

IISER Pune  
MTH401 Project Report  
**Introduction to Surfaces**

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# Summary

To study the following: classification and basic properties of compact and non-compact surfaces, surgery on manifolds, degrees of maps, homotopy equivalences and isotopies, shortest loops and arcs on surfaces with a Riemannian metric

# Contents

<b>1</b>	<b>Introduction to Surfaces</b>	<b>4</b>
1.1	Classification of compact surfaces . . . . .	4
1.1.1	Handle decomposition of a manifold . . . . .	4
1.1.2	Existence of handle decomposition for a manifold . . . . .	5
1.1.3	Reduction of number of handles . . . . .	5
1.2	Basic properties of surfaces . . . . .	9
1.3	Non-compact surfaces . . . . .	12
1.4	Simple closed curves on surfaces . . . . .	15
1.5	Powers in topology and algebra . . . . .	20
1.6	Simple closed curves on the torus and the Klein bottle . . . . .	22
<b>2</b>	<b>Homotopy of Maps on Surfaces</b>	<b>25</b>
2.1	Basics . . . . .	25
2.2	Degree of a map . . . . .	30
2.3	Homotopy Equivalences . . . . .	39
2.4	Generalizations of the Previous Section . . . . .	42
<b>3</b>	<b>Appendix - Group Theory</b>	<b>47</b>
3.1	Amalgamated Free Products . . . . .	47

3.2 Grushko's Theorem . . . . .	48
<b>References</b>	<b>51</b>

# Chapter 1

## Introduction to Surfaces

### Classification of 1-manifolds

- Any 1-manifold is homeomorphic to any one of  $\mathbb{S}^1$ ,  $[0, 1]$ ,  $[0, 1)$  or  $(0, 1)$ .
- The classification of smooth compact surfaces was given by Hirsch and uses Morse theory.

### 1.1 Classification of compact surfaces

#### 1.1.1 Handle decomposition of a manifold

Let  $X$  be an  $n$ -manifold. Attaching a  $k$ -handle to  $X$  means choosing an embedding  $f : \partial D^k \times D^{n-k} \rightarrow \partial X$ , and taking the natural identification space from the disjoint union of  $X$  and  $D^k \times D^{n-k}$ .

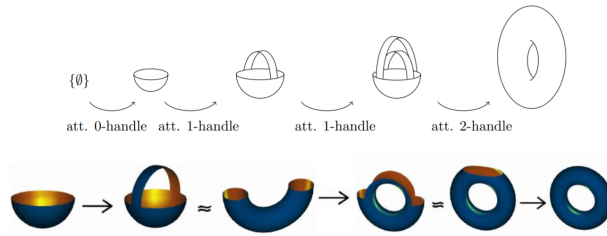


Figure 1.1: Handle decomposition of a torus

### 1.1.2 Existence of handle decomposition for a manifold

Let  $M$  be a compact triangulated surface. We construct the second derived barycentric subdivision of this triangulation of  $M$ , by adding the barycentre of each simplex, subdividing and repeating the process.

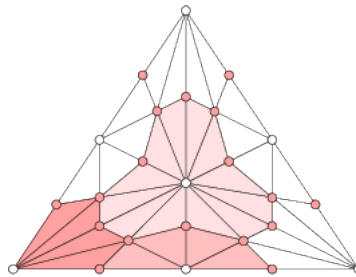


Figure 1.2: Second barycentric subdivision with stars indicated by shading

The stars of the original vertices are 0-handles, those of the original edges are 1-handles and those of the original 2-simplices are 2-handles.

### 1.1.3 Reduction of number of handles

The cores of the 0-handles and 1-handles of  $M$  form a graph in  $M$ . Let  $T$  be a maximal tree in this graph. The union of the corresponding handles of  $T$  includes all the 0-handles and some of the 1-handles, and is homeomorphic to a 2-disc. We define this union to be a 0-handle. We take this 0-handle along with the remaining 1-handles and all of the original 2-handles as a new handle decomposition. This decomposition has only one 0-handle.

If  $M$  is closed, the dual handle decomposition associated to a given handle decomposition is defined by taking the original  $k$ -handles to be  $(2 - k)$ -handles.

Repeating the process in the previous paragraph gives exactly one 0-handle in the dual decomposition, which is a 2-handle in the handle decomposition. Thus we have obtained a handle decomposition of  $M$  which has exactly one 0- and 2-handle each. This is the decomposition with the minimum possible number of handles.

If  $M$  is not closed, the above process gives a handle-decomposition with no 2-handles.

For a 1-handle attached to a 0-handle, the handle slide operation involves sliding one end of a 1-handle around the boundary of the union of the 0-handle with the other 1-handles.

There are exactly two possibilities for a single 1-handle, which are called twisted and untwisted. They are as follows:

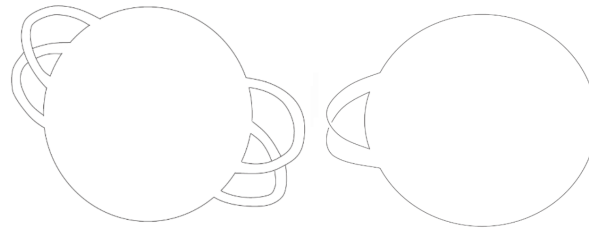


Figure 1.3: Untwisted (left) and twisted (right) handles

A surface  $M$  is the connected sum of two surfaces  $M_1$  and  $M_2$  if there is a separating simple closed curve  $C$  in  $M$  such that cutting  $M$  along  $C$  and attaching a disc to each copy of  $C$  in the two cut-out manifolds gives  $M_1$  and  $M_2$ .

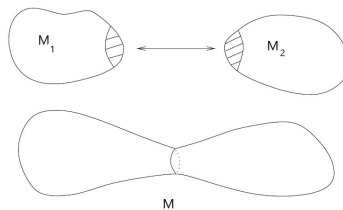


Figure 1.4: Connected sum of two manifolds

**Theorem 1.1.1** (Classification of compact surfaces). *Let  $M$  be a compact sur-*

face.

1. If  $M$  is closed, connected and orientable, then it is homeomorphic to the sphere  $S^2$  or to a connected sum of tori. The number of tori is an invariant of  $M$  called the genus.
2. If  $M$  is closed, connected and non-orientable, then it is homeomorphic to a connected sum of projective planes. The number of summands is an invariant of  $M$  called the non-orientable genus.
3. If  $M$  is not closed, then it is obtained from a closed surface by removing the interiors of a finite number of disjoint discs.

*Proof.* 1. Say the handle decomposition of  $M$  has exactly one 0-handle ( $h_0$ ) and one 2-handle. If it has no 1-handles, it is homeomorphic to  $S^2$ . Suppose  $M$  has a 1-handle, called  $k_1$ . But the boundary of the union of  $h_0$  and all the 1-handles must be connected, since  $M$  has only one 2-handle. This is not possible unless  $M$  has another 1-handle, called  $k_2$ , such that  $k_1$  and  $k_2$  are linked. From fig 1.1, we see that  $h_0 \cup k_1 \cup k_2$  is homeomorphic to the punctured torus. Thus  $M$  is a connected sum of tori. The number of tori is equal to half the rank of  $H_1(M)$  and is invariant.

2. Again, let the handle decomposition of  $M$  have exactly one 0-handle ( $h_0$ ) and one 2-handle. As it is non-orientable, it must have a twisted 1-handle called  $k_1$ .  $h_0 \cup k_1$  is homeomorphic to the Mobius band, so  $M$  is the connected sum of  $\mathbb{RP}^2$  and some closed surface. Thus  $M$  is the connected sum of projective planes. The number of summands is equal to the rank of  $H_1(M, \mathbb{Z}_2)$  and is invariant.

3. Let  $\overline{M}$  be obtained from  $M$  by gluing a disc to each boundary component. Hence it is a closed surface, and we can apply the previous.

□

Let  $T$  denote the torus,  $K$  denote the Klein bottle and  $\#$  denote a connected sum.

**Lemma 1.1.2.** *If  $N$  is a non-orientable surface, then  $N\#T$  and  $N\#K$  are homeomorphic.*

*Proof.* Both  $T$  and  $K$  are the union of two annuli  $A$  and  $B$  glued differently along their boundary components (see fig 1.5 and 1.6). We remove a disc from  $N$  and from  $B$ . If  $N'$  denotes  $N\#B$ , then  $N'$  is homeomorphic to  $N$  with the interiors of two discs removed (one corresponding to the hole of the annulus and the other corresponding to the disc cut out while making the connected sum). When making the identification space of  $N'$  and  $A$ , the maps gluing  $\partial N'$

and  $\partial A$  do not have the same orientation. But sliding one end of  $A$  around an orientation reversing loop in  $N$  makes  $N\#T$  and  $N\#K$  homeomorphic.  $\square$

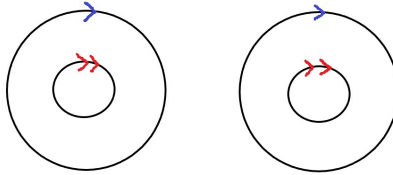


Figure 1.5: Identifications on annuli for gluing them to form a torus

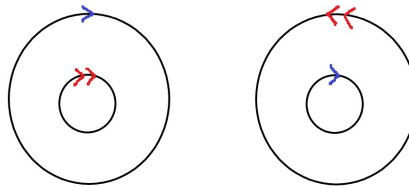


Figure 1.6: Identifications on annuli for gluing them to form a Klein bottle

**Lemma 1.1.3.** *Let  $C$  be a simple closed curve on the torus  $T$ . Then either of the following holds:*

1.  $C$  bounds a disc in  $M$ .
2. There is a homeomorphism  $h : (T, C) \rightarrow (T, S)$ , where  $S$  is the simple closed curve  $S^1 \times \{x\}$  for some  $x \in S^1$ .

*Proof.* We cut  $T$  along  $C$ . The resulting surface is named  $M$ .  $T$  is orientable, so  $C$  must be two-sided; hence,  $M$  has 2 boundary components, both homeomorphic to  $C$ .

Case I:  $M$  is disconnected.

The Euler characteristic of a compact, connected surface with non-empty boundary is  $\leq 1$ . So either both components of  $M$  have Euler characteristic 0, or one of them has value 1. But if both components have Euler characteristic 0, they must both be homeomorphic to an annulus, which is not possible as the total number of boundary components will become equal to 4. Hence, one of the components has Euler characteristic of value 1, and is homeomorphic to a disc.

Case II:  $M$  is connected.

$\chi(T) = 0 \implies \chi(M) = 0$ . Also,  $M$  is connected and orientable. Hence, it is homeomorphic to the annulus  $S^1 \times I$ , which gives the required homeomorphism.  $\square$

## 1.2 Basic properties of surfaces

**Definition 1.2.1.** A closed manifold is a manifold without boundary that is compact.

**Lemma 1.2.2.** Let  $M$  be a surface and let  $M'$  be a subsurface of  $M$  which is itself a closed surface. Then  $M'$  is a union of components of  $M$ .

*Proof.*  $M'$  is a closed surface, so it is compact. Hence,  $M'$  is a compact subspace of a Hausdorff space  $M$ , and is closed in  $M$ . But  $M'$  has no boundary, so it is open in  $M$ . Hence  $M$  is not connected (unless  $M'$  is the whole of  $M$ ), and  $M'$  is a union of components of  $M$ .  $\square$

**Lemma 1.2.3.** 1. A connected non-closed surface is homotopy equivalent to a 1-complex.

2. The fundamental group of a connected non-closed surface is a free group.

*Proof.* Case I:  $M$  is compact.

We choose some triangulation of  $M$ . Since  $M$  is non-closed, it has a non-empty boundary, so there is a 2-simplex  $\sigma$  with a free edge  $e$ . Let  $M_1$  be the 2-complex obtained from  $M$  by collapsing  $\sigma$  from  $e$  (ie, deleting the interiors of  $\sigma$  and  $e$ ).  $M_1$  is a deformation retract of  $M$ , so they are homotopy equivalent.

If possible, say none of the remaining 2-simplices has a free edge. Then their union forms a closed  $M'$  in  $M$ . So by lemma 1.2.2  $M'$  is a union of components of  $M$ . But as  $M$  is connected,  $M'$  must be the whole of  $M$ , and thus,  $M$  is closed, which is a contradiction. So there must always be a free edge on at least one of the 2-simplices.

We collapse all the remaining 2-simplices along their free edges till none are left. This gives us the required 1-complex.

Case II:  $M$  is not compact.

Again we choose a triangulation of  $M$ . However as  $M$  is not compact, so it need not have a boundary and, hence, a free edge. In such a situation we consider the dual of the triangulation and take the maximal tree  $T_d$  of the dual 1-skeleton.  $T_d$  contracts to a vertex and gives a deformation retraction of  $M$  to a 1-dimensional subcomplex of the chosen triangulation which consists of those edges which are not dual to the edges of  $T_d$ . (The deformation retraction is described in the next lemma.) This proves part(1).

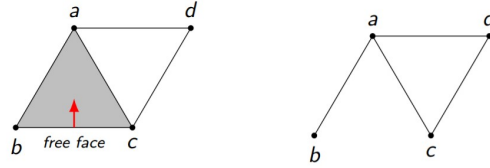


Figure 1.7: Collapsing a 2-simplex along a free edge

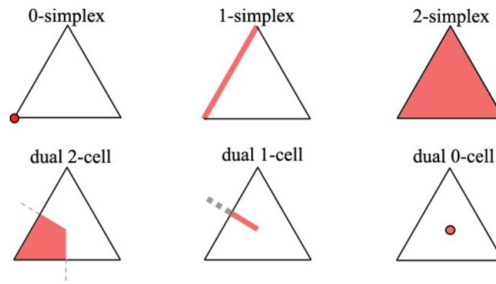


Figure 1.8: Duals of simplices

The fundamental group of any 1-complex is a free group, so part (2) follows.  $\square$

**Remark 1.2.4.** *Part (1) generalizes to manifolds of any dimension, ie, a connected non-closed  $n$ -manifold is homotopy equivalent to an  $(n - 1)$ -complex.*

**Lemma 1.2.5.** *Let  $X$  be a space which is the union of an ascending sequence of countably many subspaces  $X_n$ ,  $n \geq 1$ , such that  $X_{n-1}$  lies in the interior (in  $X$ ) of  $X_n$ . If  $X_n$  deformation retracts to  $X_{n-1} \forall n \geq 1$ , then  $X$  deformation retracts to  $X_0$ .*

*Proof.* Let  $F_n : X_n \times I \rightarrow X_n$  be the given deformation retraction from  $X_n$  to  $X_{n-1}$ , and  $f_n := F_n(X_n, 1)$  be the retraction obtained. Let  $g_n := f_n \times 1_I$ .  $\forall n \geq 1$ , define  $H_n := F_n \circ g_{n+1} \circ g_{n+2} \circ \dots : X \times I \rightarrow X_n$ . Next  $G_n : X \times [\frac{1}{n+1}, \frac{1}{n}] \rightarrow X$  using  $H_n$  by choosing a homeomorphism between  $[0, 1]$  and  $[\frac{1}{n+1}, \frac{1}{n}]$  such that 0 corresponds to  $\frac{1}{n+1}$ . Finally  $G : X \times I \rightarrow X$  is defined to be identity at time

0, and equal to  $G_n$  at time  $[\frac{1}{n+1}, \frac{1}{n}]$ ,  $\forall n \geq 1$ .

Each  $X_{n-1}$  lies in the interior of  $X_n$ , so the  $G_n$ 's are equal at points where the domains overlap; hence  $G$  is well-defined. Each  $H_n$  is a composition of continuous functions, as  $F_n$  and the  $g_i$ 's are all continuous. So  $G$  is also continuous, and is the required deformation retraction from  $X$  to  $X_0$ .  $\square$

**Corollary 1.2.6.** *If  $G$  is the fundamental group of a closed surface  $M$ , then any subgroup of  $G$  of infinite index is free.*

*Proof.* Let  $H \leq G$  such that  $H$  has infinite index. Then  $H$  corresponds to an infinite sheeted cover of  $M$ . Such a cover cannot be compact, and hence is non-closed. So by lemma 1.2.3, its fundamental group  $H$  is free.  $\square$

**Lemma 1.2.7.** *Let  $M$  be a connected surface and let  $C$  be a compact union of components of  $\partial M$  which is null homologous in  $M$ , using  $\mathbb{Z}_2$ -coefficients. Then  $M$  is compact and  $\partial M$  consists only of  $C$ .*

*Proof.* We choose a triangulation of  $M$  and take the simplicial homology using  $\mathbb{Z}_2$  coefficients. Then  $C$  is identified with the 1-chain consisting of one of each of the 1-simplices of  $C$ .  $C$  is null homologous in  $M$ , so there is a 2-chain  $\sigma$  (a finite subset of the set of 2-simplices in the triangulation of  $M$ ) such that  $\partial\sigma$  is equal to  $C$ .  $C$  lies entirely in  $M$ , so  $\sigma$  is open in  $M$ . But  $\sigma$  is also a union of finitely many 2-simplices, so it is compact and hence a closed subset of  $M$ .  $M$  is connected, so this is only possible if  $\sigma$  is the whole of  $M$ . Hence,  $M$  is compact, and  $\partial M$  is equal to  $C$ .  $\square$

**Remark 1.2.8.** *The above lemma also holds for higher dimensions, with the same proof.*

**Lemma 1.2.9.** *Let  $M$  be a connected surface, and let  $C$  be a boundary component of  $M$ , with the inclusion map denoted by  $i$ . Then either  $i_*: \pi_1(C) \rightarrow \pi_1(M)$  is injective, or  $M$  is the disc  $D^2$ .*

*Proof.* If  $C$  is homeomorphic to  $\mathbb{R}$ , then it is of the form of an open interval, so  $M$  is the disc  $D^2$ .

Let  $C$  be a circle such that  $i_*$  is not injective.  $\pi_1(C)$  is infinite cyclic, so the kernel of  $i_*$  has finite index in  $\pi_1(C)$ . Hence some finite cover  $\tilde{C}$  of  $C$  lifts to the universal cover  $\tilde{M}$  of  $M$ . But  $\tilde{M}$  is simply connected, so  $\tilde{C}$  is nullhomotopic, and hence null homologous in  $\tilde{M}$ . Then from lemma 1.2.7, we can conclude that  $\tilde{M}$  is compact and that  $\partial\tilde{M}$  consists only of  $\tilde{C}$ . From the classification of compact surfaces, we get that  $\tilde{M}$  is homeomorphic to the disc  $D^2$ . If  $g \in \pi_1(M)$ , then  $g$  acts on  $D^2$  by a covering translation. The Brouwer's fixed point theorem mandates that  $g$  must have a fixed point, and hence acts trivially. So  $\pi_1(M)$  is trivial and  $\tilde{M}$  is equal to  $M$ . Hence,  $M$  is also homeomorphic to  $D^2$ .  $\square$

**Corollary 1.2.10.** *Let  $M$  be a connected surface and let  $C$  be a simple closed curve in the interior of  $M$ . Let  $i$  be the inclusion map from  $C$  into  $M$ . Then exactly one of the following hold:*

1.  $i_*: \pi_1(C) \rightarrow \pi_1(M)$  is injective.
2.  $C$  bounds a 2-disc in  $M$ .
3.  $M$  is the projective plane  $P^2$  and  $C$  is a one-sided curve.

*Proof. Case I:*  $C$  is two-sided.

Let  $M_1$  be the surface obtained on cutting  $M$  along  $C$ , and let  $C$  and  $C'$  be the two copies of  $C$  in  $\partial M_1$ . If the inclusions of  $\pi_1(C)$  and  $\pi_1(C')$  into  $\pi_1(M)$  are injective, then so is  $i_*$ . If not, then one of these inclusion maps is not injective, and as  $C$  and  $C'$  are boundary components of  $M_1$ , so by lemma 1.2.9 one component of  $M_1$  is a disc.

*Case II:*  $C$  is one-sided.

$C$  has a neighbourhood homeomorphic to a Mobius band; let its boundary be  $C'$ . Then again from lemma 1.2.9, either  $C'$  bounds a disc  $D^2$  in  $M$ , or the inclusion map from  $\pi_1(C')$  to  $\pi_1(M)$  is injective. If the former is true, then  $M$  is the projective plane  $P^2$  (as  $P^2$  is the union of a Mobius band and a disc); otherwise, if the latter is true, then  $i_*$  is injective.  $\square$

**Lemma 1.2.11.** *If a surface  $M$  has abelian fundamental group  $G$ , then either  $G$  is trivial or it is isomorphic to one of  $\mathbb{Z}_2$ ,  $\mathbb{Z}$  or  $\mathbb{Z} \times \mathbb{Z}$ .*

*Proof. Case I:*  $M$  is not closed.

Then by lemma 1.2.3,  $G$  is free and so it is either trivial or isomorphic to  $\mathbb{Z}$ .

*Case II:*  $M$  is closed.

Then by the classification of compact surfaces,  $M$  is homeomorphic to either of  $S^2$ ,  $P^2$ ,  $T^2$  or to a connected sum of tori or projective planes. A connected sum is constructed by gluing once or twice punctured tori and projective planes along their boundary circles. From lemma 1.2.9, the inclusion map of such a boundary circle has to be injective. Also, the fundamental group of such a boundary circle cannot be equal to the fundamental group of the whole surface. So the fundamental group of  $M$  is a multiple amalgamated free product of non-trivial groups, and cannot be abelian.  $\square$

### 1.3 Non-compact surfaces

**Lemma 1.3.1.** *Let  $M$  be an open simply connected surface. Then  $M$  is homeomorphic to  $\mathbb{R}^2$ .*

*Proof.*  $M$  is paracompact, so there is an ascending sequence of nested compact subsets  $\{X_i\}$  whose union is  $M$ . Let  $N_1$  be a regular neighbourhood of  $X_1$ . Then for some  $j$ ,  $N_1 \subset X_j$ . So we can replace  $X_1, \dots, X_{j-1}$  by  $N_1$  in the original sequence. By repeating this process, we get an ascending sequence of submanifolds  $\{N_i\}$  such that  $N_i \subset \text{Int}(N_{i+1})$  and the union of  $N_i$ 's is  $M$ . We can assume each  $N_i$  is connected; if it is not, we can join its components by 1-handles.

From lemma 3.2.10, a simple closed curve in  $M$  bounds a disc. This disc must be unique, otherwise  $M$  will be the union of two discs and be compact, which is not true. So each boundary component of each  $N_i$  bounds a unique disc in  $M$ . Let  $M_i$  be the union of  $N_i$  and all the discs bounded by the boundary components of  $N_i$ . Then  $M_i$  is a compact submanifold of  $M$ , and by lemma 1.2.2, it has a non-empty boundary. So  $M_i$  is a disc. Thus we obtain an increasing sequence of discs  $\{M_i\}$  in  $M$  such that their union is  $M$  and  $M_i \subset \text{Int}(M_{i+1})$ .

$\mathbb{R}^2$  can also be expressed as the ascending union of discs of radius  $n$  centred at the origin. So we can define the required homeomorphism as taking the  $n^{\text{th}}$  disc in  $M$  to the same in  $\mathbb{R}^2$ .  $\square$

**Definition 1.3.2.** *A connected space  $X$  is called aspherical if  $\pi_i(X) = 0 \forall i \geq 2$ . For CW-complexes, this is equivalent to the condition that the universal cover of  $X$  is contractible.*

**Corollary 1.3.3.** *All connected surfaces are aspherical except for the sphere  $S^2$  and the projective plane  $P^2$ .*

*Proof.* The boundary  $\partial M$  of a manifold  $M$  has a collar neighbourhood  $X$  such that both  $M$  and  $\text{Int}(M)$  deformation retract to the closure of  $M - X$ . So  $M$  is homotopy equivalent to its interior and it suffices to prove the corollary for manifolds without boundary.

Let  $M$  be a connected surface without boundary and let  $\tilde{M}$  be its universal cover. Then either  $\tilde{M}$  is non-compact or it is closed. If  $\tilde{M}$  is non-compact, then by lemma 1.3.1 it is homeomorphic to  $\mathbb{R}^2$ . So  $M$  is aspherical as its universal cover is contractible.

If  $\tilde{M}$  is closed, then so is  $M$ . If  $\pi_1(M)$  is infinite, then  $\tilde{M}$  will be infinite sheeted and non-compact, which is a contradiction. Hence the fundamental group of  $M$  must be finite, and the classification of compact surfaces implies that  $M$  must be homeomorphic to either  $S^2$  or  $P^2$ .  $\square$

**Corollary 1.3.4.** *1. The fundamental group of a connected surface  $M$  is torsion-free, except when  $M$  is homeomorphic to the projective plane  $P^2$ .*

*2. The fundamental group of a closed connected surface  $M$  is not free, except when  $M$  is homeomorphic to the 2-sphere  $S^2$ .*

*3. If the fundamental group of a closed connected surface  $F$  contains a subgroup isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ , then  $F$  must be the torus or the Klein bottle.*

- Proof.* 1. The fundamental group of an aspherical surface is torsion-free. From corollary 1.3.3, the only other possibilities for  $M$  are for it to be homeomorphic to  $S^2$  (which is also torsion-free), or to  $P^2$ , which is not torsion-free.
2. If  $M$  is homeomorphic to  $S^2$ , then its fundamental group is free; if it is homeomorphic to  $P^2$ , its fundamental group is not free. The remaining possibility is for  $M$  to be aspherical. If possible, let  $\pi_1(M)$  be free. Then it is homotopy equivalent to a 1-complex  $K$ . But  $H_2(M, \mathbb{Z}_2)$  is isomorphic to  $\mathbb{Z}_2$ , while  $H_2(K, \mathbb{Z}_2)$  is zero, which is a contradiction.
3. Let  $F'$  be a cover of  $F$  with fundamental group isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ . By the classification of closed compact surfaces,  $F'$  is homeomorphic to a torus, and hence  $\chi(F') = 0$ . So  $F$  is also closed and has  $\chi(F) = 0$ . The only possibilities for  $F$  are for it to be homeomorphic to a torus or a Klein bottle.

□

**Lemma 1.3.5.** *Let  $M$  be an open connected surface with finitely generated fundamental group. Then  $M$  is homeomorphic to the interior of a compact surface  $N$ .*

*Proof.* Let  $X$  be a regular neighbourhood of the generator loops of  $M$ . Then  $X$  is compact and the map from  $\pi_1(X) \rightarrow \pi_1(M)$  is surjective. Let  $S$  be a component of  $\partial X$ . If  $S$  bounds a disc in  $M$ , we add this disc to  $X$ , and continue the process till either no component of  $\partial X$  bounds a disc or  $X$  itself is a disc. If  $X$  is a disc, then  $\pi_1(X) \rightarrow \pi_1(M)$  is injective, and hence an isomorphism. If no component of  $\partial X$  bounds a disc, lemma 3.2.9 says that if  $C$  is a component of  $\partial X$ , then  $\pi_1(C) \rightarrow \pi_1(M)$  is injective. Hence  $\pi_1(M)$  has the structure of the fundamental group of a graph  $\tau$  of groups, with  $\pi_1(X)$  as a vertex group and each edge group being  $\pi_1(C)$  for some component  $C$  of  $\partial X$ . The map  $\pi_1(X) \rightarrow \pi_1(M)$  is also injective, and thus an isomorphism.  $\tau$  is a tree, so if  $Y$  is the closure of some component of  $M - X$ , then  $Y \cap X$  is a single circle  $C$ . So  $\pi_1(C) \rightarrow \pi_1(Y)$  is an isomorphism. If  $\bar{Y}$  is the identification space of  $Y$  and some disc along their boundaries, then it is an open simply connected surface and hence by lemma 1.3.1 is homeomorphic to  $\mathbb{R}^2$ . So  $Y$  is homeomorphic to the half-open annulus  $S^1 \times [0, \infty)$ . Thus,  $M$  is homeomorphic to the interior of a surface obtained by adding a compact annulus to each boundary component of  $X$ . If we denote this surface by  $N$ , then clearly  $N$  is compact and homeomorphic to  $X$ . □

**Corollary 1.3.6.** *Let  $M$  be an open connected surface with finitely generated fundamental group. Then the interior of  $M$  contains a compact surface  $X$  whose inclusion in  $M$  induces an isomorphism of fundamental groups.*

*Proof.* A part of the proof of the previous lemma. □

## 1.4 Simple closed curves on surfaces

**Lemma 1.4.1.** *Let  $M$  be a connected surface, and let  $C_1$  and  $C_2$  be distinct boundary components of  $M$  which are circles and are homotopic in  $M$ . Then  $M$  is homeomorphic to the annulus  $S^1 \times I$ .*

*Proof.* Let  $M'$  be a cover of  $M$  such that  $\pi_1(M') = \pi_1(C_1)$ . The lifts of  $C_1$  and  $C_2$  to this cover are denoted as  $C'_1$  and  $C'_2$ ; they are boundary components of  $M'$ .  $C_1$  and  $C_2$  are homotopic in  $M$ , and this homotopy lifts to  $M'$ ; so  $C'_1$  and  $C'_2$  are homotopic in  $M'$  and also homologous. From lemma 1.2.7, we can conclude that  $M'$  is compact and  $\partial M'$  consists of only  $C'_1$  and  $C'_2$ .

$\pi_1(M')$  is equal to  $\pi_1(C_1)$ , so it is infinite cyclic and by the classification of compact surfaces,  $M'$  is homeomorphic to  $S^1 \times I$ . Further, if  $p: M' \rightarrow M$  is the covering map, then  $p^{-1}(C_1) = C_1$  only, so the degree of the covering map is 1 and hence  $M$  is homeomorphic to  $M'$ .  $\square$

**Definition 1.4.2.** *Two circles  $C_1$  and  $C_2$  embedded in a surface  $M$  are parallel if they are disjoint and there is an annulus  $A$  embedded in  $M$  such that  $\partial A = C_1 \cup C_2$ .*

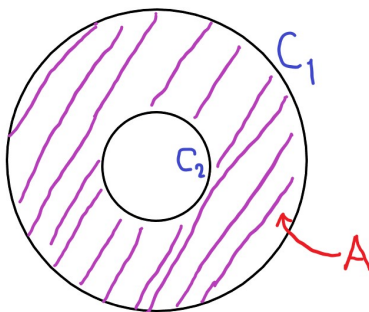


Figure 1.9: Parallel circles

**Corollary 1.4.3.** *Let  $M$  be a surface and let  $C_1$  and  $C_2$  be disjoint homotopic circles embedded in  $M$ . Then either  $C_1$  and  $C_2$  bound discs in  $M$  or they are parallel.*

*Proof.* If possible, say  $C_1$  is one sided in  $M$ . Let  $M'$  be the cover of  $M$  with fundamental group  $i_*\pi_1(C_1)$ , and let  $C'_1$  be the lift of  $C_1$  to  $M'$ . Then by lemma 1.3.5, the interior of  $M'$  is homeomorphic to an open Mobius band, with  $C'_1$  as the core. If  $C'_2$  is the lift of  $C_2$  to  $M'$  then it is homotopic to  $C'_1$ , which is not possible as they must be disjoint, but  $M' - C'_1$  contains no loop homotopic to  $C'_1$ .

If  $C_1$  and  $C_2$  are nullhomotopic in  $M$ , then  $i_*$  cannot be injective, so by corollary 1.2.10 they bound discs in  $M$ .

Say neither  $C_1$  nor  $C_2$  bound discs in  $M$ . We cut  $M$  along  $C_1$  and add discs  $D'_1$  and  $D''_1$  to the two copies of  $C_1$  in the boundary of the cut surface. Let  $X$  be the component of the cut surface which contains  $C_2$ , and  $Y$  be the other component, if any. Then we can define the 2-complex  $K$  as the union of  $X$  and  $Y$  (if it exists), or  $X$  with  $D'_1$  and  $D''_1$  identified, if  $X$  is the only component. The natural map  $\pi_1(X) \rightarrow \pi_1(K)$  is injective, and as  $C_1$  and  $C_2$  are homotopic in  $M$ ,  $C_2$  must be nullhomotopic in  $K$  and also in  $X$ . Hence  $\pi_1(C_2) \rightarrow \pi_1(X)$  is not injective, and by lemma 1.2.9,  $C_2$  bounds a disc  $D_2$  in  $X$ . But  $C_2$  does not bound a disc in  $M$  by assumption, so  $D_2$  contains  $D'_1$  or  $D''_1$  or both. If  $D_2$  contains both  $D'_1$  and  $D''_1$ , then taking  $D_2 - (D'_1 \cup D''_1)$  and gluing  $C'_1$  to  $C''_1$  gives a torus; hence  $C_1$  is non-separating in  $M$  while  $C_2$  is separating, which contradicts the hypothesis that they are homotopic. So,  $D_2$  contains one of  $D'_1$  and  $D''_1$ , and  $C_1$  and  $C_2$  are parallel.  $\square$

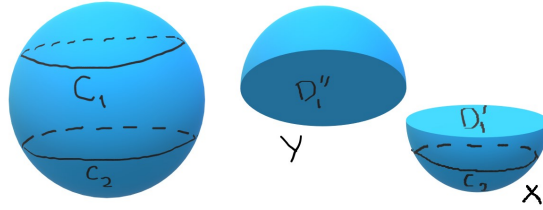


Figure 1.10: Cutting along  $C_1$  to get disjoint  $X$  and  $Y$

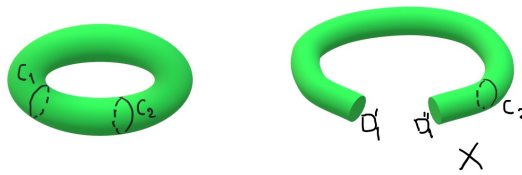


Figure 1.11: Cutting along  $C_1$  to get connected  $X$

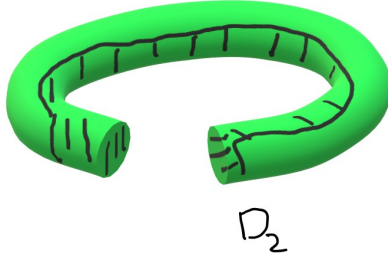


Figure 1.12:  $D_2$  containing both  $D'_1$  and  $D''_1$

**Definition 1.4.4.**  $H \leq G$  is called a free factor of  $G$  if there is a  $K \leq G$  such that  $G = H * K$ .

**Lemma 1.4.5.** Let  $M$  be a connected surface with a boundary component  $C$  which is homeomorphic to a circle. Then either  $M$  is compact and has boundary exactly  $C$ , or the natural map  $\pi_1(C) \rightarrow \pi_1(M)$  is injective and has image which is a free factor of  $\pi_1(M)$ .

*Proof.* Case I:  $M$  has a boundary component  $S$  other than  $C$ . Let  $\lambda$  be a simple path from  $C$  to  $S$ , and let  $N$  be a regular neighbourhood of  $C \cup \lambda$ .  $N$  is homeomorphic to the annulus, and its frontier has only an arc. So by the van Kampen theorem the image of the map  $\pi_1(N) \rightarrow \pi_1(M)$  is a free factor of  $\pi_1(M)$ . Also,  $\pi_1(C) \rightarrow \pi_1(N)$  is an isomorphism, so we conclude that  $\pi_1(C) \rightarrow \pi_1(M)$  is injective and its image is a free factor of  $\pi_1(M)$ .

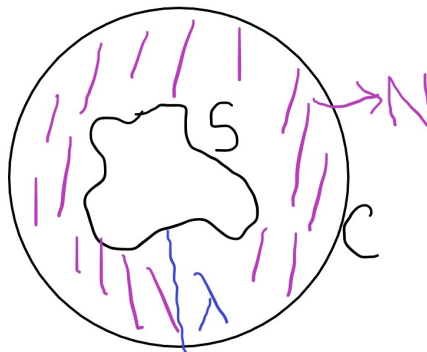


Figure 1.13:

Case II:  $M$  is not compact.

Again we choose a simple path  $\lambda$  in  $M$  from  $C$  to infinity. Then  $\lambda$  is a proper embedding of  $[0, 1)$  in  $M$ , with  $\lambda(0)$  on  $C$ . Taking  $N$  to be a regular neighbourhood of  $C \cup \lambda$ , we can use the previous process, with the modification that the frontier of  $N$  is homeomorphic to  $\mathbb{R}$  instead of a compact arc.  $\square$

**Lemma 1.4.6.** *Let  $M$  be a connected surface and let  $C$  be a union of components of the boundary of  $M$ , each homeomorphic to a circle. Then either  $M$  is compact and has boundary exactly  $C$ , or there is a submanifold  $N$  of  $M$  whose boundary contains  $C$ , whose fundamental group is freely generated by the components of  $C$ , and whose frontier in  $M$  consists of simply connected components. The natural map  $\pi_1(N) \rightarrow \pi_1(M)$  is injective and has image which is a free factor of  $\pi_1(M)$ .*

*Proof.* Case I:  $M$  is compact, and  $\partial M$  has a component  $S$  not in  $C$ .

$C$  has finitely many components. Let  $\lambda_1$  be a simple path from  $C_1$  to  $S$ ,  $\lambda_2$  be a simple path from  $C_2$  to  $S \cup \lambda_1$ , and recursively,  $\lambda_{i+1}$  be a simple path from  $C_{i+1}$  to  $S \cup (\cup_{j=1}^i \lambda_j)$ . Let  $N$  be a regular neighbourhood of  $C \cup \{\lambda_i\}$ . Then  $N$  is compact and has a free fundamental group generated by the components of  $C$ . The frontier of  $N$  in  $M$  consists of arcs which are simply connected. So by the van Kampen theorem, the natural map  $\pi_1(N) \rightarrow \pi_1(M)$  is injective and its image is a free factor of  $\pi_1(M)$ .

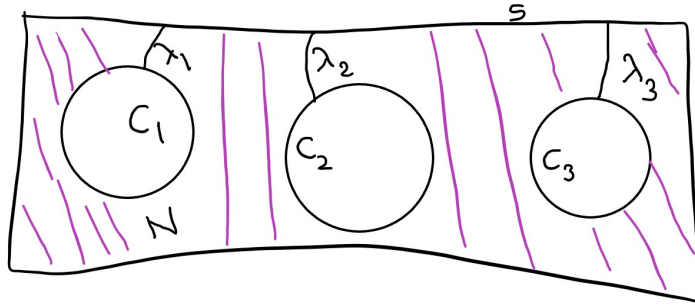


Figure 1.14:

Case II:  $M$  is not compact.

Here  $C$  may have infinitely many, but countable components. We choose a simple path  $\lambda_1$  from  $C$  to  $\infty$  which is properly embedded in  $M$ . Recursively, we choose  $\lambda_{i+1}$  from  $C_{i+1}$  to  $\cup_{j=1}^i \lambda_j$  such that the complex containing  $\cup \lambda_i$  is locally finite. Then we can define the regular neighbourhood  $N$  and argue as previously.  $\square$

**Corollary 1.4.7.** *If  $M$  is a connected surface with finitely generated fundamental group, then the number of components of  $\partial M$  which are homeomorphic to the circle is at most  $1 + \text{rank}(H_1(M))$ .*

*Proof.* Let  $C$  be the union of all but one circle components of  $\partial M$ , and  $N$  be the submanifold described in lemma 1.4.6.  $\pi_1(N)$  is a free factor of  $\pi_1(M)$ , so it must be finitely generated and its rank must be equal to that of  $\pi_1(M)$ . But the rank of  $\pi_1(N)$  is equal to the number of components of  $C$ , while the rank of  $\pi_1(M)$  is equal to that of  $H_1(M)$ .  $\square$

**Theorem 1.4.8.** *Let  $M$  be a closed connected orientable surface. Let  $\Sigma$  be a closed 1-submanifold of  $M$  with  $r$  components such that no component of  $\Sigma$  is nullhomotopic in  $M$  and no two components of  $\Sigma$  are parallel. Then an upper bound for  $r$  is given by the following formulae:*

- 0 if  $M$  is  $S^2$
- 1 if  $M$  is  $T$
- $-3\chi(M)/2$  otherwise.

*Proof.* Case I:  $M$  is  $S^2$ .

Then any circle in  $M$  is nullhomotopic, so  $r = 0$ .

Case II:  $M$  is  $T$ .

According to lemma 1.1.3, cutting  $M$  along some component of  $\Sigma$  gives an annulus. And from corollary 1.4.3, any other nullhomotopic components have to be parallel. So  $r = 1$ .

Case III:  $M$  is any other surface.

Say,  $\Sigma$  is nonempty. Then we can cut  $M$  along the various components of  $\Sigma$  into  $X_1, \dots, X_n$ , each having non-empty boundary, such that no  $X_i$  is a disc, annulus or Mobius strip. Hence,  $\chi(X_i) \leq -1 \forall i$ . From corollary 1.4.7,  $|\partial X_i| \leq 1 + \text{rank}(H_1(X_i))$ . But  $\text{rank}(H_1(X_i)) = 1 - \chi(X_i)$ . So

$$|\partial X_i| \leq 2 - \chi(X_i)$$

$M$  is obtained from the  $X_i$ 's by gluing them in pairs along the  $r$  components of  $\Sigma$ .

$$\begin{aligned} 2r &= \sum_{i=1}^n |\partial X_i| \\ \implies 2r &\leq \sum_{i=1}^n (2 - \chi(X_i)) = 2n - \chi(M) \end{aligned}$$

We know that

$$\begin{aligned} \chi(X_i) &\leq -1 \\ \implies \chi(M) &= \sum_{i=1}^n \chi(X_i) \leq -n \end{aligned}$$

$$\begin{aligned} \implies 2r &\leq 2n - \chi(M) \leq -3\chi(M) \\ \implies r &\leq -3\chi(M)/2 \end{aligned}$$

□

## 1.5 Powers in topology and algebra

Let  $F$  be a closed, connected, oriented  $k$ -manifold with fundamental homology class  $[F]$ . Then  $[F]$  is a generator of  $H_k(F, \mathbb{Z})$ . If  $F$  is PL then  $[F]$  can be represented by triangulating  $F$  and taking the sum of the  $k$ -simplices.

**Definition 1.5.1.**  $a \in G$  is said to be indivisible or primitive if it is not a proper power, ie, if  $b \in G$  is such that  $a = \overline{b^n}$ , then  $n = \pm 1$ .

**Lemma 1.5.2.** Let  $M$  be an orientable  $n$ -manifold and let  $F$  be a closed connected oriented  $(n-1)$ -manifold embedded in  $M$ . Let  $i: F \rightarrow M$  be the inclusion map. Then the class  $i_*[F]$  in  $H_{n-1}(M, \mathbb{Z})$  is either zero or it is indivisible.

*Proof.*  $M$  and  $F$  are orientable, so  $F$  is two-sided in  $M$ .

If  $F$  bounds a compact submanifold in  $M$  then  $i_*[F]$  is clearly zero. Let us suppose that it doesn't bound a compact submanifold. Let  $\lambda$  be an arc in  $M$  that meets  $F$  transversely at one point.

Case I:  $M - F$  is connected.

We extend  $\lambda$  to a circle  $\Lambda$  which meets  $F$  transversely at only one point. Then  $i_*[F]$  has homological intersection number equal to one with  $\Lambda$ , hence it is non-zero and indivisible in  $H_{n-1}(M, \mathbb{Z})$ .

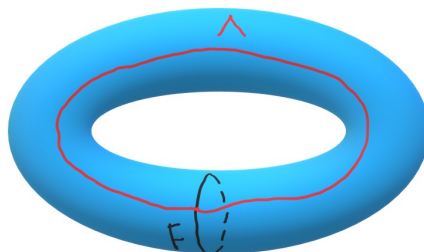


Figure 1.15:

Case II:  $M - F$  is not connected.

$F$  separates  $M$  into two pieces whose closures are called  $X$  and  $Y$ , neither of

which is compact or has boundary equal to  $F$ . Then we can extend  $\lambda$  to a path  $\Lambda$  which intersects  $F$  transversely at exactly one point, such that each end of  $\Lambda$  either lies on  $\partial M$  or goes to infinity in  $M$ . Again (by intersection theory), if  $i_*[F]$  is not indivisible in  $H_{n-1}(M, \mathbb{Z})$ , then  $\Lambda$  would intersect  $F$  more than once, which is a contradiction.  $\square$

**Definition 1.5.3.** The *core* of a manifold is a submanifold whose inclusion into the manifold is a homotopy equivalence.

**Lemma 1.5.4.** Let  $C$  be a simple closed curve on a surface  $M$  (not  $P^2$ ) such that  $C$  represents a non-trivial element  $\alpha$  of  $\pi_1(M)$  and  $\alpha = \beta^k$  for some  $k > 0$ . Then  $k$  must be equal to 1 or 2. Further, if  $k = 2$ , then  $C$  bounds a Mobius band  $X$  embedded in  $M$  whose core represents a conjugate of  $\beta$ .

*Proof.*  $M$  is not  $P^2$  and  $C$  is non-trivial, so it cannot bound a disc. Therefore by corollary 1.2.10,  $i_* : \pi(C) \rightarrow \pi(M)$  is injective, and hence  $\langle \alpha \rangle$  is infinite in  $\pi(M)$ . Let  $M_\beta$  be a cover of  $M$  with fundamental group generated by  $\beta$ . Then  $C$  lifts to a simple closed curve in  $M_\beta$  that represents  $\beta^k$ .

Case I:  $M_\beta$  is orientable.

Then we can directly apply lemma 1.5.2 and conclude that  $k = 1$ .

Case II:  $M_\beta$  is non-orientable.

We consider the orientable double cover of  $M_\beta$  and lemma 1.5.2 to it. Then if  $C$  lifts to this double cover, the lift represents  $\beta^2$ , so  $k = 2$ ; if  $C$  does not lift to the double cover, then  $k = 1$ .

Now we consider the case when  $k = 2$ . Since  $M_\beta$  is non-orientable, so  $\beta$  represents an orientation-reversing loop.  $C = \beta^2$ , so it is orientation-preserving and is two-sided in  $M$ . If  $C'$  denotes the lift of  $C$  to  $M_\beta$ ,  $C'$  is also two-sided, and represents the zero element of  $H_1(M_\beta, \mathbb{Z}_2)$ . Then by lemma 1.2.7,  $C'$  bounds a compact subsurface  $Y$  of  $M_\beta$ .  $\pi_1(M_\beta)$  is infinite cyclic, so is  $\pi_1(Y)$ .  $Y$  has a connected boundary, so by the classification of compact surfaces  $Y$  is a Mobius band.

Let  $p: M_\beta \rightarrow M$  be the covering projection.  $C$  is a closed submanifold of  $M$ , so  $p^{-1}(C)$  is also a closed submanifold of  $M_\beta$ .  $Y$  is compact, so is  $p^{-1}(C) \cap Y$ .  $p^{-1}(C) \cap Y$  obviously contains the boundary circle  $C'$  of  $Y$ , so it consists of  $C'$  and some other circles in the interior of  $Y$ . All these circles cover  $C$ , hence they are two-sided and cannot bound discs. Then by corollary 1.4.3, all these circles are parallel to  $C'$ . Let  $X$  be the closure of  $Y - (p^{-1}(C) \cap Y)$ .  $Int(X)$  is a component of  $M_\beta - p^{-1}(C)$ , so it covers  $M - C$ . Hence  $X$  is a Mobius band bounded by  $C$ . The core of  $X$  is  $\beta$ , so if we take the core of  $p(X)$  to be  $\gamma$ , then  $\gamma^n = \beta$ , where  $n$  is the degree of the covering map. Hence,  $\alpha = \gamma^{2n}$ . But the highest possible power of  $\gamma$  was proven to be 2, so  $n = 1$ . So the core of  $p(X)$  is  $\beta$ , and  $X \rightarrow p(X)$  is a homeomorphism.  $\square$

**Remark 1.5.5.** The previous lemma does not hold when  $M$  is  $P^2$  as then  $\pi_1(M)$  is  $\mathbb{Z}_2$ , so  $k$  can be any odd number as then  $k \equiv 1 \pmod{2}$ .

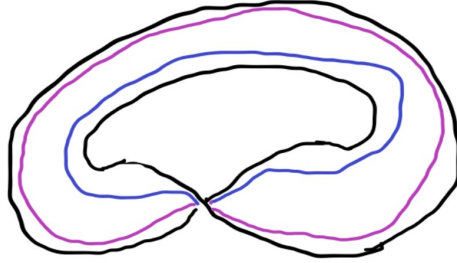


Figure 1.16: Two elements of  $p^{-1}(C) \cap Y$  on  $Y$

## 1.6 Simple closed curves on the torus and the Klein bottle

**Definition 1.6.1.**  $r, s \in \mathbb{Z}$  are said to be co-prime if either  $\gcd(r, s) = 1$ , or if one of them is zero and the other is  $\pm 1$ .

**Lemma 1.6.2.** 1.  $(r, s) \in \mathbb{Z} \times \mathbb{Z} = \pi_1(T)$  can be represented by a simple closed curve iff  $r$  and  $s$  are co-prime.

2. Any element of  $\pi_1(T)$  can be represented by a power of a simple closed curve.

*Proof.* 1. First, let us assume that  $(r, s)$  is a simple closed curve. Then lemma 1.5.2 says that it has to be indivisible, ie,  $r$  and  $s$  must be co-prime. Conversely, let  $r$  and  $s$  be co-prime. Then there exist integers  $p$  and  $q$  such that  $pr + qs = 1$ . Let  $A = \begin{pmatrix} p & q \\ -s & r \end{pmatrix}$ . Then  $A \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and it determines a linear isomorphism  $\varphi_A$  of the Euclidean plane. The torus can be identified as the quotient of the Euclidean plane modulo the integer lattice. So  $\varphi_A$  induces a homeomorphism of  $T$  such that the induced isomorphism on  $\pi_1(T)$  sends  $(r, s)$  to  $(0, 1)$ . Then  $(r, s)$  can be represented by a simple closed curve.

2. The identity on  $\pi_1(T)$  is a simple closed curve that bounds a disc. Any other non-zero element is some multiple of  $(r, s)$  where  $r$  and  $s$  are co-prime. So applying part (1) shows that it must be some power of a simple closed curve.

□

**Lemma 1.6.3.** *There are only four homotopy classes of essential simple closed curves on the Klein bottle.*

*Proof.* Let  $G$  be the group of isometries of the Euclidean plane  $\mathbb{E}^2$  generated by a translation element  $\alpha$  and a glide reflection element  $\beta$ :

$$\alpha(x, y) = (x, y + 1)$$

$$\beta(x, y) = (x + 1, -y)$$

Let  $\sigma_t$  be a reflection of  $\mathbb{E}^2$  along  $y = t$ , and  $\delta$  be the translation sending  $(x, y)$  to  $(x + 1, y)$ . Then,  $\alpha(x, y) = \sigma_{1/2}\sigma_0$  and  $\beta = \sigma_0\delta$ .  $\beta\alpha\beta^{-1} = \alpha^{-1}$ , so  $\alpha$  generates an infinite cyclic subgroup of  $G$ . Also,  $\beta^2$  is a translation, so  $\langle \alpha, \beta^2 \rangle$  is a subgroup of  $G$  isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ . This subgroup is the fundamental group of the orientable double cover of  $K$ .

We now identify  $K$  as the quotient of  $\mathbb{E}^2$  by  $G$ . Let  $\gamma$  be some non-trivial element of  $G$ . If it is a glide reflection, then it has a unique invariant line called the axis; this projects down to a closed curve on  $K$  which represents  $\gamma$ . If  $\gamma$  is a translation, it has an infinite number of parallel invariant lines, each of which is called an axis; as in the previous case, each projects down to a closed curve in  $K$  that represents  $\gamma$ .

Any element of  $G$  is of the form  $\alpha^k\beta^l$ .

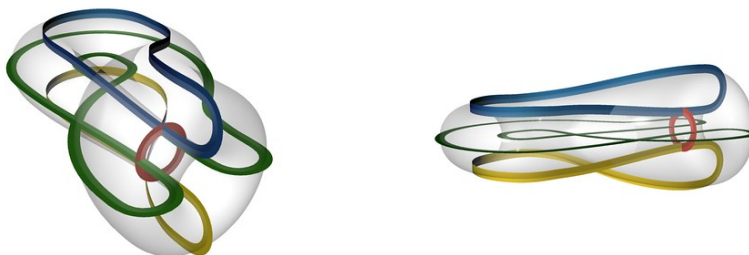
$$\alpha^k = \sigma_{k/2}\sigma_0$$

$$\begin{aligned} \beta^l &= \sigma_0\delta^l \text{ when } l \text{ is odd} \\ &= \delta^l \text{ when } l \text{ is even (ie, it is a translation)} \end{aligned}$$

So if  $l$  is odd,  $\alpha^k\beta^l = \sigma_{k/2}\delta^l$ , which is a glide reflection with axis  $y = \frac{k}{2}$ ,  $k \in \mathbb{Z}$ . If  $l$  is even,  $\alpha^k\beta^l$  is a translation, and will have a horizontal axis iff  $k = 0$ . So an element of  $G$  has a horizontal axis if it is a translation with an even power of  $\beta$ , or is a glide reflection in the line  $y = \frac{k}{2}$  for some integer  $k$ . The line  $y = \frac{k}{2}$  can be sent to  $y = 0$  or  $y = \frac{1}{2}$  using a suitable power of  $\alpha$ , and  $\alpha\beta$  and  $\beta$  are glide reflections on those lines. So any glide reflection of  $G$  (and thus any element of  $G$  with a horizontal axis) is conjugate to some power of  $\alpha\beta$  or  $\beta$ .

If  $\gamma$  is the power of some curve  $C$  on  $K$ , then each component of the pre-image of  $C$  in  $\mathbb{E}^2$  is homeomorphic to  $\mathbb{R}$ . Let  $L$  be one such component that is  $\gamma$ -invariant, and  $M$  be an axis for  $\gamma$ . Then  $L$  and  $M$  are at bounded distances from each other as they are both  $\gamma$ -invariant and the quotient by  $\gamma$  is compact.  $\forall g \in G$ ,  $gM$  is either parallel to  $M$  or coincided with  $M$ . If  $M$  has slope  $m$ , the slope of  $\beta M$  is  $-m$ . For  $\beta M$  to be parallel to or coincide with  $M$ , it must either be horizontal or vertical. If  $M$  is vertical,  $\gamma$  must be a power of  $\alpha$ , and if it is horizontal,  $\gamma$  must be conjugate to a power of  $\beta$  or  $\alpha\beta$ .

As previously discussed, each of  $\alpha$ ,  $\beta$ ,  $\beta^2$  and  $\alpha\beta$  is represented by a simple closed curve on  $K$ , represented as  $C_\alpha$ ,  $C_\beta$ ,  $C_{\beta^2}$  and  $C_{\alpha\beta}$ .  $\alpha$  is a translation, so  $C_\alpha$  is two-sided and non-separating.  $\beta$  and  $\alpha\beta$  are glide reflections with disjoint axes, so  $C_\beta$  and  $C_{\alpha\beta}$  are disjoint and one-sided.  $C_{\beta^2}$  is a separating simple closed curve that splits  $K$  into two Mobius bands, each of which is a regular neighbourhood of  $C_\beta$  and  $C_{\alpha\beta}$ .  $(\alpha\beta)^2 = \beta^2$  and from lemma 1.5.4, we know that no higher powers are possible. So the above are the four homotopy classes of simple closed curves on a Klein bottle.  $\square$



**Lemma 1.6.4.** *Let  $\gamma$  be a non-trivial element of  $\pi_1(K)$  which generates a normal subgroup. Then  $\gamma$  must be a power of  $\alpha$  or of  $\beta^2$ .*

*Proof.* Case I:  $\gamma$  is a glide reflection.

Let  $l$  be the axis of  $\gamma$ . Then as  $\gamma$  generates a normal subgroup in  $\pi_1(K)$ , so every element of  $K$  must preserve  $l$ , which is not necessarily true as translations may not preserve it. so  $\gamma$  **cannot** be a glide reflection.

Case II:  $\gamma$  is a translation.

In this case,  $\gamma$  has many axes, all of which are parallel to each other. So any element of  $\pi_1(K)$  sends an axis  $l$  to a parallel line. As the translations under consideration are all horizontal and vertical translations, so  $l$  must either be a horizontal line (power of  $\beta^2$ ) or a vertical line (power of  $\alpha$ ).  $\square$

**Lemma 1.6.5.** *Let  $M$  be a surface and let  $\gamma$  be a non-trivial element of  $\pi_1(M)$  such that  $\gamma$  is conjugate to its inverse. Then either  $M$  is  $P^2$ , or it is the Klein bottle and  $\gamma$  is some power of  $\alpha$ .*

*Proof.* Suppose  $\delta\gamma\delta^{-1} = \gamma^{-1}$ . Then  $\gamma$  and  $\delta^2$  commute and generate an abelian subgroup  $H$ , and this is the fundamental group of a cover  $N$  of  $M$ . By lemma 1.2.11,  $H$  is isomorphic to either of  $\mathbb{Z}$  or  $\mathbb{Z}_2$  or  $\mathbb{Z} \times \mathbb{Z}$ .

Case I:  $H$  is isomorphic to  $\mathbb{Z}$ .

Then  $\delta$  commutes with some non-zero power of  $\gamma$ , and this contradicts the fact that conjugation by  $\delta$  inverts  $\gamma$ . Hence this case is **not** possible.

Case II:  $H$  is isomorphic to  $\mathbb{Z}_2$  or  $\mathbb{Z} \times \mathbb{Z}$ .

Then by the classification of compact surfaces,  $N$  is isomorphic to either  $P^2$  or to  $T$ .  $P^2$  can only cover  $P^2$ , so  $N$  is  $P^2$  implies that  $M$  is also  $P^2$ . If  $N$  is  $T$ , then it is the orientable double cover of the Klein bottle, and hence  $M$  is  $K$ .

If  $M$  is  $K$ , then if  $\gamma$  is a glide reflection with axis  $l$ , we must have  $\delta l = l$ . This implies that  $\delta$  commutes with  $\gamma$ , which is a contradiction. So  $\gamma$  is a translation and  $\delta$  is a glide reflection, as they cannot commute. Further,  $\gamma$  has to be a vertical translation, otherwise it would be a power of  $\beta^2$  and commute with  $\delta$ . So  $\gamma$  is a power of  $\alpha$ .  $\square$

## Chapter 2

# Homotopy of Maps on Surfaces

### 2.1 Basics

**Lemma 2.1.1** (The Alexander Trick). 1. Let  $f: D^n \rightarrow D^n$  be a map which restricts to a homeomorphism on  $\partial D^n$ . Then  $f$  is homotopic to a homeomorphism rel  $\partial D^n$ .

2. Let  $h: D^n \rightarrow D^n$  be a homeomorphism which restricts to the identity on  $\partial D^n$ . Then  $h$  is isotopic to the identity rel  $\partial D^n$ .

*Proof.* 1. We want a homotopy  $F: \mathbb{D}^n \times I \rightarrow \mathbb{D}^n$  with  $F_0 \equiv f$ ,  $F_1$  is a homeomorphism and  $F_t|_{\partial \mathbb{D}^n} \equiv f|_{\partial \mathbb{D}^n}$ . These restrictions tell us how to define  $F_t|_{\mathbb{D}^n \times \{0\} \cup \partial \mathbb{D}^n \times I}$ . We can also extend the homeomorphism  $F_t|_{\partial \mathbb{D}^n \times \{1\}}$  to a homeomorphism from  $\mathbb{D}^n \times \{1\}$  to  $\mathbb{D}^n$ . After this we finally extend  $F_t|_{\partial(\mathbb{D}^n \times I)}$  conically over  $\mathbb{D}^n \times I$  as in the following diagram.

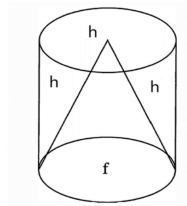


Figure 2.1:

2. We use the exact same procedure as in the previous case, except we take  $F_0 \equiv h$ , a homeomorphism, and  $F_1 \equiv Id_{\mathbb{D}^n}$ . Each  $F_t$  is a homeomorphism and  $F$  is level-preserving, ie,  $F_t(\mathbb{D}^n \times \{t\}) = \mathbb{D}^n \times \{t\}$ . So  $F$  is an isotopy.

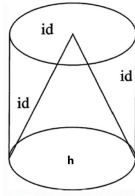


Figure 2.2:

□

**Definition 2.1.2.** Let  $X$  be a cell complex and  $Y$  be a subcomplex with a neighbourhood homeomorphic to  $Y \times \mathbb{R}^k$  for some  $k$ . Let  $M$  be an  $n$ -manifold. Then a map  $f: M \rightarrow X$  is transverse to  $Y$  if  $f^{-1}(Y)$  is empty or is a properly embedded codimension- $k$  submanifold of  $M$  such that there are neighbourhoods  $U$  and  $V$  of  $f^{-1}(Y)$  and  $Y$  respectively, and homeomorphisms of  $U$  and  $V$  with  $f^{-1}(Y) \times \mathbb{R}^k$  and  $Y \times \mathbb{R}^k$  such that the diagram below commutes, where the diagonal maps are simply projections.

$$\begin{array}{ccccc}
 f^{-1}(Y) \times \mathbb{R}^k & \cong U & \rightarrow & V & \cong Y \times \mathbb{R}^k \\
 \searrow & & & & \swarrow \\
 & & \mathbb{R}^k & & 
 \end{array}$$

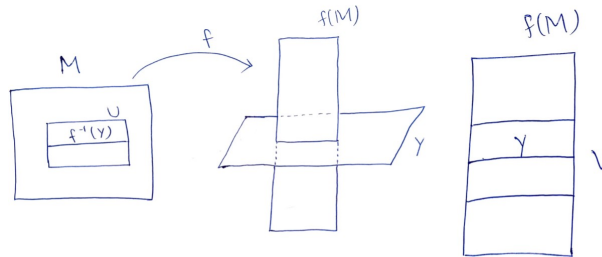


Figure 2.3:

**Remark 2.1.3.** If  $M$  has non-empty boundary and  $f: M \rightarrow X$  is transverse to  $Y$ , then  $f|_{\partial M}$  is also transverse to  $Y$ .

**Lemma 2.1.4.** Let  $M$  be an  $n$ -dimensional PL manifold. If  $f: M \rightarrow I$  is a PL map, then  $f$  is transverse to all points of  $I$  except for a countable number.

*Proof.* Case I:  $M$  is compact.

We triangulate  $M$  and  $I$  so that  $f$  becomes simplicial. Then  $f$  is transverse to all points of  $I$  which are not vertices of this triangulation. As  $M$  and  $I$  are compact, so the number of such vertices is countable. Hence,  $f$  is transverse to all but a countable number of points of  $I$ .

Case II:  $M$  is not compact.

Since every point of  $M$  has a neighbourhood that is locally homeomorphic to  $\mathbb{R}^n$ , and  $\mathbb{R}^n$  is second countable and locally compact, so  $M$  can be covered by a countable number of compact submanifolds,  $M_i$ ,  $i \geq 1$ . As each  $M_i$  compact, so  $f|_{M_i}$  is transverse to all but a finite number of points of  $I$ , which form the set  $W_i$ . So  $f$  is transverse to all points of  $I$ , except those in  $\cup W_i$ .  $\square$

**Definition 2.1.5.** A subcomplex  $Y$  of  $X$  is said to be PL bi-collared in  $X$  if there is a PL embedding  $k: Y \times [-1, 1] \rightarrow X$  such that  $k(x, 0) = x \forall x \in Y$  and  $k(Y \times (-1, 1))$  is open in  $X$ .  $k$  is called the PL bi-collar.

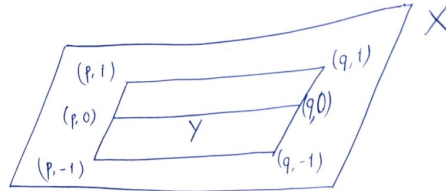


Figure 2.4:

**Corollary 2.1.6.** Let  $X$  be a cell complex with a bi-collared subcomplex  $Y$ .

1. If  $f: M \rightarrow X$  is a PL map, then  $f$  is transverse to  $Y \times \{t\}$ ,  $\forall t \in I$  except for a countable number.
2. Any map  $f: M \rightarrow X$  is  $\epsilon$ -homotopic to a map transverse to  $Y$ .

*Proof.* 1. We take a neighbourhood  $V$  of  $Y$  such that it is homeomorphic to  $Y \times I$ . Let  $p: V \rightarrow I$  be the projection map. Then we can consider the

map  $p \circ f: f^{-1}(V) \rightarrow I$ . This is a PL map and hence by lemma 2.1.4, it is transverse to all points of  $I$ , except a countable number. This means  $f$  is transverse to all but a countable number of points in  $Y \times \{t\}$ .

2. Again, we choose a neighbourhood  $V$  of  $Y$  such that  $V$  is homeomorphic to  $Y \times I$  and the homeomorphism identifies  $Y$  with  $Y \times \{\frac{1}{2}\}$ . Then for some  $t \in I$  within an  $\epsilon$  distance from  $\frac{1}{2}$ ,  $f$  is transverse to  $Y \times \{t\}$ . Let  $h$  be a homeomorphism of  $I$  such that  $h(0) = 0$ ,  $h(1) = 1$  and  $h(t) = \frac{1}{2}$  and moves each point of  $I$  by an  $\epsilon$  distance at most. There is also an ambient isotopy  $H$  from  $Id_I$  to  $h$ . Then  $\overline{H} = Id_Y \times H$  is the ambient isotopy on  $Y \times I$ . Then the required  $\epsilon$ -homotopy is:

$$\begin{aligned} F &= Id \ \forall x \notin f^{-1}(Y \times I) \\ &= \overline{H} \circ f \ \forall x \in f^{-1}(Y \times I) \end{aligned}$$

□

**Definition 2.1.7.** Let  $\lambda: I \rightarrow M$  be an immersion of the unit interval in the surface  $M$ .

1.  $\lambda$  is self-transverse if given any pair of disjoint sub-intervals  $U$  and  $V$  of  $I$  such that  $\lambda|_U$  and  $\lambda|_V$ , then  $\lambda|_U$  is transverse to  $\lambda(V)$ .
2.  $\lambda$  is in general position if it is self-transverse and has no triple points, ie,  $\lambda$  does not map three distinct points of  $I$  to a single point of  $M$ .

**Remark 2.1.8.** Any path  $\lambda: I \rightarrow M$  is  $\epsilon$ -homotopic to a general position immersion by moving the vertices of some triangulation by  $\epsilon$  radius at most, and keeping  $\partial I$  fixed.

**Definition 2.1.9** (Surgery on a 1-manifold along an arc). Let  $C$  be a 1-manifold properly embedded in a surface  $M$ , and there is an arc  $\mu$  in the interior of  $M$  such that  $\mu \cap C = \partial\mu$ . Let  $N$  denote a regular neighbourhood of  $\mu$  in  $M$  relative to  $C$ . Thus  $N$  is a disc containing  $\mu$  such that  $N \cap C$  equals  $\partial N \cap C$  and is a regular neighbourhood  $A$  of  $\partial\mu$  in  $C$  consisting of two intervals. Let  $B$  be the closure of  $\partial N - A$ , and let  $D$  denote the 1-manifold  $(C - A) \cup B$ , ie, it is obtained from  $C$  by replacing two sub-arcs with two new arcs. Then  $D$  is said to be obtained from  $C$  by performing surgery on  $C$  along  $\mu$ .

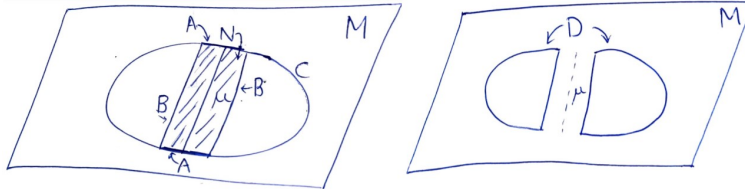


Figure 2.5:

**Lemma 2.1.10** (Surgery on a map). *Let  $X$  be a cell complex with a bi-collared subcomplex  $Y$ , such that  $\pi_1(Y)$  injects into  $\pi_1(X)$  and  $\pi_2(X) = 0$ . Let  $M$  be a surface and  $f: M \rightarrow X$  be transverse to  $Y$ . Suppose there is an arc  $\mu$  embedded in the interior of  $M$  such that  $\mu \cap f^{-1}(Y) = \partial\mu$  and  $f|_{\mu}$  is homotopic rel  $\partial\mu$  into  $Y$ . Then there is a homotopy of  $f$  to a map  $g$ , supported on a small neighbourhood of  $\mu$ , such that  $g$  is also transverse to  $Y$ , and  $g^{-1}(Y)$  is obtained from  $f^{-1}(Y)$  by surgery along  $\mu$ .*

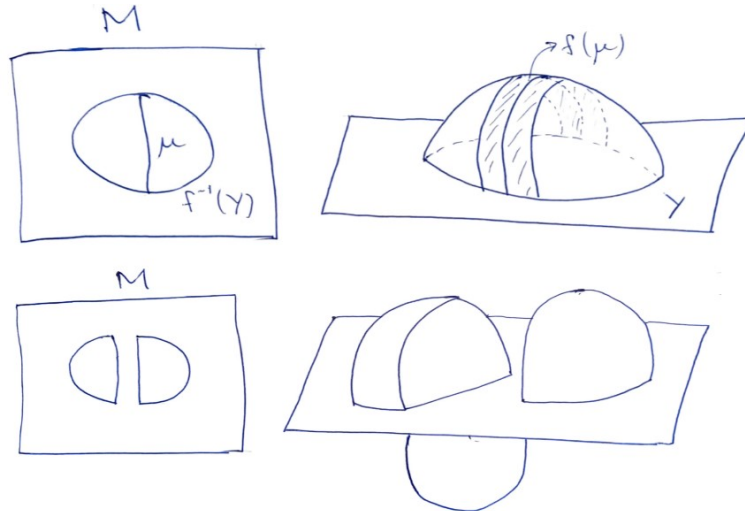


Figure 2.6: Depiction of what happens during surgery on a map

*Proof.* Let  $U$  and  $V$  be regular neighbourhoods of  $\mu$  and  $Y$  respectively.  $U$  is taken to be  $[-2, 2] \times [-2, 2]$ , with  $\mu$  being  $[-1, 1] \times \{0\}$  and  $U \cap f^{-1}(Y)$  being  $\{-1, 1\} \times [-2, 2]$ . The required map  $g: M \times X$  is defined as follows.  $g \equiv f$

outside  $U$ , on the boundary of  $U$  and on the arcs  $\{-1, 1\} \times ([-2, -1] \cup [1, 2])$ .  $f|_\mu$  can be homotoped rel  $\partial\mu$  into  $Y$ , and the same holds if we take either of the arcs  $[-1, 1] \times \{-1, 1\}$  in place of  $\mu$ . So  $[-1, 1] \times \{-1, 1\}$  can be sent into  $Y$  by a map that is homotopic to the restriction of  $f$  to the arc rel the boundary. From how we have defined  $g$  so far, its restriction to the boundary of the discs  $[-1, 1] \times [-2, -1]$ ,  $[-1, 1] \times [1, 2]$  and  $U$  minus the union of these two discs is homotopic to the restriction of  $f$  to the same, and hence is nullhomotopic. So we can extend  $g$  over the remaining parts of  $U$ , ie, the interiors of the previously mentioned discs. The boundary of each disc is mapped by  $g$  into  $V_1$  or  $V_2$ . Also,  $\pi_1(V_1) \cong \pi_1(V_2) \cong \pi_1(Y)$ , so  $\pi_1(V_1)$  and  $\pi_1(V_2)$  inject into  $\pi_1(X)$ . So we can choose some extension of  $g$  to each disc that maps the entire disc into either  $V_1$  or  $V_2$ , and then push the interior of each disc off  $Y$  using its collar in  $V_1$  or  $V_2$ . Thus,  $g$  is transverse to  $Y$  and  $g^{-1}(Y) \cap U = (\{-1, 1\} \times ([-2, -1] \cup [1, 2])) \cup ([-1, 1] \times \{-1, 1\})$ , ie,  $g^{-1}(Y)$  is obtained from  $f^{-1}(Y)$  by surgery along  $\mu$ .  $\square$

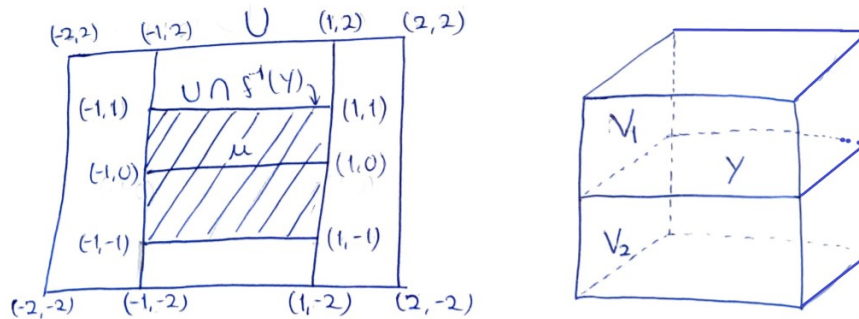


Figure 2.7:

## 2.2 Degree of a map

**Definition 2.2.1.** A map  $f: M \rightarrow N$  between  $n$ -manifolds is orientation true if  $f$  maps any orientation preserving loop in  $M$  to an orientation preserving loop in  $N$ , and any orientation reversing loop in  $M$  to an orientation reversing loop in  $N$ .

**Definition 2.2.2.** A map  $f: M \rightarrow N$  be a proper map between two compact connected oriented  $n$ -manifolds. Suppose  $f$  is transverse to a point  $y$  in the

interior of  $N$ , so that there is a disc neighbourhood  $D$  of  $y$  in  $N$  whose pre-image in  $M$  consists of a finite number of discs  $D_1, \dots, D_r$ , each mapped by a homeomorphism to  $D$ . We assign the number  $\epsilon_i$  to each  $D_i$ , where  $\epsilon_i$  equals 1 or  $-1$  according to whether  $f|_{D_i}$  is orientation preserving or not. Then the degree of  $f$ , denoted  $d(f)$  is the sum of the  $\epsilon_i$ 's.

**Definition 2.2.3.** Let  $M$  and  $N$  be compact  $n$ -manifolds, and let  $f: M \rightarrow N$  be any proper map. If  $f$  is transverse to a point  $y$  of the interior of  $N$ , there is a disc neighbourhood  $D$  of  $y$  in  $N$  whose pre-image in  $M$  consists of a finite number of discs, each mapped to  $D$  by a homeomorphism. The geometric degree of  $f$  at  $y$  is the number of these discs. The geometric degree of  $f$  is the minimal value of this number over all points  $y$  in  $Y$  to which  $f$  is transverse.

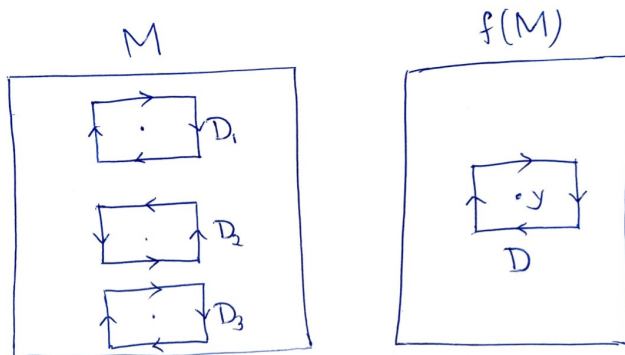


Figure 2.8: A map with algebraic degree 1 and geometric degree 3

**Theorem 2.2.4.** Let  $f: M \rightarrow N$  be an orientation true proper map between two compact connected  $n$ -manifolds. Then:

1. The geometric degree of  $f$  is at least as large as its algebraic degree  $d(f)$ .
2. If  $f$  has non-zero algebraic degree, then it must be a surjective map from  $M$  to  $N$ .

*Proof.* 1. This is obvious from the definitions of both terms.

2. Let  $y \in Y$ .  $f$  has non-zero algebraic degree, so from part (1), it also has non-zero geometric degree. But the geometric degree of  $f$  is the least value of the geometric degree at any point on  $Y$ , so there is a disc neighbourhood  $D$  of  $y$  such that  $f^{-1}(y)$  is contained in some non-empty pre-image of  $D$ . Hence  $f$  is surjective.

□

**Theorem 2.2.5.** *Let  $f: M \rightarrow N$  be an orientation-true proper map between two compact, connected  $n$ -manifolds, where  $n \neq 2$ . Then  $f$  is properly homotopic to a map  $g$  whose geometric degree equals  $d(f)$ .*

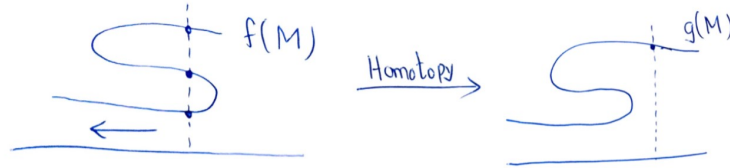


Figure 2.9: Basic idea of the proof

*Proof.* When  $n = 1$

$M$  and  $N$  are homeomorphic to either  $S^1$  or to the unit interval  $I$ . If  $N$  is homeomorphic to  $S^1$ , then so is  $M$ , since  $f$  is a proper map. We can think of  $S^1$  as the set of all unit complex numbers. Then  $f$  is homotopic to a map  $f_k$  taking  $z$  to  $z^k$ , for some integer  $k$ . When  $k = 0$ ,  $f_k$  is a point map, and when  $k \neq 0$ , then it is a covering map of degree  $k$ .

If  $N$  is homeomorphic to  $I$ , then  $f$  is properly homotopic to a homeomorphism (Alexander's trick) or to a point map.

When  $n \geq 3$

We assume that  $M$  and  $N$  are closed, and work with the following cases separately:

Case I: When both  $M$  and  $N$  are orientable

First, we consider the case  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is onto. Let  $f$  be transverse to  $y \in N$  and  $D$  be a disc containing  $y$ .  $D_1, D_2, \dots, D_r$  are the pre-images of  $D$ , such that  $f(x_i) = y$  for  $x_i \in D$ . If  $r = d$ , we are done. If  $r > d$ , then at least a pair of discs in  $f^{-1}(D)$  is of opposite sign, say  $D_1$  and  $D_2$ . Let  $\lambda$  be a path in  $M$  joining  $D_1$  and  $D_2$  such that it does not intersect any other  $D_i$ 's and its endpoints both project to the same point  $z$  of  $\partial D$ . We are working with  $n \geq 3$ , so we can "raise" and homotop  $\lambda$  to be simple without moving its endpoints.  $f \circ \lambda$  is a loop based at  $z$ . Suppose this loop is non-trivial in  $\pi_1(N)$ . Then as

$f_*$  is onto, there is a loop  $\mu$  in  $M$  at one endpoint of  $\lambda$  such that  $f \circ \mu$  and  $f \circ \lambda$  are homotopic. Replacing  $\lambda$  by  $\bar{\mu} * \lambda$  gives a new path that projects to a null-homotopic loop in  $N$ . So we can assume that  $f\lambda$  is trivial in  $\pi_1(N)$ . Again, as  $n \geq 3$ , so  $f \circ \lambda$  is actually trivial in  $\pi_1(N - D)$ . Hence, we can homotop  $f \circ \lambda$  to  $z$  without moving the endpoints of  $\lambda$ .

Let  $V$  be a neighbourhood of  $\lambda$  in  $M$ , and  $Z$  be a neighbourhood of  $z$  in  $N$ . Then we can homotop  $f$  so that  $f(V)$  is contained in  $\partial D \cap Z$ , without altering the  $f^{-1}(D - Z)$ . If  $U = D_1 \cup D_2 \cup V$ , it is homeomorphic to an  $n$ -disc, and  $f$  maps  $(U, \partial U) \rightarrow (D, \partial D)$ . So if  $w$  is a point of  $\partial D$  not in  $Z$ , then  $\partial U \rightarrow \partial D$  has algebraic degree zero and is nullhomotopic, and we can homotop  $f$  fixed outside  $U$  so that  $f(U) \subset \partial D$ . This removes  $x_1$  and  $x_2$  from  $f^{-1}(y)$  and reduces the geometric degree by 2, while keeping the new map transverse to  $w$ . Repeating the process gives the required homotopy.

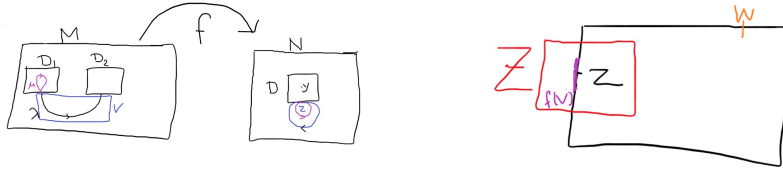


Figure 2.10: Rough sketch of the loops and neighbourhoods used in the proof

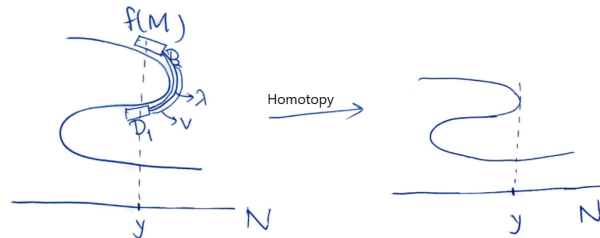


Figure 2.11: A more realistic view of how the proof works

If  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is not onto, then we take the cover  $N_1$  of  $N$  such that  $\pi_1(N_1) = f_*(\pi_1(M))$ .  $f_1$  is the lift of  $f$  to  $N_1$  and  $p: N_1 \rightarrow N$  is the covering map. If  $p$  is finite sheeted, then geometric and algebraic degrees are equal, and  $f_{1*}: \pi_1(M) \rightarrow \pi_1(N_1)$  is onto by the suitable choice of  $N_1$ . So using the previous arguments, we can homotop  $f_1$  to  $g_1$  such that geometric degree and algebraic

degree are the same for the latter. Then  $g = p \circ g_1$  is the required map. If  $p$  is an infinite sheeted cover, then  $H_n(N_1) = 0$ , so  $f$  must have degree 0. According to Remark 1.2.4,  $N_1$  deformation retracts to an  $(n - 1)$ -dimensional subcomplex. So  $p$  is homotopic to a map  $q$  which is not surjective, and  $q \circ f_1$  is the required map of geometric degree zero.

Case II:  $M$  is orientable but  $N$  is not.

We consider the orientable double cover  $N'$  of  $N$ , along with the induced map  $f': M \rightarrow N'$ . Then this is similar to the previous case, and we get a map  $g'$  homotopic to  $f'$ , with geometric degree equal to algebraic degree. If  $p': N' \rightarrow N$  is the covering map, then  $p' \circ g'$  is the required map.

Case III: Both  $M$  and  $N$  are non-orientable.

Again, we assume  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is onto. Let  $f$  be transverse to  $y \in N$  and  $D$  be a disc containing  $y$ .  $D_1, D_2, \dots, D_r$  are the pre-images of  $D$ , such that  $f(x_i) = y$  for  $x_i \in D_i$ . If  $r = d$ , we are done, so we take  $r > d$ .  $M'$  and  $N'$  are the orientable double covers of  $M$  and  $N$ , with the induced map  $f'$ . We take  $y'$  to be one of the points on  $N'$  going to  $y$ , and  $D'$  to be the lift of  $D$  containing  $y'$ . The pre-image of  $D'$  consists of the discs  $D'_1, D'_2, \dots, D'_r$ , each mapping to  $D'$  by a homeomorphism, with  $x'_i \in D'_i$  mapping to  $y'$ . A choice of orientations on  $M'$  and  $N'$  gives a sign for each  $D'_i$ .  $r > d$ , so there must be at least a pair of discs of opposite sign; say these are  $D'_1$  and  $D'_2$ . We take a path  $\lambda'$  joining  $D'_1$  and  $D'_2$  such that its endpoints project to the same point  $z$  on  $\partial D$ , and  $f' \circ \lambda'$  is trivial in  $\pi_1(N')$ . So we get a path  $\lambda$  joining  $D_1$  and  $D_2$  such that its endpoints project to  $z$  on  $\partial D$ , and  $f \circ \lambda$  is trivial in  $\pi_1(N)$ . We homotop  $\lambda$  to be simple and not intersecting any other  $D_i$ 's. We take a neighbourhood  $V$  of  $\lambda$  in  $M$  and  $Z$  of  $z$  in  $N$ , and some point  $w$  of  $\partial D$  not in  $Z$ . Then  $U = D_1 \cup D_2 \cup V$  is an  $n$ -disc, and the map  $(U, \partial U) \rightarrow (D, \partial D)$  is transverse to  $w$ . So we can homotop  $f$  to reduce the geometric degree by 2 and repeat to obtain the required homotopy.  $\square$

**Theorem 2.2.6.** *Let  $f: M \rightarrow N$  be an orientation-true proper map between two compact, connected 2-manifolds. Then  $f$  is properly homotopic to a map  $g$  whose geometric degree equals  $d(f)$ .*

*Proof.* If  $f$  has geometric degree zero, then algebraic degree is also zero, and this case is trivial. So we can assume that  $f$  is not surjective. We will use the classification of compact surfaces to go case-wise in the proof.

Case I:  $N$  is  $S^2$

Say  $f$  is transverse to  $y$  on  $S^2$ . Then there is a disc  $D$  containing  $y$  and having pre-images  $D_1, D_2, \dots, D_r$ , each mapping to  $D$  by a homeomorphism. Let  $x_i \in D_i$  such that  $f(x_i) = y$ . Again, we assume  $r > d$ , and that  $D_1$  and  $D_2$

have opposite signs. Say  $\lambda$  joins  $D_1$  and  $D_2$  such that it does not meet any other  $D_i$  and its endpoints both project to the same point  $z \in \partial D$ . Then  $f \circ \lambda$  is a 2-disc in  $(N - D)$  and is nullhomotopic. We use the same arguments as in the previous theorem after this point to remove  $x_1$  and  $x_2$  from  $f^{-1}(y)$  and reduce the geometric degree by 2. We repeat this process till we get the desired result.

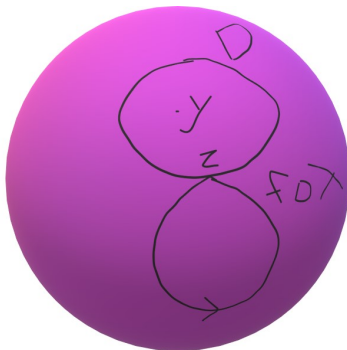


Figure 2.12:

Case II:  $N$  is  $P^2$

If  $M$  is orientable, then we work with the map  $f': M \rightarrow S^2$ , which is the lift of  $f$ , to obtain  $g'$  which is homotopic to  $f'$  and has geometric degree equal to algebraic degree. Then if  $p: S^2 \rightarrow P^2$  is the covering map, the required map homotopic to  $f$  is  $p' \circ g'$ .

Suppose  $M$  is also non-orientable. Then we homotop  $f$  to be transverse to  $y \in P^2$ . Then there is a disc neighbourhood  $D$  of  $y$  whose pre-image consists of the discs  $D_1, \dots, D_r$ , each mapping to  $D$  by a homeomorphism. Again, we take a simple path  $\lambda$  in  $M$  joining  $D_1$  and  $D_2$ , not meeting any other  $D_i$  and whose endpoints go to the same point of  $\partial D$ . Then  $f \circ \lambda$  is a loop in the Mobius band  $\overline{P^2 - D}$ . If  $f \circ \lambda$  represents an odd integer, then we replace  $\lambda$  by another simple arc which takes  $x_1$  around an orientation-reversing simple closed curve in  $M$  so that  $f \circ \lambda$  represents an even integer. We further alter  $\lambda$  by moving one end around  $\partial D_1$  so that  $f \circ \lambda$  becomes null-homotopic in  $\overline{P^2 - D}$ . We take a regular neighbourhood  $V$  of  $\lambda$  and  $X = D_1 \cup D_2 \cup V$ . Since  $X$  is a disc, we can check orientations of  $D_1$  and  $D_2$ . If they have opposite orientations, we homotop  $f$  to remove them from the pre-image of  $D$ ; if they have the same orientation, we homotop  $f$  till  $f|_X$  is a branched cover of  $D$  of degree 2. We repeat this till the pre-image of  $y$  consists of a single point and there is a disc  $D$  containing  $y$  such that  $f$  restricted to the pre-image of  $D$  is a branched cover of some degree  $s$ . For any point in  $D - \{y\}$ ,  $f$  has geometric degree at most  $s$ .

Now let  $f'$  be a map such that the following diagram commutes:

$$\begin{array}{ccc}
M' & \xrightarrow{f'} & S^2 \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & P^2
\end{array}$$

Then there is  $y' \in S^2$  and a disc  $D'$  containing  $y'$  such that  $f'$  restricted to the pre-image of  $D'$  is a branched cover of degree  $s$ . Then for any point on  $D' - \{y'\}$ ,  $f'$  has geometric and algebraic degrees both equal to  $s$ . But  $f$  and  $f'$  have the same algebraic degree, so  $s = d$ .

Case III:  $N$  is not  $S^2$  or  $P^2$

Then there is an essential two-sided simple closed curve  $S$  on  $N$ . The following lemma then says that we can homotop  $f$  so that  $f$  is transverse to  $S$  and  $f^{-1}(S)$  is either empty or a single circle  $C$ . In the latter case,  $f$  can be further homotoped so that it is either a point map or a covering map. If  $M$  and  $N$  are orientable, we are done; otherwise, we consider their orientable covers.  $\square$

**Definition 2.2.7.** *A map is a branched covering if it is a covering map everywhere except on a nowhere dense set.*

**Lemma 2.2.8.** *Let  $M$  and  $N$  be closed connected surfaces, not  $S^2$  or  $P^2$ , and let  $S$  be an essential two-sided simple closed curve on  $N$ . Let  $f : M \rightarrow N$  be a map that induces a surjection from  $\pi_1(M)$  to  $\pi_1(N)$ . Then we can homotop  $f$  so that  $f$  becomes transverse to  $S$  and  $f^{-1}(S)$  is either empty or a single circle.*

*Proof.* By corollary 2.1.6, we can homotop  $f$  to be transverse to  $S$ .  $f^{-1}(S)$  consists of disjoint, two-sided simple closed curves in  $M$ . If it is empty we are done; hence we assume that it is non-empty. We will use Stallings' method of binding ties to reduce the number of components of  $f^{-1}(S)$ . Let  $C_1$  and  $C_2$  be distinct components of  $f^{-1}(S)$ , and  $\lambda$  be some arc joining them such that its endpoints map to the same point  $y$  of  $S$ . Then  $f \circ \lambda$  is a loop in  $N$ . If it is non-trivial, we take another loop  $\nu$  based at one endpoint of  $\lambda$  in  $M$  so that  $f \circ \nu$  is homotopic to  $f \circ \lambda$  (possible since  $f_*$  is onto). Then if we take  $\lambda * \bar{\nu}$  as our new  $\lambda$ ,  $f \circ \lambda$  is now null-homotopic in  $N$ . This property is not preserved, so we need to work with the fact that  $f \circ \lambda$  is based homotopic into  $S$ .  $\lambda \cap f^{-1}(S)$  cuts  $\lambda$  into finitely many subarcs  $\lambda_1, \lambda_2, \dots, \lambda_n$ . We define the length of  $\lambda$  to be this number of subarcs. Each  $\lambda_i$  maps to some loop  $g_i$  in  $\pi_1(N)$ , and  $g_1 g_2 \dots g_n$  is in  $\pi_1(S)$ . Now  $\pi_1(N)$  splits over  $\pi_1(S)$ , so some  $g_i$  must be in  $\pi_1(S)$ . Let the endpoints of the corresponding  $\lambda_i$  lie on  $C$  and  $C'$  of  $f^{-1}(S)$ . If  $C$  and  $C'$  are the same component, then we replace  $\lambda_i$  by a path on  $C$  and push this off  $C$ . The new arc  $\lambda'$  is still such that  $f \circ \lambda'$  is based homotopic into  $S$ , and its length is strictly less than that of  $\lambda$ . By repeating this, we get an arc  $\mu$  joining  $C$  and  $C'$  (which are distinct), not meeting  $f^{-1}(S)$  at any points other than its endpoints and  $f(\mu)$  is homotopic rel  $\partial\mu$  into  $S$ .

This is the required binding tie.

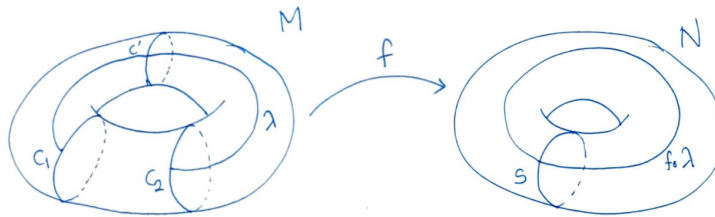


Figure 2.13:

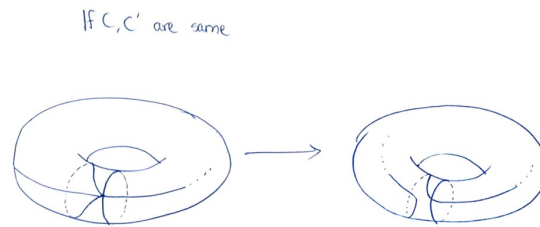


Figure 2.14:

Next, we will replace  $\mu$  by an embedded arc. First, we homotop  $\mu$  to be in general position, so that there are no triple points. After that, we choose the double point of  $\mu$  nearest to an endpoint (say, the one on  $C$ ), and push it off that end. We obtain a sub-arc  $\nu$  in the interior of  $\mu$  which has both endpoints on  $C$ , and is homotopic rel  $\partial\nu$  into a sub-arc  $\gamma$  of  $C$ . Then we replace  $\gamma$  by  $C - \gamma$  and homotop this off  $C$  to reduce the number of double points by one. As the new  $\mu$  differs from the old one by a copy of  $C$ , hence its image is still homotopic rel boundary into  $S$ .

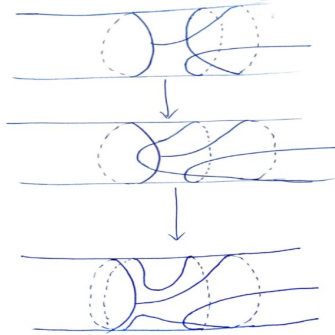


Figure 2.15:

Now we can perform surgery on  $C$  and  $C'$  along  $\mu$  to replace them by their connected sum. If there are more components, we repeat the process till  $f^{-1}(S)$  consists of a single circle.

□

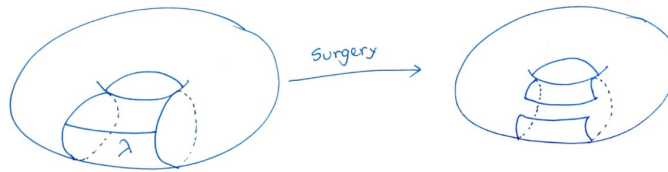


Figure 2.16:

**Corollary 2.2.9.**  *$M$  and  $N$  are closed surfaces, not  $S^2$  or  $P^2$ , and  $S$  is an essential two-sided simple closed curve on  $N$ .  $f : M \rightarrow N$  is a map of non-zero geometric degree  $G(f)$  which induces a surjection of fundamental groups. Then we can homotop  $f$  so as to arrange that  $f$  is transverse to  $S$  and  $f^{-1}(S)$  is a single circle which maps to  $S$  by a covering map of degree  $G(f)$ .*

*Proof.* We pick some  $z \in S$  and homotop  $f$  to have geometric degree  $G(f)$  at  $z$ . If  $Z$  is a small neighbourhood of  $z$ , then we can apply the same arguments as in lemma 2.2.8 to obtain a homotopy of  $f$  such that  $f^{-1}(S)$  is a single circle, all without altering  $f$  on  $f^{-1}(Z)$ . This is done by choosing a  $\lambda$  that does not meet  $f^{-1}(Z)$ . □

## 2.3 Homotopy Equivalences

**Theorem 2.3.1.**  *$M$  and  $N$  are compact connected surfaces with non-empty boundary and  $f : M \rightarrow N$  is a homotopy equivalence that induces a homeomorphism  $\partial M \rightarrow \partial N$ . Then  $f$  is homotopic to a homeomorphism rel  $\partial M$ .*

*Proof.* If we consider the handle decomposition of a compact surface  $N_0$  which is not  $\mathbb{D}^2$ , then each 1-handle has a transverse, non-separating arc. Hence any compact surface with non-empty boundary must either be a disc or admit a non-separating, properly embedded arc.

Now let  $N_1$  be the surface obtained by cutting  $N_0$  along the non-separating embedded arc. The rank of  $H_1(N_1)$  is one less than the rank of  $H_1(N_0)$ . Repeating this gives a sequence of surfaces  $\{N_k\}$ ,  $0 \leq k \leq l$ , so that  $H_1(N_k)$  has strictly decreasing rank.

The proof of the theorem proceeds by induction on the rank  $r$  of  $H_1(N)$ .



Figure 2.17:

If  $r = 0$ , then  $M$  and  $N$  have zero first homology group and are discs. So the theorem follows by the Alexander's trick.

Now let us assume that the hypothesis holds for all  $N$  with  $H_1(N) < r$ ,  $r > 0$ . Say  $N$  has  $H_1(N) = r$ . Then  $M$  and  $N$  are not discs, so  $N$  has a properly embedded non-separating arc  $\lambda$ . By corollary 2.1.6, we can assume  $f$  is transverse to  $\lambda$ .  $M$  is compact and  $f^{-1}(\lambda)$  is closed in  $M$ , so  $f^{-1}(\lambda)$  is a compact bi-collared submanifold of  $M$ , and its components must be arcs or circles.

Let  $C$  be a circle component of  $f^{-1}(\lambda)$ .  $f(C) \subset \lambda$  and is null-homotopic in  $N$ . As  $f_*$  is an isomorphism, so  $C$  must be null-homotopic in  $M$ . Hence the inclusion of  $\pi_1(C)$  in  $\pi_1(M)$  is not an isomorphism and by corollary 1.2.10,  $C$  bounds a disc  $D$  in  $M$ .  $f|_D : (D, C) \rightarrow (N, \lambda)$  is an element of  $\pi_2(N, \lambda)$ .  $\lambda$  is contractible and  $N$  is aspherical, so from the following long exact sequence, we get that  $\pi_2(N, \lambda)$  is trivial:

Hence we can homotop  $f|_D$  rel  $C$  to a map whose image lies in  $\lambda$ . This gives a homotopy of  $f$  with support  $D$  to a map  $f_1$  with  $f_1(D) \subset \lambda$ . There is a

$$\dots \longrightarrow \pi_2(N) \longrightarrow \pi_2(N, \lambda) \longrightarrow \pi_1(\lambda) \longrightarrow \dots$$

neighbourhood  $U$  of  $D$  such that  $f_1(U - D)$  is disjoint from  $\lambda$  and lies in a collar  $V$  of  $\lambda$ , ie, on one side of  $\lambda$ , which is denoted as  $V_1$ . We can then push  $D$  into the interior of  $V_1$ , giving a homotopy of  $f$  to a map which does not contain the circle component  $C$ . Repeating the process removes all circle components of  $f^{-1}(\lambda)$ , ie, it consists only of arcs.

$\partial\lambda$  consists only of 2 points. As  $f$  restricts to a homeomorphism along the boundaries of  $M$  and  $N$ , so  $\partial f^{-1}(\lambda)$  also consists of only 2 points. And hence  $f^{-1}(\lambda)$  is a single arc,  $\mu$ . Using the Alexander's trick, we can homotop  $f|_\mu$  so that it gives a homeomorphism from  $\mu$  to  $\lambda$ , and extend this to the whole of  $M$ , preserving the fact that  $f^{-1}(\lambda) = \mu$ . We cut  $M$  along  $\mu$  to get  $M'$ , and  $N$  along  $\lambda$  to get  $N'$ , with rank of  $H_1(N') < r$ . This gives a map  $f' : M' \rightarrow N'$ , which also induces a homeomorphism  $\partial M' \rightarrow \partial N'$ .  $\pi_1(M) = \pi_1(M') * \mathbb{Z}$  and  $\pi_1(N) = \pi_1(N') * \mathbb{Z}$ . Further,  $(f')_* * 1 : \pi_1(M') * \mathbb{Z} \rightarrow \pi_1(N') * \mathbb{Z}$  is  $f'_*$ , so  $f'_*$  also induces an isomorphism of fundamental groups. hence, we can use the induction hypothesis to conclude that  $f'$  is homotopic rel  $\partial M'$  to a homeomorphism. Since the homotopy of  $f'$  does not move the endpoints of  $\partial M'$ , so  $f$  is also homotopic rel  $\partial M$  to a homeomorphism.  $\square$

**Theorem 2.3.2.** *Let  $M$  and  $N$  be connected surfaces such that  $M$  is closed, but not  $S$  or  $P^2$ . Let  $f : M \rightarrow N$  be a homotopy equivalence. Then  $f$  is homotopic to a homeomorphism.*

*Proof.* In this case,  $H_2(N, \mathbb{Z}_2) \cong H_2(M, \mathbb{Z}_2)$ , which is isomorphic to  $\mathbb{Z}_2$ ; hence  $N$  must also be closed. Since  $f$  is a homotopy equivalence, so  $N$  cannot be  $S^2$  or  $P^2$  either. Using  $\mathbb{Z}$  coefficients shows that  $M$  is orientable iff  $N$  is, and similarly for their covers. Hence  $f$  is orientation true and has degree 1. Then by theorem 2.2.6, we can homotop  $f$  to also have geometric degree 1.

Let  $S$  be an essential simple closed curve on  $N$ . Then by corollary 2.2.9, we can homotop  $f$  so that it is transverse to  $S$  and  $f^{-1}(S)$  is a single circle  $C$  which maps to  $S$  by a homeomorphism.

Like in theorem 2.3.1, we cut  $M$  along  $C$  and  $N$  along  $S$  to obtain  $M_1$  and  $N_1$ , and the induced map  $f_1 : M_1 \rightarrow N_1$ . Again using the same arguments as in the previous proof, we claim that  $f_1$  is a homotopy equivalence and induces a homeomorphism  $\partial M_1 \rightarrow \partial N_1$ . Hence by theorem 2.3.1,  $f_1$  is homotopic to a homeomorphism rel boundary, and so is  $f$ .  $\square$

**Theorem 2.3.3.** *Let  $f : S^2 \rightarrow S^2$  be a homotopy equivalence. Then  $f$  is homotopic to a homeomorphism.*

*Proof.* A homotopy equivalence has degree 1, and  $f$  is trivially orientation true as we are working with orientable surfaces. So by theorem 2.2.6, we can homotop  $f$  so that it also has geometric degree 1. Hence,  $f$  is transverse to  $y \in S^2$ ,

$f^{-1}(y) = x$  and  $f$  is a homeomorphism on a small disc  $D$  with  $x \in D$ . If  $D' = S^2 - D$ , then  $y \notin f(D')$ . We can then homotop  $f|_{D'}$  rel  $\partial D'$  so that  $f(D') = S^2 - f(D)$ . Next we use the Alexander's trick to homotop  $f|_{D'}$  into a homeomorphism rel  $\partial D'$ .  $\square$



Figure 2.18:

**Theorem 2.3.4.** *Let  $f : P^2 \rightarrow P^2$  be a homotopy equivalence. Then  $f$  is homotopic to a homeomorphism.*

*Proof.* Again, it is obvious that  $f$  is orientation true, as both surfaces are non-orientable. Further,  $f$  has degree 1, so by theorem 2.2.6, we can homotop it to have geometric degree 1 as well. Then  $f$  is transverse to  $y \in P^2$ ,  $f^{-1}(y) = x$  and  $f$  is a homeomorphism on a small disc  $D$  with  $x \in D$ . Further,  $y \notin f(P^2 - D)$ . Using the ambient isotopy, we homotop  $f$  so that  $f(D)$  is  $D$ , and then homotop  $f$  rel  $D$  so that  $f(P^2 - D)$  is also  $P^2 - D$ . Now, along the Mobius band  $P^2 - D$ ,  $f$  is a self map which is a homeomorphism along its boundary. Hence it induces an isomorphism of fundamental groups, and by using theorem 2.3.1 on  $f|_{\overline{P^2 - D}}$ , we conclude that  $f$  is homotopic to a homeomorphism.  $\square$

By combining the theorems 2.3.1 to 2.3.4, we can conclude the following:

**Theorem 2.3.5.**  *$M$  and  $N$  are compact connected surfaces and  $f : M \rightarrow N$  is a homotopy equivalence that induces a homeomorphism  $\partial M \rightarrow \partial N$ . Then  $f$  is homotopic to a homeomorphism rel  $\partial M$ .*

**Remark 2.3.6.** *If we look at the proofs involved in theorem 2.3.5, we note that we do not really need  $f$  to be a homotopy equivalence per se; it only needs to induce an isomorphism of the subgroups  $\pi_1(M)$  and  $\pi_1(N)$ .*

## 2.4 Generalizations of the Previous Section

Removing the requirement that  $f$  induces a homeomorphism from  $\partial M$  to  $\partial N$ .

**Theorem 2.4.1.**  *$M$  and  $N$  are compact connected surfaces that are not  $S^2$ ,  $P^2$  or  $D^2$ , and  $f : M \rightarrow N$  is a proper map inducing an isomorphism  $\pi_1(M) \rightarrow \pi_1(N)$ . Then  $f$  is properly homotopic to a map  $g$  such that one of the following hold:*

1.  $g$  is a homeomorphism.
2.  $g(M)$  is contained in  $\partial N$ ,  $M$  is the Mobius band or annulus, and  $N$  is the annulus.

*Proof.* If  $\partial M$  is empty, then we get case (1) by theorem 2.3.2. For the rest of the proof, we assume non-empty boundary.

Let  $C$  be a component of  $\partial M$  and  $S$  be the component of  $\partial N$  containing  $f(C)$ .  $M$  is not a disc, hence by lemma 1.2.9,  $\pi_1(C) \rightarrow \pi_1(M)$  is injective. We let  $C$  represent  $\alpha \in \pi_1(M)$ ,  $S$  represent  $\gamma \in \pi_1(N)$  and  $\beta$  be  $f_*^{-1}(\gamma) \in \pi_1(M)$ . If  $f : C \rightarrow S$  has degree  $k$ , then clearly  $\alpha = \beta^k$  in  $\pi_1(M)$ . By 1.5.4,  $k = 1$  or  $2$ .

If  $k = 2$ , then lemma 1.5.4 further states that  $C$  bounds a Mobius band in  $M$ . But  $C$  is a boundary component of  $M$ , hence  $M$  itself is a Mobius band.  $f_*$  is an isomorphism of fundamental groups, hence  $\pi_1(N)$  is infinite cyclic, so  $N$  is either an annulus or a Mobius band.  $f$  is not a homeomorphism from  $\partial M$  to  $\partial N$  as  $k \neq 1$ , so  $N$  is not a Mobius band; hence it is an annulus.  $N$  deformation retracts to  $S$  by the retraction  $H_r$ . So the required homotopy of  $f$  into  $\partial N$  is  $H_r \circ f$ . This gives case (2).

Now let  $k = 1$ . A map of degree 1 can be homotoped to a homeomorphism. Hence each component of  $\partial M$  is mapped to a component of  $\partial N$  by a homeomorphism. If  $f|_{\partial M}$  is injective, then  $f(\partial M)$  is non-empty in  $\partial N$  and zero in  $H_1(N)$ , meaning  $f(\partial M) = \partial N$ . This gives case (1).

If  $f|_{\partial M}$  is not injective, then there are components  $C_1$  and  $C_2$  of  $\partial M$  such that  $f(C_1)$  and  $f(C_2)$  are both equal to the same component  $S$  of  $\partial N$ . But as  $f$  induces an isomorphism of fundamental groups, so  $C_1$  and  $C_2$  must be homotopic, and by lemma 1.4.1,  $M$  is an annulus. Hence,  $\pi_1(N)$  is infinite cyclic, so  $N$  is either a Mobius band or an annulus. But  $\pi_1(C_1) \hookrightarrow \pi_1(M)$  is an isomorphism, and  $f_*$  takes  $\pi_1(C_1)$  into  $\pi_1(S)$  isomorphically, hence  $\pi_1(M) \cong \pi_1(S)$ . This is possible only if  $N$  is an annulus. Hence, we again case (2).  $\square$

Weakening the requirement that  $f$  induces an isomorphism of fundamental groups.

**Theorem 2.4.2.**  *$M$  and  $N$  are compact connected surfaces that are not  $S^2$ ,  $P^2$  or  $D^2$ , and  $f : M \rightarrow N$  is a proper map inducing an injection  $\pi_1(M) \rightarrow \pi_1(N)$ . Then  $f$  is properly homotopic to a map  $g$  such that one of the following hold:*

1.  $g$  is a covering map.
2.  $g(M)$  is contained in  $\partial N$ , and  $M$  is the Mobius band or annulus.

*Proof.* Let  $N'$  be the cover of  $N$  such that  $\pi_1(N') = f_*\pi_1(M)$ , and  $f' : M \rightarrow N'$  be the lift of  $f$ .

Case I:  $N'$  is compact.

Then we can use theorem 2.4.1 to conclude that  $f'$  is properly homotopic to a map  $g'$  which is either a homeomorphism or maps  $M$  to  $\partial N'$  such that  $M$  is either a Mobius band or an annulus. If  $p : N' \rightarrow N$  is the covering map, then  $f$  is properly homotopic to  $g = p \circ g'$  which is a covering map and we get (2).

Case II:  $N'$  is non-compact.

$f'$  induces an isomorphism of fundamental groups, as  $M$  and  $N'$  are aspherical, and  $M$  is not  $S^2$  or  $P^2$ . Hence, if  $M$  is closed, so is  $N'$ , and by theorem 2.3.2,  $f'$  is homotopic to a homeomorphism. This gives (1).

Next, say  $M$  is not closed. Let  $C$  be a component of  $\partial M$  and  $S$  be the component of  $\partial N'$  which contains  $f'(C)$ . As  $f'$  is  $\pi_1$ -injective, so  $S$  must be a circle. Again, if the map  $C \rightarrow S$  has degree  $k$ , then  $k$  must be either 1 or 2.

If  $k = 2$ , then  $M$  is a Mobius band.  $\pi_1(C)$  has index 2 in  $\pi_1(M)$  and  $f'_*(\pi_1(C))$  has index 2 in  $\pi_1(S)$ . As  $f'_* : \pi_1(M) \rightarrow \pi_1(N')$  is an isomorphism, so  $\pi_1(S) \cong \pi_1(N')$ . Thus,  $N'$  deformation retracts to  $S$  via the homotopy  $H_r$  and  $H_r \circ f'$  is a homotopy of  $f'$  into  $\partial N'$ . Taking projection gives a proper homotopy of  $f$  into  $\partial N$ . This gives (2).

If  $k = 1$ , then for injective  $f'|_{\partial M}$ , similar to the proof of theorem 2.4.1, we have that  $N'$  is compact and  $f'(\partial M) = \partial N'$ , giving (1). If  $f'$  is not injective, again we have two components  $C_1$  and  $C_2$  mapping to the same component of  $\partial N'$ . Hence,  $M$  is an annulus and we obtain a proper homotopy of  $f$  into  $\partial N$ , giving (2).  $\square$

**Lemma 2.4.3.**  *$M$  and  $N$  are closed connected surfaces, and  $f : M \rightarrow N$  is an orientation true map of non-zero degree. Then  $\chi(M) \leq \chi(N)$ .*

*Proof.* It suffices to assume that  $M$  and  $N$  are orientable, as we can take their orientable double covers otherwise.

If  $N$  is  $S^2$ , then for any closed surface,  $\chi(M) \leq 2 = \chi(N)$ , so the result trivially holds.

If  $N$  is not  $S^2$ , then  $\chi(N) \leq 0$ . Let  $N_1$  be the cover of  $N$  such that  $\pi_1(N_1) = f_*\pi_1(M)$ . Let  $f_1 : M \rightarrow N_1$  be the lift of  $f$ .

If  $N_1$  is non-compact, then  $H_2(N_1) = 0$  and hence  $f_* : H_2(M) \rightarrow H_2(N)$  is zero, which is a contradiction. Hence,  $N_1$  must be closed.

$N_1$  covers  $N$ , so we must have  $\chi(N_1) \leq \chi(N)$ . Further,  $f_1$  induces a surjection of fundamental groups and hence a surjection  $H_1(M) \rightarrow H_1(N_1)$ . So  $\chi(M) \leq \chi(N_1)$ , and thus,  $\chi(M) \leq \chi(N)$ .  $\square$

**Theorem 2.4.4.** *1.  $F$  is a compact surface with non-empty boundary and  $f : F \rightarrow F$  sends  $\partial F$  to  $\partial F$  by a homeomorphism. Then  $f$  is homotopic*

rel  $\partial F$  to a homeomorphism.

2.  $F$  is a closed orientable surface and  $f : F \rightarrow F$  has degree 1. Then  $f$  is homotopic to a homeomorphism.

*Proof.* 1. If possible, let  $C$  be an essential simple closed curve in  $F$  such that  $f(C)$  is null-homotopic. Then we can perform surgery on  $F$  over  $C$  to get a new surface  $F'$  and a new map  $f' : F' \rightarrow F$  such that  $f'$  also sends  $\partial F'$  to  $\partial F$  by a homeomorphism. Lemma 1.2.7 says that no proper subset of the components of  $\partial F$  can bound  $F$ . Hence, either  $F'$  is connected, or it has two components, one of which is closed. Let  $F_1$  be the non-closed component of  $F'$  and  $f_1 : F_1 \rightarrow F$  be a map sending  $\partial(F_1)$  to  $\partial(F)$  by a homeomorphism. We repeat the process to get a series of connected surfaces with boundary,  $F_1, F_2, \dots, F_n, \dots$ , with  $f_n : F_n \rightarrow F$  sending  $\partial F_n$  to  $\partial F$  by a homeomorphism. Since this construction is done by surgery, the Euler number  $\chi(F_i)$  is strictly increasing and hence the sequence must terminate as Euler number has an upper limit. We get a surface  $F_n$  which has no essential simple closed curve  $C_n$  such that  $f_n(C_n)$  is null-homotopic. Then we can apply theorem 2.3.1 to conclude that  $f_n$  is homotopic to a homeomorphism rel  $\partial(F_n)$ . But  $\chi(F_n) \neq \chi(F)$ , hence contradiction. So there was never such a  $C$ , and we can apply theorem 2.3.1 to conclude that  $f$  is homotopic rel  $\partial F$  to a homeomorphism.

2. If  $F$  is  $S^2$ , this follows directly from theorem 2.3.3. If  $F$  is not  $S^2$ , we consider an essential two-sided embedded non-separating circle  $S$  in  $F$ . We homotop  $f$  to also have geometric degree 1; then  $f_* : \pi_1(F) \rightarrow \pi_1(F)$  is surjective and  $f_* : H_1(N) \rightarrow H_1(N)$  is an isomorphism. By corollary 2.2.9, we can homotop  $f$  further so that it is transverse to  $S$  and  $f^{-1}(S)$  is a single circle mapping to  $S$  by a homeomorphism. We cut  $F$  along  $S$  to obtain the connected surface  $F_1$  (as  $S$  is non-separating). By the classification of compact surfaces, cutting  $F$  along  $C$  gives a surface homeomorphic to  $F_1$ . This gives us a map  $f_1 : F_1 \rightarrow F_1$  which sends  $\partial F_1$  to  $\partial F_1$  by a homeomorphism. By part (1), this is homotopic rel  $\partial F_1$  to a homeomorphism, and hence  $f$  is homotopic to a homeomorphism.  $\square$

**Theorem 2.4.5** (Simple Loop Theorem).  *$M$  and  $N$  are closed surfaces, and  $f : M \rightarrow N$  is a map. If  $f_* : \pi_1(M) \rightarrow \pi_1(N)$  is not injective, then there is an essential simple closed curve  $C$  on  $M$  such that  $f(C)$  is null-homotopic in  $N$ .*

*Proof.* Case I:  $M$  or  $N$  is  $S^2$ .

In this case, the result trivially holds as  $\pi_1(S^2)$  is trivial.

Case II:  $M$  is  $P^2$ .

$P^2$  has only two elements, hence as  $f_*$  is not injective, both these elements are mapped to zero. The non-zero element of  $P^2$  is the required simple closed curve.

Case III:  $N$  is  $P^2$  and  $M$  is not  $S^2$  or  $P^2$ .

$\pi_1(P^2)$  is abelian, so any null-homologous loop in  $M$  lies in the kernel of  $f_*$ .

If  $M$  is not the torus, then it has a separating essential simple closed curve  $C$  (hence, null-homologous). Hence,  $f(C)$  is null-homotopic in  $N$ .

If  $M$  is the torus, then the kernel of any non-trivial homomorphism  $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_2$  contains one of  $(1, 0)$ ,  $(0, 1)$  or  $(1, 1)$ , each of which is given by a simple closed curve according to lemma 1.6.2.

Case IV: All other cases.

We will assume that such a simple closed curve does not exist.

Since  $f$  need not be orientation-true, we will work with geometric degree, which is well defined even in the absence of properly determined orientations.

If  $f$  has geometric degree zero, then  $f(M)$  lies in a subsurface  $N'$  of  $N$  which is obtained by removing a disc from  $N$ . Let  $\Lambda$  be the union of a finite family of disjoint arcs properly embedded in  $N'$  such that cutting along  $\Lambda$  gives a disc.

We homotop  $f$  to be transverse to  $\Lambda$ . Then  $f^{-1}(\Lambda)$  consists of disjoint simple closed curves, each going to a null-homotopic loop in  $N$ . Then according to our assumption, each of the curves in  $f^{-1}(\Lambda)$  is null-homotopic too, hence they bound a disc in  $M$ . Using the same techniques as in the proof of theorem 2.3.1, we can remove all these circle components of  $f^{-1}(\Lambda)$ ; hence  $f^{-1}(\Lambda)$  becomes empty. But then  $M$  gets pushed to one side of  $\Lambda$  and mapped to a disc in  $N$ , which is a contradiction.

If  $f_*(\pi_1(M))$  has infinite index in  $\pi_1(N)$ , then  $f$  lifts to an infinite sheeted cover of  $N$ . Such a cover is homotopy equivalent to a 1-complex, meaning  $f$  is homotopic to a map of geometric degree 0. Hence, the theorem holds in this case.

Next we assume that  $f_*(\pi_1(M))$  has finite index in  $\pi_1(N)$ ,  $\tilde{N}$  is the finite cover of  $N$  corresponding to  $f_*(\pi_1(M))$ , and  $\tilde{f}$  is the lift of  $f$  to this cover. Thus it suffices to prove the theorem for  $\tilde{f}$ , meaning we can assume that  $f_*$  is surjective. We will also assume that  $f$  has the minimum geometric degree in its homotopy class.

Let  $S$  be an essential, two-sided, non-separating, simple closed curve in  $N$ . Then by corollary 2.2.9, we can homotop  $f$  so that  $f^{-1}(S)$  is a single circle  $C$  which maps to  $S$  by a covering map of degree  $G(f)$ . We cut  $M$  along  $C$  and  $N$  along  $S$  to get the surfaces  $M_1$  and  $N_1$ , each having two boundary components, along with the map  $f_1 : M_1 \rightarrow N_1$ , which is a covering map along the boundaries. Again we consider a family  $\Lambda$  of disjoint arcs on  $N_1$  such that cutting along  $\Lambda$  gives a disc. As previously, we can homotop  $f_1$  and remove all the circle components of  $f_1^{-1}(\Lambda)$ . Since  $f_1$  also has non-zero geometric degree, so there is some arc  $\mu$  in  $f_1^{-1}(\Lambda)$ , and  $\lambda$  is an arc in  $\Lambda$  which contains  $f_1 \circ \mu$ . If  $f_1$  maps  $\partial\mu$  to a single point of  $\partial\lambda$ , we can homotop  $f_1 \circ \mu$  rel  $\partial\mu$  to this point.  $f_1$  maps  $\partial\mu$  to one component of  $\partial N_1$ , so  $\partial\mu$  itself lies in a single component  $D$  of  $\partial M_1$ .  $f_1$  is a covering map along boundaries, so  $D$  covers a component of  $\partial N_1$ . Hence a regular neighbourhood of  $D \cup \mu$  is non-orientable and is homeomorphic to the doubly punctured projective plane.

In the original surfaces  $M$  and  $N$ ,  $\mu$  is embedded in  $M$ , so  $\mu \cap C = \partial\mu$ . Further,

$f \circ \mu$  is based homotopic to a point of  $S$ . Hence by lemma 2.1.10, we can homotop  $f$  to a map  $g$  which is transverse to  $S$  and  $g^{-1}(S)$  is obtained by performing surgery on  $f^{-1}(S)$ . Thus  $g$  is a map such that  $g^{-1}(S)$  is a single circle mapping to  $S$  with degree strictly less than that of  $f$ . This contradicts the minimality of geometric degree for  $f$ .

Hence the two endpoints of  $\mu$  must map to the two endpoints of  $\lambda$ , so that  $f$  is homotopic to a homeomorphism. We repeat this for all the arcs in  $f_1^{-1}(\Lambda)$ . Then cutting  $M_1$  along  $f^{-1}(\Lambda)$  and  $N_1$  along  $\Lambda$  gives the surfaces  $M_2$  and  $N_2$  respectively such that the induced map  $f_2$  from  $\partial M_2$  to  $\partial N_2$  is a covering map.  $N_2$  is clearly a disc as that is how we have defined  $\Lambda$ . So by our assumption, each component of  $M_2$  is also a disc.

Next we claim that the boundary of each component of  $M_2$  maps to the circle  $\partial N_2$  by a homeomorphism. If this were not correct, there would be a component  $D'$  of  $M_2$  such that  $\partial D'$  covers  $\partial N_2$  with a degree of more than 1. We take an arc on  $S$  in  $\partial N_2$  and two arcs of  $\partial D'$  which lie in the same copy of  $C$  and map to the same arc of  $S$ . We join these by an arc  $\gamma$  embedded in  $D'$  such that both endpoints of  $\gamma$  map to the same point of  $S$ . Then  $f_1 \circ \gamma$  can be homotoped rel  $\partial\mu$  to this point. Thus we can homotop  $f$  to a new map  $h$  which is transverse to  $S$ , and  $h^{-1}(S)$  is obtained from  $f^{-1}(S)$  by performing surgery along  $\gamma$ . Again, similar to  $g$ , this new map  $h$  contradicts minimality of geometric degree for  $f$ . So we get a contradiction.

As the boundary of each component of  $M_2$  maps to the circle  $\partial N_2$  by a homeomorphism, so we can homotop  $f_2$  rel  $\partial M_2$  to be a homeomorphism on each component of  $M_2$ . This gives a homotopy of  $f$  to a covering map, inducing an injection of fundamental groups. This is a contradiction, so we conclude that the theorem holds.  $\square$

**Lemma 2.4.6.**  *$F$  is a closed orientable surface, not  $S^2$ , and  $f : F \rightarrow F$  is a map of degree  $k > 1$ . Then  $F$  is the torus and  $f$  is homotopic to a covering map of degree  $k$ .*

*Proof.* Case I:  $f$  is  $\pi_1$ -injective.

Then by theorem 2.4.2,  $f$  is properly homotopic to a covering map of degree  $k$ . This gives  $\chi(F) = k\chi(F)$ , ie,  $\chi(F) = 0$  as  $k \neq 1$ . Hence,  $F$  is a torus.

Case II:  $f$  is not  $\pi_1$ -injective.

Then by the Simple Loop Theorem, there is an essential simple closed curve  $C$  in  $F$  such that  $f(C)$  is nullhomotopic in  $F$ . Then we can perform surgery on  $F$  along  $C$  to obtain a surface  $F'$ ; the map  $f' : F' \rightarrow F$  is also of degree  $k$ . As  $F'$  is obtained by performing surgery on  $F$ , so  $\chi(F') \geq \chi(F)$ . This contradicts lemma 2.4.3.  $\square$

## Chapter 3

# Appendix - Group Theory

### 3.1 Amalgamated Free Products

**Definition 3.1.1.** Let  $A, B$  be groups,  $C \subset A, B$ . A reduced word is a sequence  $a_1 b_1 \dots a_n b_n c$  such that  $c \in C$ ,  $a_i$  is in a chosen coset of  $A/C$ ,  $b_i$  is in a chosen coset of  $B/C$  and  $a_i = 1 \implies i = 1, b_i = 1 \implies i = n$ .

**Definition 3.1.2.** The collection of such reduced words as defined above is called the free product of  $A$  and  $B$  amalgamated along  $C$ , denoted by  $A *_C B$ .

**Definition 3.1.3.** The amalgamated free product of  $A$  along  $C$ , denoted by  $A *_C$  is defined as the collection of reduced words of the form  $a_1 a_2 \dots a_n c$ , where  $a_i$  is a coset representative of  $a_i C$  in  $A/C$ .

**Theorem 3.1.4.** The maps  $A \rightarrow A *_C B, B \rightarrow A *_C B$  are injective and each element can be represented by a unique reduced word.

*Proof.* Let  $W$  be the set of reduced words. We define an action of  $A$  on  $W$  as:

$$(a_1 b_1 \dots a_n b_n c)^a = \begin{cases} a_1 \dots b_n a' c' & \text{if } b_n \neq 1, ca = a' c' \text{ with } a' \in T_A \\ a_1 \dots b_{n-1} a'' c'' & \text{if } b_n = 1, a_n ca = a'' c'' \text{ with } a'' \in T_A \end{cases}$$

Next we define an action of  $B$  on  $W$  as

$$(a_1 b_1 \dots a_n b_n c)^b = a_1 \dots a_n b' c', \text{ if } b_n cb = b' c' \text{ in } B \text{ with } b' \in T_B.$$

These define maps  $A \rightarrow W, B \rightarrow W$  which agree on  $C$ , giving the map  $A *_C B \xrightarrow{\phi} W$ .  $\square$

**Theorem 3.1.5.** The map  $A \rightarrow A *_C$  is injective and each element can be represented by a unique reduced word.

*Proof.* The proof is similar to the previous.  $\square$

## 3.2 Grushko's Theorem

**Definition 3.2.1.** Let  $K$  be a based space and  $f : K \rightarrow X$  be a based map. Let  $F$  be a finitely generated free group, and  $\phi : F \rightarrow G$  be an epimorphism (surjective morphism), where  $G = \pi_1(X)$ . We say that  $f$  represents  $\phi$  if there is an isomorphism of  $\pi_1(K)$  with  $F$  such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(K) & \xrightarrow{\cong} & F \\ & \searrow f_* & \downarrow \phi \\ & & G \end{array}$$

**Definition 3.2.2.** Let  $C \leq G$ .  $G$  splits over  $C$  if  $G = A *_C$  or  $G = A *_C B$  for some  $A, B \leq G$  with  $C \subset A, B$

**Definition 3.2.3.** A graph of groups is an abstract connected graph  $\Gamma$  and a function  $\mathcal{G}$  assigning to each vertex  $v$  of  $\Gamma$  a group  $G_v$  and to each edge  $e$  another group  $G_e$ , with  $G_{\bar{e}} = G_e$ , and an injective homomorphism  $f_e : G_e \rightarrow G_{\partial_0 e}$

**Theorem 3.2.4** (Grushko's Theorem). Let  $F$  be a finitely generated free group,  $G = G_1 * G_2$  and  $\phi : F \rightarrow G$  is an epimorphism (surjective morphism). Then  $\exists F_1, F_2 \leq F$  such that  $F = F_1 * F_2$  and  $\phi(F_i) = G_i$ .

*Proof.* We take two CW-complexes with fundamental groups  $G_1$  and  $G_2$  and join them using an interval  $E$ . The new CW-complex  $X$  has  $\pi_1(X) = G_1 * G_2$ . If  $v$  is the midpoint of  $E$ , then we subdivide it so that  $v$  is a vertex of  $E$ . We take  $v$  as the base-point, and denote  $X_i$  as the closure of the component of  $X - \{v\}$  which has fundamental group  $G_i$ .

We take a CW-complex  $K$  such that the map  $f : K \rightarrow X$  represents  $\phi$ . This is done by first taking  $K_0$  to be the wedge of  $n$  circles, where  $n$  is the rank of  $F$ , and picking  $f_0 : K_0 \rightarrow X$  representing  $\phi$  so that  $f_0^{-1}(v)$  is a finite number of 0-cells in  $K_0$ , ie, it is forest with  $\alpha_0$  components. If  $f_0^{-1}(v)$  is a single point, then we are done. Otherwise, we use the following lemma 3.2.5 to obtain the spaces  $K_1, K_2, \dots, K_n$  such that the map  $f_n : K_n \rightarrow X$  represents  $\phi$  and  $f_n^{-1}(v)$  is a tree.

Finally, we let  $L_i = f_n^{-1}(X_i)$  and take  $F_i = \pi_1(L_i)$ . Clearly  $L_1 \cup L_2 = K$  and  $L_1 \cap L_2 = f_n^{-1}(v)$ , which is a tree, and hence simply connected. Hence,  $\pi_1(K) = F_1 * F_2$ . Further,  $f_*(F_i) \subset G_i$ , hence Grushko's theorem follows as  $f$  represents  $\phi$ , and  $\phi$  is an epimorphism.  $\square$

**Lemma 3.2.5.** *Let  $K$  be a based CW-complex and  $f : K \rightarrow X$  be a map representing  $\phi : f \rightarrow G$  such that  $f^{-1}(v)$  is a forest with  $\alpha$  components. If  $\alpha \geq 2$ , then there is a based CW-complex  $K'$  and a map  $f' : K' \rightarrow X$  representing  $\phi$  such that  $f'^{-1}(v)$  is a forest with  $\alpha - 1$  components.*

*Proof.* Let  $l$  be a path in  $K$  joining two distinct components of  $f^{-1}(v)$ . Then we can attach the 1-cell  $e$  to  $\partial l$  and the 2-cell  $B$  to  $e \cup l$  to construct the space  $K'$ . If we can extend  $f : K \rightarrow X$  to  $f' : K' \rightarrow X$ ,  $f'^{-1}(v)$  will have one component less than  $f^{-1}(v)$ . Further,  $f'$  is simply an extension of  $f$ , and  $K'$  deformation retracts to  $K$ , so  $f'$  also represents  $\phi$ . To construct this extension  $f'$ , we require that  $f \circ l$  has an image in  $X_1$  or  $X_2$  and that this image is contractible in  $X$ .

We denote two distinct components of  $f^{-1}(v)$  as  $A$  and  $B$ . Let  $L$  be a path

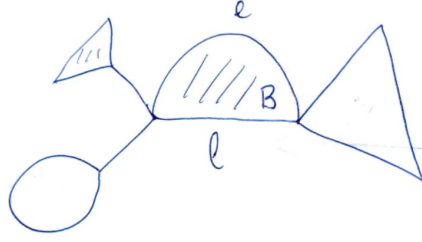


Figure 3.1:

in  $K$  joining them. As  $f_* : \pi_1(K) \rightarrow \pi_1(X)$  is onto so there is a loop  $\gamma$  in  $K$  based at  $L(0)$  such that  $f \circ \gamma$  is homotopic to  $f \circ L$ . We take  $l = \bar{\gamma} * L$ ; then  $f \circ l$  is nullhomotopic in  $X$ .

We can divide  $l$  into  $l_1, l_2, \dots, l_n$  such that the endpoints of each  $l_i$  lie on  $f^{-1}(v)$ ; then  $f \circ l_i$  is a loop in  $X_1$  or  $X_2$ . We can also assume that the  $f \circ l_i$ 's are alternatively in  $X_1$  or  $X_2$ . (If they are not, we combine  $l_i$  and  $l_{i+1}$  both lying in the same  $X_i$  to be a single component; then the new choice of  $l_i$ 's is as required.) We define  $n$  to be the length of  $l$ .

Let  $g_i = [f \circ l_i] \in \pi_1(X, v)$ . Let  $l_j$  be such that  $g_j = 1$  and both endpoints of  $l_j$  lie in the same component of  $f^{-1}(v)$ . Then we can alter  $l$  to  $l'$  by replacing  $l_j$  with a path  $l'_j$  in  $f^{-1}(v)$  which joins the endpoints of  $l_j$ . As this is contractible,  $l'$  has length less than  $n$ . We repeat the process till there are no components  $l_i$  of  $l$  with both endpoints on the same component of  $f^{-1}(v)$  and having trivial  $g_i$ .

Now we have the path  $l = l_1 l_2 \dots l_r$  which becomes  $1 = g_1 g_2 \dots g_r$  in  $\pi_1(X)$ . The  $g_i$ 's lie alternatively in  $G_1$  and  $G_2$ . Further,  $G = G_1 * G_2$ , so  $G_1 \cap G_2 = 1$ . So for the  $g_i$ 's to cancel out and give 1, we must have some  $g_i = 1$ . The corresponding  $l_i$  joins two distinct components of  $f^{-1}(v)$  and contractible  $f \circ l_i$ . This is the required path.  $\square$

**Corollary 3.2.6.** *If  $G = G_1 * G_2$  and  $\mu(G)$  denotes the minimum number of generators of  $G$ , then  $\mu(G) = \mu(G_1) + \mu(G_2)$ .*

*Proof.*  $\mu(G) \leq \mu(G_1) + \mu(G_2)$  is obvious.

From Grusko's theorem,  $\phi(F_i) = G_i$ , so  $\mu(G_i) \leq \mu(F_i)$ . If we take  $F$  to be the free group on  $n$  generators ( $n = \mu(G)$ ), then  $\mu(G_1) + \mu(G_2) \leq \mu(F_1) + \mu(F_2) = \mu(G)$ .

Hence, by combining the two inequalities, we get the desired result.  $\square$

**Corollary 3.2.7.** *If  $G$  is a finitely generated group, then  $G = G_1 * \dots * G_n$  for some  $n$ , where each  $G_i$  is indecomposable; ie,  $G_i = A * B \implies$  either  $A$  or  $B = 1$ .*

# References

1. *An Introduction to 3-Manifolds* by G Peter Scott (primary reference)
2. *Introduction to 3-Manifolds* by Jennifer Schultens (for looking up certain topics)
3. *Riemann Surfaces* by Simon K Donaldson (for classification theorem of surfaces)