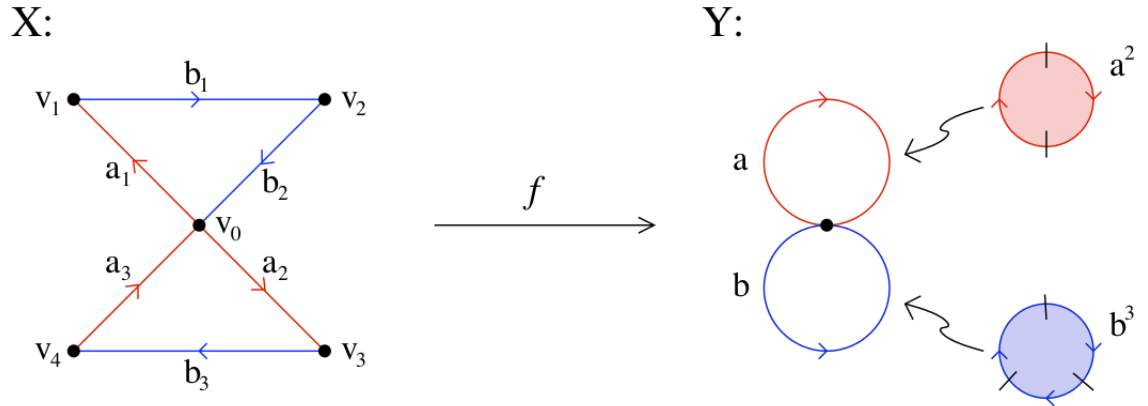


Figure 2: The spaces X and Y .

is equivalent to our original surjection ϕ . Let A_i denote the union of all the vertices of X along with segments which are mapped into B_i . Then $f(A_i) \subseteq B_i$ and the intersection of A_i is the set of vertices of X as shown in figure 2.

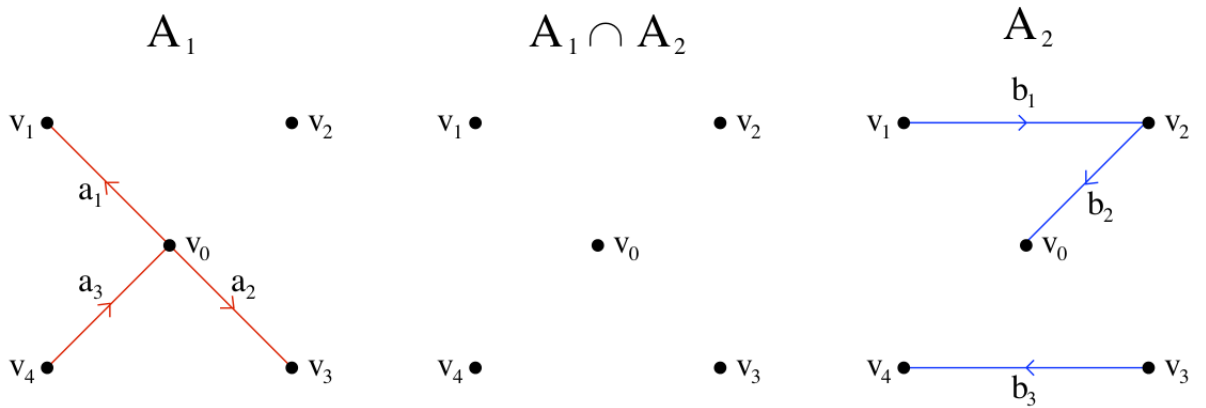


Figure 3: The subcomplexes A_1 and A_2 and their intersection

We now outline a method with which to decompose F_2 , and we illustrate this with our example. Our strategy is to alter X and the map f so that the domain space reflects the free product structure promised in Grushko's theorem. First we construct a finite, connected, 2-dimensional CW complex X' that deformation retracts to X , and a map $f' : X' \rightarrow Y$, which is an

extension of f . It follows that $\pi_1(X') \simeq \pi_1(X) = F_2$, and thus

$$f'_* : \pi_1(X') \rightarrow \pi_1(Y)$$

is the given surjection ϕ .

We build X' such that it is the union of connected subcomplexes A'_i so that the following are satisfied,

1. $A_i \subset A'_i$
2. $f'(A'_i) \subset B_i$
3. There exists a tree T containing all the vertices of X' such that $T \subset A'_i$ for each i .
4. $A'_1 \cap A'_2 = T$

It follows as a corollary to the Seifert-Van Kampen theorem that $\pi_1(X')$ is the free product of the groups $\pi_1(A'_i)$. Therefore, we let

$$\Gamma_i = \pi_1(A'_i).$$

We construct X' by adding 2-cells whose boundary is divided into two arcs that meet only at their endpoints. For each additional 2-cell, we identify one of the arcs with a path in X while leaving the other free. The 2-cells are added by an attaching map on one of the arcs in an algorithmic manner so that each free arc connects two components of the intersection $\bigcap A_i$.

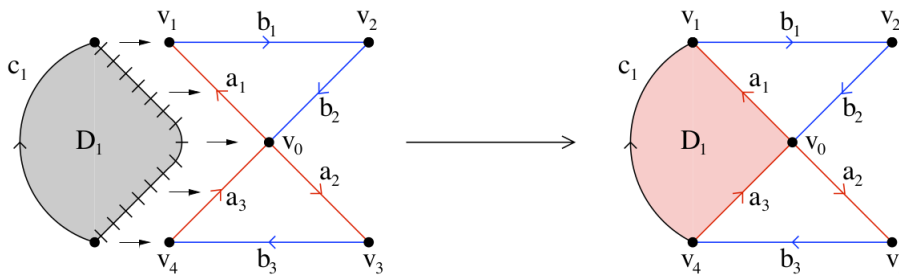


Figure 4: Adding a extra 2-cell via the half attaching map

In the figures below we illustrate the construction of the complex X' . Letting c_i denote the free arc in each new 2-cell, we are able to describe the

boundary word of the additional 2–cells as follows.

Note that as it is difficult to draw all the 2–cells we shall add, the diagram only depicts the new arcs introduced by the 2–cell. Each new 2–cell is added so that its free arc connects two components of the intersection. The first 2–cells added are attached along a_3a_1 and a_3a_2 , respectively. For the other 2–cells, we add the third 2–cell such that one of the one cells is attached to the space along the path given by $c_2b_3c_1b_1b_2$, while the fourth 2–cell is attached along the path $b_3^{-1}c_2^{-1}c_1b_1$.

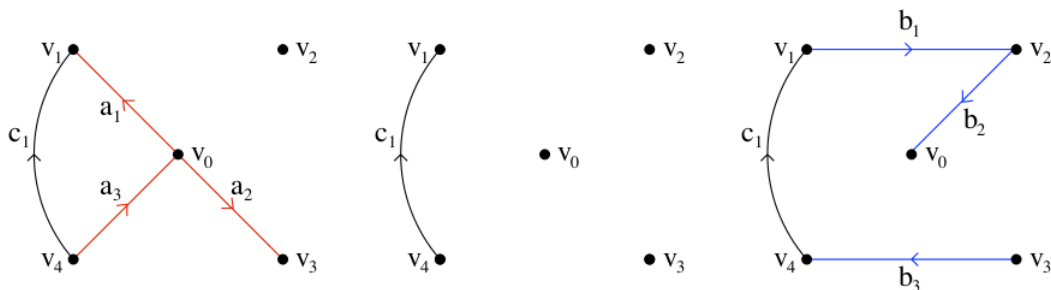


Figure 5: Adding the 2–cell D_1 as in Figure 3 with only the boundary of D_1 illustrated

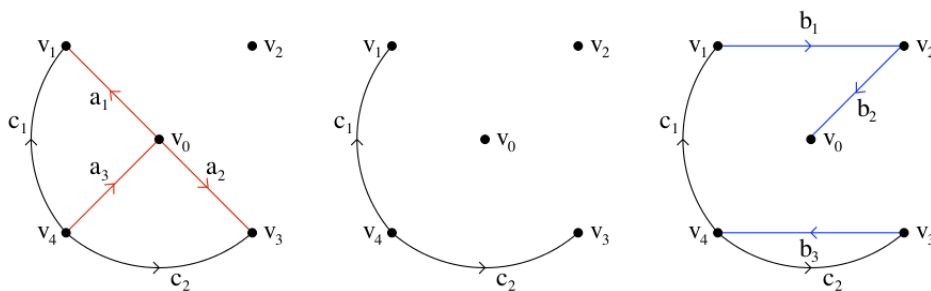


Figure 6: Adding another 2–cell

We have now built the complex X' . Since each c_i is a free edge in the boundary of a newly added 2–cell, it follows that X' contains X as a deformation retract simply by collapsing the new 2–cells to their boundary edge identified with one of the A_i . Define the subcomplexes A'_i of X' to be the

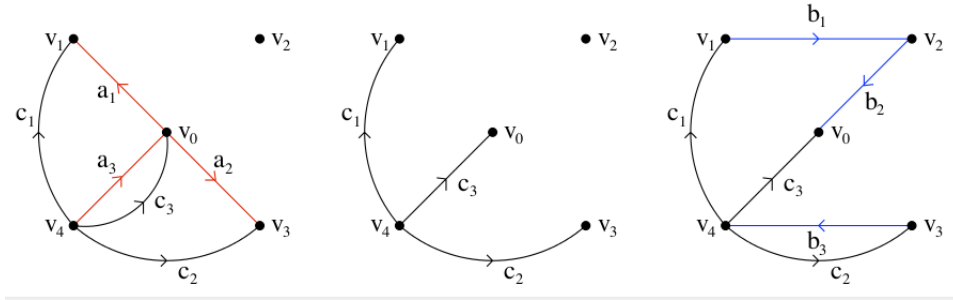


Figure 7: Adding a third 2–cell

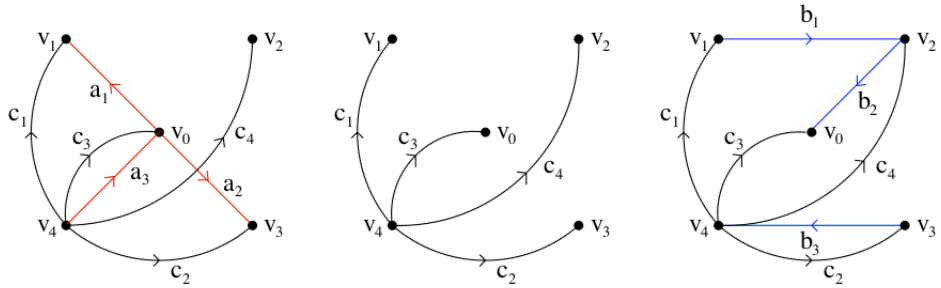


Figure 8: Adding a fourth 2–cell

union of A_i along with all the new edges and 2–cells that were attached to segments in A_i . Following our coloring scheme, we could view all the new arcs as both red and blue, the 2–cells corresponding to c_1 and c_2 as red, and those corresponding to c_3 and c_4 as blue.

We now extend the map $f : X \rightarrow Y$ continuously to a map

$$f' : X' \rightarrow Y$$

as follows. Map all the new c_i arcs into the common base point of B_1 and B_2 , while mapping the red 2–cells into B_1 and blue 2–cells into B_2 . Thus we have a new CW complex X' along with a set of subcomplexes such that the following is satisfied:

1. $A_i \subset A'_i$
2. $f'(A'_i) \subset B_i$
3. There exists a tree T containing all the vertices of X' such that $T \subset A'_i$ for each i . The tree T is given by the middle figure in the last diagram.

4. $A'_1 \cap A'_2 = T$

From here we see that conditions required above of X' are satisfied and we have a continuous extension of f . Thus,

$$F_2 \simeq \pi_1(X) \simeq \pi_1(X') \simeq \pi_1(A'_1) * \pi_1(A'_2) \simeq \Gamma_1 * \Gamma_2.$$

To find a basis for $F_1 * F_2$, view the complex X' as the union of two sub-complexes which are identified along the tree, T , consisting of all the c_i arcs and vertices in X' which are the same as those in X . Each subcomplex will only contain elements that get mapped into one of the B_i or their common basepoint. This is easily visualized by first drawing T as a Figure 8 and then adding the segments from A_i to each diagram.

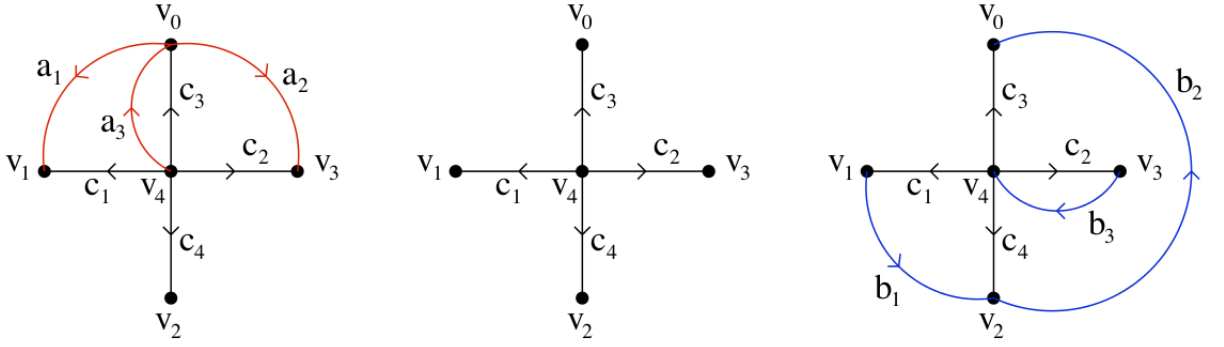


Figure 9: Adding a fourth 2-cell

After a deformation retract, pick a generator for each side. Let

$$u = c_3^{-1}a_3 \quad \text{and} \quad v = c_3^{-1}c_2b_3c_3.$$

Writing c_3^{-1} in terms of a and b , we have

$$u = c_3^{-1}a_3 = b_2^{-1}b_1^{-1}a_1^{-1}a_3^{-1}b_3^{-1}a_2^{-1}a_3^{-1}a_3.$$

After removing the trivial subword $a_3^{-1}a_3$, and observing that

$$x_{-1} = b_2^{-1}b_1^{-1}a_1^{-1} \quad \text{and} \quad y_{-1} = a_3^{-1}b_3^{-1}a_2^{-1},$$

we have $u = x^{-1}y^{-1}$. A similar expansion and reduction process gives $v = x^{-1}yx$. Let u generate F_1 and v generate F_2 . Thus we have a basis for $F_1 * F_2$ such that $\phi(F_i) \subset B_i$. Since $vu = x^{-1}$ and $u^{-1}v^{-1}u = y^{-1}$, we see that $\{u, v\}$ is indeed a basis for F_2 . Therefore we have a desired decomposition of F_2 .

4 The Theorem

For heading towards the main theorems from the paper we will proceed step by step. The first of the steps is the following lemma. This lemma plays an important role in the proof of Grushko's Theorem and enables us to construct a space X' which has the same fundamental group as the given space X .

Lemma 1: Let $f : (X; \{A_\alpha\}) \rightarrow (Y; \{B_\alpha\})$ be a map of $J - ads$. Then there is a $J - ad$ $(X'; \{A'_\alpha\})$, containing $(X; \{A_\alpha\})$, such that X is a deformation retract of X' , and such that each component of $\bigcap_{\gamma \in J} A'_\gamma$ consists of components of $\bigcap_{\gamma \in J} A_\gamma$ joined by arcs. And there is a map of $J - ads$, $f' : (X'; \{A'_\alpha\}) \rightarrow (Y; \{B_\alpha\})$ which extends f , such that with respect to f' there are no binding ties.

Proof: We use transfinite induction on the number of components of $\bigcap_{\gamma \in J} A_\gamma$. If there is only one component of $\bigcap_{\gamma \in J} A_\gamma$, then we can choose $X' = X$ and $f' = f$, since there are no ties in X and hence no binding ties which implies that statement is true if $\bigcap_{\gamma \in J} A_\gamma$ has only one component.

Suppose the statement is true for all ordinals $< \beta$.

Let $\bigcap_{\gamma \in J} A_\gamma$ have β components. Suppose P is a binding tie with color α i.e. $P[0, 1] \subseteq A_\alpha$ and fP is homotopic in B_α to a path in $\bigcap_{\gamma \in J} B_\gamma$.

Δ will denote a abstract 2-complex such that the boundary of the complex is the union of 2 one cells, Γ_1 and Γ_2 , which intersect only in the end points. Identify $[0, 1]$ with Γ_1 so that P can be thought of as a map from Γ_1 to $A_\alpha \subseteq X$.

Denote X' to be the CW complex obtained by the union of X and Δ and identifying $P(t)$ with $t \in \Gamma_1$. Then X' is a CW complex with 2 extra cells, which are Γ_2 and Δ . Observe that the new CW complex X' deformation retracts onto X . Also the set of path classes of X and X' are the same, since Γ_1 is homotopic to P .

Now define A'_α to be the union of A_α and Δ in X' . For, $\gamma \neq \alpha$, define A'_γ to be the union of A_γ and Γ_2 in X' . Then with these definitions, we have, $(X', \{A'_\gamma\})$ is a $J - ad$ containing $(X, \{A_\gamma\})$. Now observe that $\bigcap_{\gamma \in J} A'_\gamma$ contains $\bigcap_{\gamma \in J} A_\gamma$ and the path Γ_2 .

This also implies that $\bigcap_{\gamma \in J} A'_\gamma$ has one component less than the number of components of $\bigcap_{\gamma \in J} A_\gamma$. Now we construct the map f' .

We already have a map from X to Y which is a map of $J - ads$. We want to define a map from Δ so that the total map is a map of $J - ads$, $f' : (X'; \{A'_\alpha\}) \rightarrow (Y; \{B_\alpha\})$. For this we will use the homotopy of P with the path in $\bigcap_{\gamma \in J} B_\gamma$.

We have a homotopy H between fP and a path in $\bigcap_{\gamma \in J} A_\gamma$. Thus we have $H : \Delta \rightarrow B_\alpha$ such that on Γ_1 we have $H = fP$ and $H(\Gamma_2) \subseteq \bigcap_{\gamma \in J} B_\gamma$. Then the map $f \cup H : X \cup \Delta \rightarrow Y$ factors through X' , and hence we have a map $f' : X' \rightarrow Y$. Clearly, $f'(A'_\gamma) \subset B_\gamma$ for all $\gamma \in J$. And hence the map f' is a map of $J - ads$.

Now $\bigcap_{\gamma \in J} A'_\gamma$ has components less than β . Then by our induction hypothesis we can obtain a space and a function such that there are no binding ties with respect the new function. This is possible since taking direct limit of CW complexes which is the same as taking union of CW complexes. This is done when we arrive at a limit ordinal. Hence we see that the theorem holds for any CW complex X and any map of $J - ads$ $f : X \rightarrow Y$.

Remark: Observe that if $\bigcap_{\gamma \in J} A_\gamma$ is a union of 0-cells then the resulting space will be such that the components of $\bigcap_{\gamma \in J} A'_\gamma$ will be trees. Also the fundamental group of the components of $\bigcap_{\gamma \in J} A'_\gamma$ will be trivial. Then by Van Kampen Theorem, we will have that the fundamental group of A'_γ will be a free factor of the fundamental group of X .

This brings us to an end of our first step in proving Grushko's Theorem. Next we will observe how any loop or tie in a $J - ad$ looks like. Also we will prove the existence of a binding tie in a CW complex under certain hypothesis.

Lemma 2: Each loop or tie in a $(X; \{A_\alpha\})$, is homotopic to a product of monochromatic loops and ties $P_1 P_2 \dots P_n$, whose end points are among the set of base points, one per component of $\bigcap_{\gamma \in J} A_\gamma$.

Proof: Let $(X; \{A_\alpha\})$ be a $J - ad$. We first note that given any loop or tie in X , it is homotopic to a product of paths which run through one cells in the complex.

Then we can group the product into maximal monochromatic blocks, (i.e call $R_i R_{i+1}$ to be one path if the have the same color) so that we have that the original path P is homotopic to $R_1 R_2 \dots R_n$ such that R_i and R_{i+1} are of different colors for all i . Then all the end points of R_i are in $\bigcap_{\gamma \in J} A_\gamma$. This is because we have $A_\alpha \cap A_\beta = \bigcap_{\gamma \in J} A_\gamma$ and P is a tie or a loop.

Now, in each component of $\bigcap_{\gamma \in J} A_\gamma$, choose a base point such that the starting point and the end point of the original path is included in this set. Now, for each right end point of $R_i, i < n$, choose a path Q_i in $\bigcap_{\gamma \in J} A_\gamma$, such that the path Q_i starts at the end point of R_i and ends at the selected base point of the component of $\bigcap_{\gamma \in J} A_\gamma$. Then consider the path,

$$(R_1 Q_1)(Q_1^{-1} R_2) \dots (Q_{n-1}^{-1} R_n).$$

This path is homotopic to P . Also, each block in the product is a monochromatic path. Call P_i the i^{th} component of the product. If the end points of P_i are in different components of $\bigcap_{\gamma \in J} A_\gamma$ then P_i is a tie. If the end points are in the same component then P_i is a loop. Thus we have that given a tie or a loop in X it is homotopic to a product of monochromatic paths or loops.

Now using this above fact we give the existence of a binding tie.

Theorem 1: Let $f : (X; \{A_\alpha\}) \rightarrow (Y; \{B_\alpha\})$ be a map of J -ads and let X be connected. Suppose $\bigcap_{\gamma \in J} B_\gamma$ is a single point, so that $\pi_1(Y)$ is naturally the free product of the set of groups $\{\pi_1(B_\alpha)\}$. Suppose that the induced map $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is onto. Then if $\bigcap_{\gamma \in J} A_\gamma$ is not connected, then there is a binding tie.

Proof: Step 1: We claim that there is a tie Q in X , whose path class η is such that $f_*(\eta) = 1$, the trivial element of $\pi_1(Y)$.

We know that X is connected and locally path connected which says that X is path connected. Then there is a tie P in X such that the left end point of the path P is the base point of X . Let ϑ be the path class represented by P . Since we have $\bigcap_{\gamma \in J} B_\gamma$ is a single point, $f_*(\vartheta)$ is a loop in Y .

We know that $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is onto. Then choose a loop $\lambda \in \pi_1(X)$ such that $f_*(\vartheta) = f_*(\lambda)$. Let L be the loop which is represented by λ . Then $P^{-1}L$ is the required tie Q , such that $f_*(\vartheta^{-1}\lambda) = 1$, since f_* is a homomorphism.

Now by the previous lemma, we have

$$Q = Q_1 Q_2 \dots Q_n.$$

Step 2: Now we show that the path $Q = Q_1 Q_2 \dots Q_n$, representing $\eta = \eta_1 \eta_2 \dots \eta_n$, can be chosen such that for all i , Q_i and Q_{i+1} have different colors, and if for any i , Q_i is a loop then $f_*(\eta_i) \neq 1$.

Suppose that for some i , Q_i and Q_{i+1} have the same colors then we can write $Q = Q_1 Q_2 \dots Q_{i-1} (Q_i Q_{i+1}) \dots Q_n$, which is again a product of monochromatic ties and loops but with a fewer number of terms than the original decomposition.

Also, if there is a loop Q_i such that $f_*(Q_i) = 1$, then we can as well skip that factor Q_i , so as to get $Q' = Q_1 Q_2 \dots Q_{i-1} Q_{i+1} \dots Q_n$ and if η' is a class representing Q' then we have $f_*(\eta') = 1$ reducing the length of the decomposition. Thus after a finite number of steps, we obtain the required path Q which is of length at least 1 since we started with a tie.

Third and final step: Now we have a path $Q = Q_1 Q_2 \dots Q_n$ represented by $\eta = \eta_1 \eta_2 \dots \eta_n$ as described in the above step. Thus we have the following equation in $\pi_1(Y)$.

$$1 = f_*(\eta) = f_*(\eta_1) f_*(\eta_2) \dots f_*(\eta_n).$$

Now the terms $f_*(\eta_i)$ and $f_*(\eta_{i+1})$ lie in different factors $\pi_1(B_\alpha)$ for all i and we have that $n \geq 1$. Since we are in a free product of groups, there is some i such that $f_*(\eta_i) = 1$. Now, Q_i cannot be a loop since we have removed all such loops in step 2. This implies that Q_i is a tie, and $f_*(\eta_i) = 1$. Thus we have that fQ_i is null homotopic in Y . But since $\pi_1(Y)$ is a free product, fQ_i is null homotopic in the factor B_α in which it is mapped. Hence Q_i is a binding tie.

Grushko's Theorem: Let $\phi : \Gamma \rightarrow *_\alpha \Pi_\alpha$ be a homomorphism of the free group Γ onto the free product of groups $\{\Pi_\alpha\}$. Then Γ is itself a free product, $\Gamma = *_\alpha \Gamma_\alpha$, such that $\phi(\Gamma_\alpha) \subseteq \Pi_\alpha$.

Proof: Let $\{B_\alpha\}$ be the 2-dimensional CW complex which is determined by Π_α , i.e. $\pi_1(B_\alpha) = \Pi_\alpha$. Let Y be the one point union of all the B_α 's. Then by Van Kampen Theorem we have that $\pi_1(Y) = *_\alpha \Pi_\alpha$.

Let $\{\gamma_\tau\}$ be a free basis of Γ with τ running over some indexing set. For a fixed τ , suppose

$$\phi(\gamma_\tau) = a_1 a_2 \dots a_n,$$

where each a_i belongs to some Π_α .

Let S_τ denote the 1-circle, divided into n 1-cells, starting at the base point, going around S_τ and label them W_1, W_2, \dots, W_n . Note that this n is the same n that appears in the product $a_1 a_2 \dots a_n$. Now define f_τ on W_i to be a path in B_α such that $f_\tau|_{W_i}$ represents a_i . Then f_τ defines a map from S_τ to Y .

Let X denote the one point union of S_τ for all τ . Observe that all the f'_τ 's sends base points of S_τ to base point of Y . Thus we get a map f from X to Y . We can identify Γ with $\pi_1(X)$. Also by construction of the map f we have that the given map ϕ is the same as the map $f_* : \pi_1(X) \rightarrow \pi_1(Y)$.

Now, we want to construct $J - ads$ from the spaces X and Y . We already have a natural $J - ad$ structure on Y given by $(Y, \{B_\alpha\})$. Now we will give an appropriate $J - ad$ structure on X so that X becomes a $J - ad$ and the map f becomes a map of $J - ads$.

To each of the 1-cell W_i of S_τ , associate the index α such that $f|_{W_i}$ is a loop in B_α . Define A_α to be the union of all the one cells to which we have associated the color α and all the 0-cells of X . Then $(X; \{A_\alpha\})$ and $(Y; \{B_\alpha\})$ are $J - ads$ with the same indexing set as that for the free product $*_\alpha \Pi_\alpha$, and f is a map of $J - ads$. We also note that $\bigcap_{\gamma \in J} A_\gamma$ is just the 0-skeleton of X .

Now by Lemma 1, we know that there exists a $J - ad$, $(X', \{A'_\alpha\})$ and an extension f' of the map f , such that X is a deformation retract of X' and there is no binding tie with respect to f' in X' . Even after this construction we have that, $f'_* = \phi$ since the maps f and f' are homotopic.

Thus we do not have a binding tie in X' with respect to f' . This implies by Theorem 1 that $\bigcap_{\gamma \in J} A'_\gamma$ is connected. Also, as we saw earlier that if $\bigcap_{\gamma \in J} A_\gamma$ is a union of points, $\bigcap_{\gamma \in J} A'_\gamma$ is a tree and hence contractible. Thus by Van Kampen Theorem, we have that $\pi_1(X')$ is the free product of $\pi_1(A'_\alpha)$. Now, define $\Gamma_\alpha = \pi_1(A'_\alpha)$. Then we have that, $\Gamma = *_\alpha \Gamma_\alpha$, and $\phi(\Gamma_\alpha) \subseteq \pi_\alpha$.

5 Wagner's Theorem

Now we will prove a similar result, which is known as Wagner's Theorem, but the result is weaker than Grushko's Theorem. Before going to the result we will prove a lemma which will be used in the proof.

Lemma 3: Let $f : (X; \{A_\alpha\}) \rightarrow (Y; \{B_\alpha\})$ be a map of J -ads and let X be connected. Suppose that $C = \bigcap_{\alpha} B_\alpha$ has only one 0-cell, and for each α , $\pi_1(C) \rightarrow \pi_1(B_\alpha)$ be an embedding; so that $\pi_1(Y)$ is naturally isomorphic to a free product of the set of groups $\{\pi_1(B_\alpha)\}$ with amalgamated subgroup $\pi_1(C)$. Suppose that the induced map $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is onto. In addition, suppose that each component of A_α is trivial fundamental group. Then if $\bigcap_{\alpha} A_\alpha$ is not connected, there is a binding tie.

Proof: The important extra hypothesis is that each component of A_α has trivial fundamental group. This says that each monochromatic loop in X is null homotopic. Thus if $\bigcap_{\alpha} A_\alpha$ is connected the $\pi_1(X)$ is trivial. Thus the only interesting case is when $\bigcap_{\alpha} A_\alpha$ is not connected.

The proof of this lemma is similar to the proof of Theorem 1.

First we claim that there is a tie Q in X , whose path class η is such that $f_*(\eta) = 1$, the trivial element of $\pi_1(Y)$.

Now we know that X is connected and locally path connected which says that X is path connected. Then there is a tie P in X such that the left end point of the path P is the base point of X . Let ϑ be the path class represented by P . Since we have $\bigcap_{\gamma \in J} B_\gamma$ is a single point, $f_*(\vartheta)$ is a loop in Y .

We know that $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is onto. Then choose a loop $\lambda \in \pi_1(X)$ such that $f_*(\vartheta) = f_*(\lambda)$. Let L be the loop which is represented by λ . Then $P^{-1}L$ is the required tie Q , such that $f_*(\vartheta^{-1}\lambda) = 1$, since f_* is a homomorphism.

Now by the previous lemma, we have

$$Q = Q_1 Q_2 \dots Q_n.$$

Now observe that the path $Q = Q_1 Q_2 \dots Q_n$, representing $\eta = \eta_1 \eta_2 \dots \eta_n$, has only ties for all i , since monochromatic loops are trivial. Also Q_i and Q_{i+1} have different colors, if not we can call (Q_i, Q_{i+1}) one tie and have a decomposition of Q into fewer components.

Then we have the following:

$f_*(\eta) = 1$ and that $f_*(\eta_i)$ and $f_*(\eta_{i+1})$ lie in different factors $\pi_1(B_\alpha)$. Then we have

$$1 = f_*(\eta) = f_*(\eta_1)f_*(\eta_2) \cdots f_*(\eta_n).$$

If none of the $f_*(\eta_i)$ lie in the amalgamated subgroup $\pi_1(C)$, then such a product where adjacent terms lie in different factors of the amalgamated product cannot be trivial. Hence there exists an i such that $f_*(\eta_i)$ belongs to $\pi_1(C)$. This is the same as saying that fQ_i is homotopic to a loop in $C = \bigcap_{\gamma \in J} B_\gamma$. But then we know that $\pi_1(C) \rightarrow \pi_1(B_\alpha)$ is an embedding, hence fQ_i is homotopic in B_α to a loop in C . Thus we have that Q_i is a binding tie.

Wagner's Theorem: Let $\phi : \Gamma \rightarrow (*_\alpha \Pi_\alpha)_\Sigma$ be a homomorphism of free groups onto a free product of groups $\{\Pi_\alpha\}$ with amalgamated subgroup Σ . Then there is an element x of some free basis of Γ , and an index α such that $\phi(x) \in \Pi_\alpha$.

Proof: Let C_α be a complex with $\pi_1(C_\alpha) = \Pi_\alpha$. Also, let D be a 2 dimensional CW complex such that $\pi_1(C_\alpha) = \Sigma$ and D has only one 0 cell. Then we can find maps $g_\alpha : D \rightarrow C_\alpha$ which induces the inclusion $\Sigma \subset \Pi_\alpha$. Now, let B_α be the mapping cylinder of g_α . These mapping cylinders intersect exactly in D ; their union will be Y . We can identify $\pi_1(Y)$ with $(*_\alpha \Pi_\alpha)_\Sigma$.

We then construct a one dimensional complex X whose π_1 is Γ , and a J -ad structure $(X, \{A_\alpha\})$ and a map $f : (X, \{A_\alpha\}) \rightarrow (Y, \{B_\alpha\})$ inducing ϕ , just as in the proof of **Grushko's Theorem**. Again as in the previous proof we see that $\bigcap_\alpha A_\alpha$ is a discrete set of points.

By the Lemma 1, we can assume that X does not have any binding ties with respect to f , and that the components of $\bigcap_\alpha A_\alpha$ are trees which will say that the fundamental group of the components is trivial. Now by the previous lemma, we observe that one of the A_α 's have non-trivial fundamental group. Otherwise we will have a binding tie in X which is not possible. Also by Van Kampen, $\pi_1(A_\alpha)$ is a free factor of $\pi_1(X)$ conjugated by some path class in X .

Let z be an element of the free basis of one of the non-trivial fundamental groups of A_α . Let ρ be a path in X connecting the base point of A_α to the base point of X . Then $\rho^{-1}z\rho$ belong to a basis of $\pi_1(X)$.

Now since by construction D has only one 0 cell, $f_*(\rho) \in \pi_1(Y)$. Since, $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is onto, there exists a $\sigma \in \pi_1(X)$ such that $f_*(\sigma) = f_*(\rho)$.

Now $x = \sigma(\rho^{-1}z\rho)\sigma^{-1}$ belongs to a basis of $\pi_1(X)$, and $f_*(x) = f_*(z) \in \pi_1(B_\alpha)$, since z belongs to the fundamental group of some A_α . Thus x is a element such that $f_*(x) \in B_\alpha$ since $z \in \pi_1(A_\alpha)$ and f is a map of J -ads.

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