

IISER Pune
MTH402 Project Report
Introduction to 3-Manifolds

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Summary

To study the following: irreducibility of a manifold, normal surfaces, their construction and finiteness, normal immersions, barriers, smooth and PL-areas, the Disc theorem, the Sphere theorem and its generalizations

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Chapter 1

Normal Surfaces in 3-manifolds

1.1 Assumptions and Preliminary Definitions

- Boundaries have collars.
- A regular neighbourhood of an embedded circle in a 3-manifold is homeomorphic to the solid torus ($S^1 \times D^2$) or to the solid Klein bottle ($S^1 \tilde{\times} D^2$).
- If M is a 3-manifold with a given disc D in its boundary, the union of M with $D^2 \times I$ obtained by gluing D to $D \times \{0\}$ is homeomorphic to M .

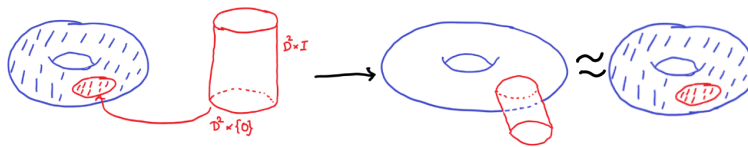


Figure 1.1:

Definition 1.1.1. A 3-manifold M is called *irreducible* if any 2-sphere embedded in M bounds a 3-ball in M .

Definition 1.1.2. A 3-manifold M is called prime if whenever $M = M_1 \# M_2$, then either M_1 or M_2 is S^3 .

Lemma 1.1.3. Any irreducible manifold is prime.

Proof. Let M be an irreducible manifold. Say $M = M_1 \# M_2$. This connected sum is obtained by removing a ball from M_1 and M_2 , and then gluing the resulting 2-spheres together. These 2-spheres form a 2-sphere in M . As M is irreducible, so the 2-sphere bounds a ball. Now if we look at M_1 and M_2 by undoing the gluing, this either of the two is formed by adding this 3-ball to it, and hence is S^3 . Hence, M is prime. \square

Theorem 1.1.4. \mathbb{R}^3 is irreducible.

Proof. Let S be a PL embedded 2-sphere in \mathbb{R}^3 . We triangulate \mathbb{R}^3 using linear simplices such that S is a subcomplex. A direction θ in \mathbb{R}^3 is called good if projection along a line parallel to θ induces an injection of the vertices of S .

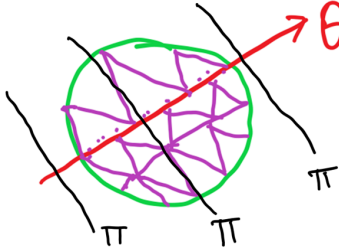


Figure 1.2:

Π is a plane perpendicular to θ . If no vertex of S lies in Π , then $S \cap \Pi$ is either empty or is a closed 1-manifold. If this is not the case, then Π contains exactly one vertex of S so that $S \cap \Pi$ is a 1-complex with at most one non-manifold point, which is either isolated or the whole of $S \cap \Pi$. Π is called singular if it is neither a 1-manifold nor a single point. Further, we define $n(S, \Pi)$ to be the rank of $H_1(S \cap \Pi)$ and $n(S, \theta)$ to be $\sum n(S, \Pi)$, summing over all the singular planes perpendicular to θ . Finally, the complexity $n(S)$ is defined as the minimum value of $N(S, \theta)$, with the minimum being taken over all good directions.

We will prove the theorem by induction on $n(S)$. Suppose $n(S) = 0$. We choose some direction θ for which $n(S, \theta)$ is zero. For any plane Π perpendicular to θ , $\Pi \cap S$ is either a single point, or a circle or is empty. Without loss of generality, we assume that θ is the vertical direction. S bounds a compact 3-manifold B which is the union of the discs in the horizontal planes bounded by the circle

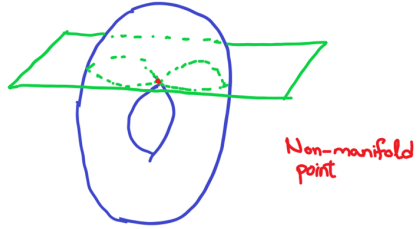


Figure 1.3:

of intersection of S with that plane. We choose a collection of planes $\Pi_1, \Pi_2, \dots, \Pi_{2n+1}$ perpendicular to θ such that Π_{i+1} is above Π_i and the vertices of S lie along the odd-numbered planes. Let R be a region between two consecutive planes; then it is a union of pieces of tetrahedra from the chosen triangulation of \mathbb{R}^3 . Each of these pieces B_j is obtained from a tetrahedron by slicing it by two parallel planes, and for one B_j one of these planes passes through a vertex. So the B_j 's are polyhedral convex subsets of \mathbb{R}^3 ; hence they are all 3-balls. Let Π be some plane not through a vertex and B_j . Using the lemma after this theorem, we re-arrange the B_j 's forming R such that if $Y_r = \cup_{j=1}^r B_j$, then Y_r is a disjoint union of 3-balls and B_{r+1} meets each component of Y_r in a 2-disc or the empty set. Then as R is formed from gluing 3-balls along discs on their boundaries, it is also a 3-ball. Further, B is the union of such regions R glued along discs on their boundaries, so B is also a 3-ball, and the basic step of the induction holds.

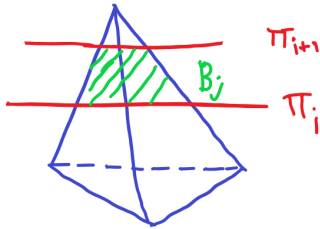


Figure 1.4:

Now we move to the induction step. Let S be a 2-sphere in \mathbb{R}^3 that has non-zero complexity $n(S)$. We also assume that any other 2-sphere with smaller

complexity will bound a 3-ball. Now let θ be a good direction with $n(S, \theta) = n(S)$. Now we want to choose Π such that the non-manifold point of $\Pi \cap S$ is not isolated. To do this, we choose some plane Π disjoint from S and slide it in the direction θ towards S . At some point, $\Pi \cap S$ becomes a single point and immediately after that it becomes a circle C . We continue along this till C either becomes an isolated point again, or a connected 1-complex containing a non-manifold point. If C becomes a single point, then S is a sphere, and must have $n(S) = 0$, which is a contradiction. Hence, C becomes a connected 1-complex at some singular plane Π' where the non-manifold point is not isolated.

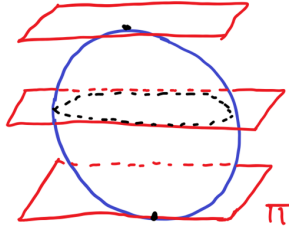


Figure 1.5:

Now we take a singular plane Π such that the non-manifold point v is not an isolated point of $\Pi \cap S$. We take a circle C which is such that the disc D in Π bounded by C intersects $\Pi \cap S$ only in C .

Case I: v lies on C

Let C cut S into the discs D_1 and D_2 . Let $S_i = d_i \cup D$ for $i = 1, 2$. Then S_1 and S_2 are embedded 2-spheres in \mathbb{R}^3 and $S_1 \cap S_2 = D$. We claim that S_i bounds a 3-ball in \mathbb{R}^3 .

First, we need to perform an isotopy on S so that D is a strictly convex subset of Π . All circles of Π are ambient isotopic, so we can ambiently isotop C to be the boundary of a convex disc, and then extend this ambient isotopy to the whole of \mathbb{R}^3 so that each plane parallel to Π is preserved. The image of S under this isotopy is denoted S' . $S' \cap P$ is homeomorphic to $S \cap P$ for each plane P parallel to Π . So $n(S') \leq n(S', \theta) = n(S, \theta) = n(S)$. If $n(S') < n(S)$, then by the induction hypothesis, S' bounds a 3-ball in \mathbb{R}^3 , and hence so does S . So we consider only the non-trivial case of $n(S') = n(S)$. Further, we can assume that S is a convex disc while dealing with this case.

Both S_1 and S_2 contain the disc D along Π , so θ cannot be a good direction as the vertices of the triangulation of D do not inject onto it. So we will have

to choose a new good direction θ_i for each S_i . First, we assume WLOG that Π is the horizontal direction, and that S_1 lies below Π and S_2 lies above it. D is strictly convex, so we find a straight line L in Π which intersects D in the non-manifold point v . Then we slightly rotate \mathbb{R}^3 about L so that the points of Π on the side of D move up. The image of θ after this rotation is denoted as θ_1 and the image of Π as Π_1 . The circle C will not appear in $\Pi_1 \cap S_1$, so $n(S_1, \Pi_1) < n(S, \Pi)$. Similarly for any other plane P parallel to Π and passing through a vertex of the triangulation of S , we let P_1 be the plane perpendicular to Π_1 and passing through the same vertex. Then $P \cap S$ is homeomorphic to $P_1 \cap S$ by rotation through a small angle. Thus, if the rotation is through a sufficiently small angle, then as C does not occur in $H_1(S \cap \Pi)$ and we do not add any new circles, so $n(S_1, \theta_1) < n(S, \theta) = n(S)$, giving $n(S_1) < n(S)$. Similarly, via rotation in the other direction about L and using the same arguments, we get $n(S_2) < n(S)$.

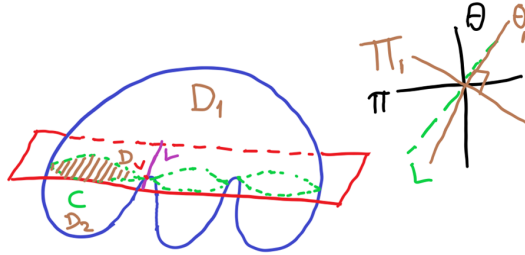


Figure 1.6:

So both S_1 and S_2 bound balls B_1 and B_2 . If $B_1 \cap B_2 = D$, then S bounds the ball $B_1 \cup B_2$. Otherwise, either $B_1 \subset B_2$ or vice versa. Then S bounds the 3-ball $B_1 - B_2$ or $B_2 - B_1$.

Case II: v does not lie on C

In this case, v will lie on exactly one of S_1 and S_2 ; WLOG, we assume it lies on S_2 . As in the previous case, we will assume that Π is horizontal, and that S_1 lies below Π . Again, we let L be a straight line which meets D at a single point, so that D lies on one side of L . Now v does not lie on S_1 , so $n(S_1) < n(S)$ and hence S_1 bounds a ball. Hence we can isotop D_1 across this ball to D and then off Π upwards. Let the resulting sphere be S' . Then $n(S', \Pi) = n(S, \Pi) - 1$, as $S' \cap \Pi$ is $S \cap \Pi$ with C removed. Hence, $n(S', \theta) < n(S, \theta) = n(S)$, so S' bounds a ball. But S' is isotopic to S , so S also must bound a ball. This completes the induction step. \square

Lemma 1.1.5 (Shelling). *If the 2-disc D is expressed as the union of the 2-discs D_j , $1 \leq j \leq n$, such that the interiors of the D_j 's are disjoint, then we*

can re-order the D_j 's so that if X_k denotes the union of the first k of the D_j 's then X_k has each component a 2-disc and further D_{k+1} meets each component of X_k in one arc or in the empty set.

Proof. We will prove the lemma by using induction on n . For $n = 1$, the result holds trivially.

As D is the union of D_j 's, so some D_j must meet ∂D in an arc; we take this to be our D_n . Now the components of $\overline{D - D_n}$ are all discs and D_n meets each of these components in an arc, so X_{n-1} satisfies all the conditions of the lemma. Hence we can apply the induction hypothesis to each component of X_{n-1} to obtain an ordering of the D_j 's in each component. \square

Lemma 1.1.6. *If M is a compact prime 3-manifold, then M is either irreducible or it is homeomorphic to one of the two bundles over S^1 with fiber S^2 . Further each of these bundles is prime.*

Proof. M is prime, hence any separating 2-sphere embedded in M bounds a 3-ball. Thus, M is either irreducible or it has a non-separating 2-sphere S embedded in its interior. We take a regular neighbourhood $V = S^2 \times I$ of S . Let Γ be an embedded path in $M - V$ joining a point on $S^2 \times \{-1\}$ to a point on $S^2 \times \{1\}$. Then the connected sum of $S^2 \times \{-1\}$ and $S^2 \times \{1\}$ along the boundary of a regular neighbourhood U of Γ is a sphere S' which bounds $U \cup V$ in M . $U \cup V$ is not simply connected, hence $M = U \cup V \cup D^3$ and it is homeomorphic to one of the two bundles over S^1 with fiber S^2 .

So it only remains to show that these bundles are prime. Let M denote one these bundles, and Σ denote a separating 2-sphere in M . $\pi_1(M)$ is infinite cyclic, so one of the pieces, denoted X , into which Σ separates M is simply connected. X lifts to the universal cover \tilde{M} of M . But \tilde{M} is homeomorphic to $S^2 \times \mathbb{R}$, so it embeds in \mathbb{R}^3 as a neighbourhood of an embedded sphere. This induces an embedding of X in \mathbb{R}^3 , but \mathbb{R}^3 is irreducible, so X is a 3-ball. \square

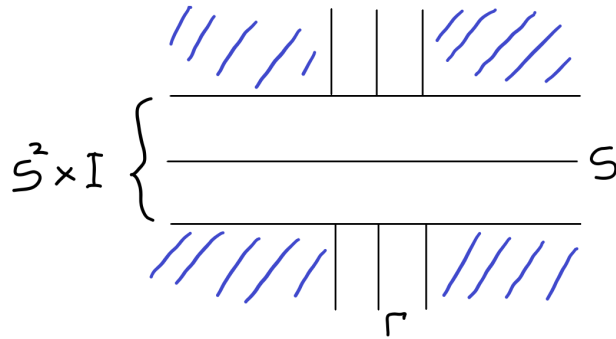


Figure 1.7:

Remark 1.1.7. The two bundles over S^1 with fiber S^2 are obtained by taking $S^2 \times I$ and gluing $S^2 \times \{0\}$ to $S^2 \times \{1\}$ by a homeomorphism. But any homeomorphism of S^2 is isotopic to either the identity or to a reflection. Hence there are two possible bundles: the orientable manifold $S^2 \times S^1$ and a non-orientable manifold $S^1 \tilde{\times} D^2$.

Remark 1.1.8. In all the following definitions and proofs, we fix a triangulation of the 3-manifold M and do NOT change it.

Definition 1.1.9. A tetrahedron is a 3-simplex in the given/chosen triangulation.

Remark 1.1.10. M is generally compact, hence the triangulation has only finitely many simplices. Non-compact manifolds will mostly be encountered as coverings of compact manifolds.

Definition 1.1.11. A surface F is called transverse in the triangulation of a 3-manifold if it meets each simplex transversely. It does not meet any vertices, meets each 1-simplex in finitely many points, meets each 2-simplex in a compact 1-manifold, and meets each 3-simplex in a compact 2-manifold.

Remark 1.1.12. Any 3-simplex can be given a linear structure obtained from an embedding in \mathbb{R}^3 as the convex hull of four non-coplanar points.

Definition 1.1.13. Let τ be a 3-simplex in \mathbb{R}^3 and Π be a plane meeting τ but not containing any of its vertices. The Π meets τ in a triangle or quadrilateral, which is called a plane triangle or plane quadrilateral respectively.

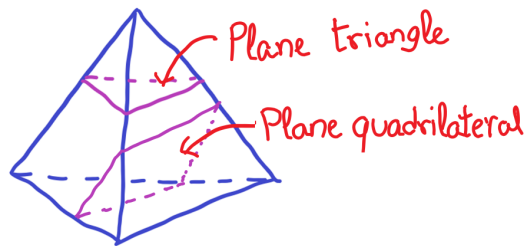


Figure 1.8:

Remark 1.1.14. Given a vertex v of a 3-simplex τ in \mathbb{R}^3 , choosing a point on each of the three edges of τ which end at v determines a plane triangle in τ . Given two vertices v_1 and v_2 of τ , choosing a point on each of the edges v_1v_3 , v_2v_3 , v_1v_4 and v_2v_4 determines a simple closed curve with four edges lying on the boundary of τ . To get a plane quadrilateral from this, we let e be the centre of mass of the four points and take the required spanning disc to be the cone on the circle with cone point e . This quadrilateral consists of four plane triangles.

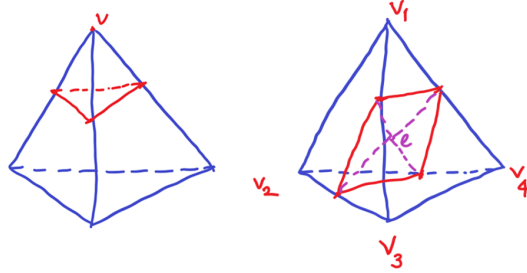


Figure 1.9:

Definition 1.1.15. A disc in τ which is either a plane triangle or the canonical disc for four points on the edges of τ as constructed in remark 1.1.14 above is called a standard disc, or a standard triangle or standard quadrilateral, as appropriate.

Definition 1.1.16. Two surfaces properly embedded in τ such that neither meets a vertex of τ and the boundary of each is transverse to the 1-skeleton of τ are called normally isotopic if they are isotopic through such surfaces. The number of points of intersection of the surface with each edge of τ must be constant during such an isotopy.

Definition 1.1.17. Any disc in τ which is normally isotopic to a standard disc is called a normal disc, or a normal triangle or normal quadrilateral, as appropriate.

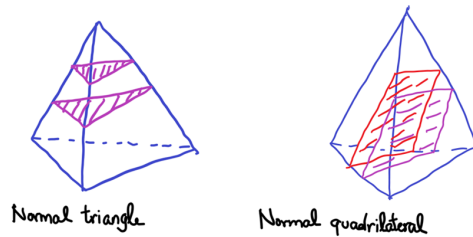


Figure 1.10:

Remark 1.1.18. There are four normal isotopy classes of normal triangles and three normal isotopy classes of normal quadrilaterals, giving seven normal isotopy classes of normal discs.

Definition 1.1.19. An arc properly embedded in a 2-simplex is normal if its endpoints lie on the interiors of distinct edges.



Figure 1.11:

Definition 1.1.20. An embedded surface Σ in M is normal if Σ is transverse to the triangulation of M and, for each tetrahedron τ of M , $\Sigma \cap \tau$ consists of a disjoint union of finitely many normal discs.

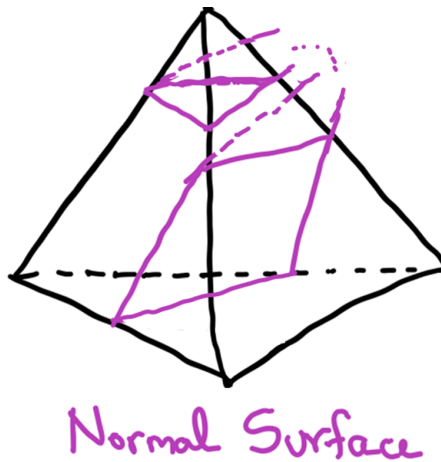


Figure 1.12:

Definition 1.1.21. Two surfaces properly embedded in M and transverse to the triangulation of M are normally isotopic if they are isotopic through such surfaces. The number of points of intersection of the surface with each edge of M must be constant during such an isotopy.

Definition 1.1.22. A map $f : X \rightarrow Y$ of manifolds is called proper if $f^{-1}(\partial Y) = \partial X$ and for any compact subset C of Y , $f^{-1}(C)$ is also compact.

Definition 1.1.23. Let C be a simple closed curve on the boundary of the tetrahedron τ such that C meets the 1-skeleton of τ transversely. The length of C is defined as the number of points of $C \cap \tau^{(1)}$, where $\tau^{(1)}$ is the 1-skeleton of τ .

Lemma 1.1.24. Let C be a simple closed curve on the boundary of a tetrahedron τ such that C meets the 1-skeleton of τ transversely and meets every 2-simplex σ of τ in normal arcs. Then C bounds a normal disc in τ iff it has length at most 4.

Proof. First, say C bounds a normal disc. This is either a normal triangle or a normal quadrilateral. Hence, the length of C can be 3 or 4.

Conversely, let C be a simple closed curve on $\partial\tau$ of length at most 4. Then C must bound a disc D properly embedded in τ as it is simple closed. Hence, C must have length 3 or 4, as it cannot bound a disc if it has length 1 or 2. That is, $C \cap \tau^{(1)}$ cuts C into 3 or 4 arcs. If we consider one of these arcs, all such are equivalent up to normal isotopy under the action of the full symmetry group of τ . On inspection, we realize there is only one possibility for C if it has three such arcs (ie, a normal triangle) and only one possibility for it if it has four such arcs (ie, a normal quadrilateral). Thus, D is a normal disc. \square

Lemma 1.1.25. Let X be any space and let α denote an element of $H_2(X)$. Then there is a closed orientable surface F , not necessarily connected, and a map $f : F \rightarrow X$ such that $f_*([F]) = \alpha$, ie, f represents α .

Proof. Using singular homology theory, we obtain a singular cycle σ which represents α . σ can be written as $\sum \epsilon_i \sigma_i$, where each σ_i is a singular 2-simplex in X , and $\epsilon_i = 1$ or -1 and the summation is finite. Let N be the number of σ_i 's. σ is a cycle, so we can pair up all the singular 1-simplices in the boundary of the σ_i 's such that they cancel out in $\sum \epsilon_i \sigma_i$. We get a surface F and a map $f : F \rightarrow X$ by taking N 2-simplices τ_i each mapped into X by σ_i and gluing the boundary edges of the τ_i 's in pairs as above. This construction yields a closed orientable surface whose fundamental class is mapped by f_* to α . \square

Lemma 1.1.26. Let M be a 3-manifold and α be an element of $H_2(M)$. Then there is a closed orientable surface F , not necessarily connected, and an embedding of F in M which represents α .

Proof. From the previous lemma, we get a closed orientable surface F and a map $f : F \rightarrow M$ which represents α . All that remains to be done is to alter F and f , without altering the homology class $f_*([F])$, to get an embedding. We assume that F and M are triangulated, so that f is simplicial. We then homotope f so that the restriction of f to any simplex of F is injective, possibly by a subdivision of M and F . But this could destroy the fact that f is simplicial. Hence we go case-by-case.

Case I: 2-simplices of F .

Say the 2-simplices σ and τ of F have the same image. Then we homotope f on the interior of one of them so as to arrange that their interiors in the image are disjoint.

Case II: 1-simplices of F .

Let the two 1-simplices e and e' of F have the same image. Let x and x' be points in the interiors of e and e' such that $f(x) = f(x')$. Let U and U' be small neighbourhoods of x and x' in F ; f embeds both U and U' in M . If f maps them such that they do not cross at x , then we homotope f on the interior of one of e and e' so that their interiors become disjoint, while also preserving disjointness of the interiors of the 2-simplices of F .

If f maps U and U' such that they cross at x , then we first perform cut-and-paste along the common image E of e and e' in M , to get a new surface F_1 and a new map $f_1 : F_1 \rightarrow M$ which also represents α but does not have a crossing along E . Then we homotope f_1 so that the two 1-simplices with image E no longer meet in their interiors.

Thus we have arranged that f is injective except possibly on the vertices of F . Say f maps the vertices v and v' of F to the same point of M . If U and U' are small neighbourhoods of v and v' respectively, as f is injective away from the vertices of F , so $f(U)$ and $f(U')$ intersect only at $f(v)$. Thus performing a small homotopy of f makes it embed F . \square

1.2 Finiteness for Normal Surfaces

The aim of this section is to find a result similar to theorem 2.0.2, using normality instead of nullhomotopicity.

Definition 1.2.1. *Two surfaces F and F' in M are called parallel if they bound a submanifold X in M such that X is homeomorphic to $F \times I$.*

Bound on the number of disjoint one-sided surfaces in a compact 3-manifold

Lemma 1.2.2. *1. If F is a connected surface properly embedded in a 3-manifold M such that F is one-sided in M , then there is a surjective homomorphism $\pi_1(M) \rightarrow \mathbb{Z}_2$ such that the composite map $\pi_1(F) \rightarrow \pi_1(M) \rightarrow \mathbb{Z}_2$ is also surjective.*

*2. If F_1, \dots, F_n are disjoint properly embedded connected surfaces in a 3-manifold M such that each is one-sided in M , then there is a surjective homomorphism $\pi_1(M) \rightarrow *_{i=1}^n \mathbb{Z}_2$ so that for each of the projection maps $p_i : *_{i=1}^n \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ the composite map $\pi_1(F_i) \rightarrow \pi_1(M) \rightarrow \mathbb{Z}_2$ is also surjective.*

Proof. 1. Let N be a regular neighbourhood of F in M , and let X be the space obtained from N by identifying the frontier of N in M , denoted by

S , to a single point x_0 . $\pi_1(N)$ is isomorphic to $\pi_1(F)$, and the image of $\pi_1(S)$ in $\pi_1(N)$ has index 2. Hence, $\pi_1(X)$ is isomorphic to \mathbb{Z}_2 . We define $f : M \rightarrow X$ by using the identification map $N \rightarrow X$ and sending $M - X$ to x_0 . $Id : N \rightarrow X$ maps $\pi_1(N)$ onto $\pi_1(X)$ and $i : F \hookrightarrow M$ maps $\pi_1(F)$ isomorphically to $\pi_1(N)$. Thus $f_* : \pi_1(M) \rightarrow \pi_1(X)$ and the composite map $\pi_1(F) \rightarrow \pi_1(M) \rightarrow \pi_1(X)$ are both surjective.

2. Similar to the above, let N_i be a regular neighbourhood of F_i in M and X_i be the space obtained from N_i by identifying the frontier of N_i in M , denoted S_i , to a single point x_i . Let X be the space obtained by identifying all the x_i 's to a single point x_0 . $f : M \rightarrow X$ is defined by using identification maps $N_i \rightarrow X_i$ and sending $M - \cup N_i$ to x_0 . From part (1), $\pi_1(X) \cong \ast_{i=1}^n \mathbb{Z}_2$ and the composite map $\pi_1(F_i) \rightarrow \pi_1(M) \rightarrow \pi_1(X)$ is surjective. The projection map $p_i : X \rightarrow X_i$ is surjective, so $\pi_1(F_i) \rightarrow \pi_1(M) \rightarrow p_{i1}(X) \rightarrow p_{i1}(X_i)$ is surjective. This implies that $f_* : \pi_1(M) \rightarrow p_{i1}(X)$ is also surjective.

□

Corollary 1.2.3. *If M admits n disjoint properly embedded connected surfaces such that each is one-sided in M , then $n \leq \text{rank} H_1(M, \mathbb{Z}_2)$.*

Proof. $\ast_{i=1}^n \mathbb{Z}_2$ abelianises to $\times_{i=1}^n \mathbb{Z}_2$, so applying this to part (1) of the lemma says there is a surjective homomorphism $H_1(M, \mathbb{Z}_2) \rightarrow \times_{i=1}^n \mathbb{Z}_2$. □

Bound on the number of two-sided non-separating disjoint surfaces

Lemma 1.2.4. *Let F_1, \dots, F_n be disjoint properly embedded connected surfaces in a 3-manifold M such that each is two-sided in M , and suppose that $M - \cup_{i=1}^n F_i$ has r components. Then $r \geq n + 1 - \text{rank} H_1(M, \mathbb{Z})$.*

Proof. We construct a graph Γ embedded in M such that it has a vertex v_j for each component X_j of $M - \cup_{i=1}^n F_i$ and an edge e_i for each F_i . v_j is placed in X_j , and e_i is embedded such that it crosses F_i transversely at one point and does not cross any other F_j . So e_i joins vertices corresponding to components of $M - \cup_{j=1}^n F_j$ that meet F_i . Some such embedding of Γ in M is chosen and denoted by g

Next we define $f : M \rightarrow \Gamma$. Let C be a collar of F , the union of the F_i 's, C_i be the component of C which is a collar of F_i and Y_j be the collar of $X_j - C$. Y_j is mapped to v_j and C_i is mapped to e_i by identifying C_i with $F_i \times I$ and projecting C_i onto the I factor which is identified with e_i . The composite map $f \circ g : \Gamma \rightarrow \Gamma$ is homotopic to identity. Hence, $f_* : \pi_1(M) \rightarrow \pi_1(\Gamma)$ is onto, and so is $f_* : H_1(M) \rightarrow H_1(\Gamma)$. Γ has n edges and r vertices, so $\chi(\Gamma) = r - n$. Hence, $H_1(\Gamma) = 1 - (r - n) = 1 - r + n$. Thus $\text{rank} H_1(M) \geq 1 - r + n$. □

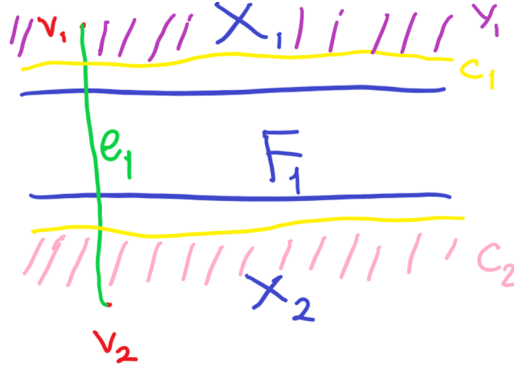


Figure 1.13:

Lemma 1.2.5. *Let M be a compact triangulated 3-manifold. There is a number $k(M)$ such that if Σ is a normal surface properly embedded in M with more than $k(M)$ components, then two components Σ are parallel.*

Proof. Let N be the number of 3-simplices in M , β_1 be rank $H_1(M, \mathbb{Z})$ and β_2 be rank $H_1(M, \mathbb{Z}_2)$. We will show that $k(M) = 6N + \beta_1 + 2\beta_2 - 1$.

Σ cuts τ into some regions, and we call such a region as a product region if it is bounded by two components of $\Sigma \cap \tau$ which are in the same normal isotopy class. We can have at most 6 non-product regions.

Say Σ has n components. By corollary 1.2.3, at most β_2 of them can be one-sided in M . The remaining must cut M into at least $(n - \beta_2) - \beta_1 + 1$ components, according to lemma 1.2.4. If Σ cuts M into r regions such that $r > 6N$, then at least $r - 6N$ regions made entirely of product regions in the tetrahedra of M will be present. Each such region will be homeomorphic to a possibly twisted I -bundle over some surface obtained by taking the union of the middle regions in each product region. If $r - 6N > \beta_2$, there will be some region homeomorphic to a trivial I -bundle. That is, two components of Σ will be parallel under this condition. This is equivalent to having $n > 6N + \beta_1 + 2\beta_2 - 1$. \square

Corollary 1.2.6. *Let M be a compact triangulated 3-manifold. There is a number $k(M)$ such that if Σ is a normal surface properly embedded in M with r components and $r > 6N$, then at least $r - k(M)$ components of the closure of $M - N(\Sigma)$ are trivial I -bundles.*

1.3 Constructing Normal Surfaces

Definition 1.3.1 (Surgery on an embedded surface along a circle). *Let F be a surface properly embedded in a 3-manifold M and suppose that there is a circle*

C in F which bounds a disc D in M such that $D \cap F = C$. Assume also that either D lies in the interior of M or that D lies in ∂M . Let N denote a regular neighbourhood of D in M relative to F . Thus N intersects F in a regular neighbourhood A of C in F and A must be an annulus. Let B denote $\text{fr}_M(N) - A$, and let F' denote the surface which is the closure of $F \cup B - A$. Thus F' is obtained from F by removing an annulus and adding either one or two discs, according to whether D lies in ∂M or not. We say that F' is obtained from F by performing surgery on F along C using the disc D .

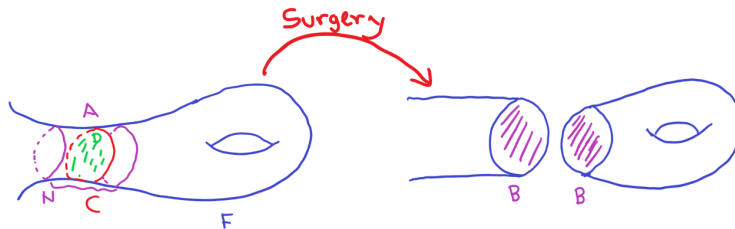


Figure 1.14:

Definition 1.3.2 (Surgery on an embedded surface along an arc). Let F be a surface properly embedded in a 3-manifold M . Suppose that there is an arc λ properly embedded in F , an arc μ embedded in ∂M with $\partial\lambda = \partial\mu$, and a disc D embedded in M such that $\partial D = \lambda \cup \mu$, $D \cap F = \lambda$ and $D \cap \partial M = \mu$. Let N denote a regular neighbourhood of D in M relative to F . Thus N intersects F in a regular neighbourhood of λ in F . Let F' denote the surface which is the closure $F \cup \text{fr}_M(N) - (\partial N \cap F)$. Thus F' is obtained from F by removing a collar neighbourhood of λ and adding two discs. We say that F' is obtained from F by performing surgery on F along λ using the disc D .

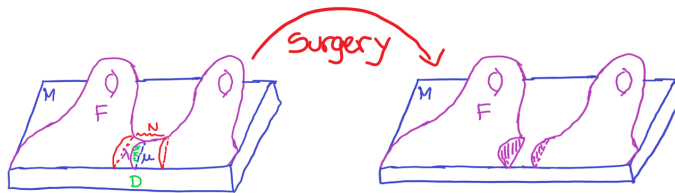


Figure 1.15:

Lemma 1.3.3. *F and F' (obtained from F by performing some sort of surgery) have different topological types, but both represent the same homology class in M .*

Proof. A compact surface F has a fundamental class represented by the 2-cycle in F obtained by triangulating F and taking the sum of all the 2-simplices of F ; abusing notation, we can assume that F itself is a 2-cycle. If F is orientable, this cycle represents a generator of the infinite cyclic group $H_2(F, \partial F; \mathbb{Z})$. Further, in general, it represents a non-zero element of $H_2(F, \partial F; \mathbb{Z}_2)$. If F is closed, we replace the relative homology groups by absolute groups $H_2(F)$ and $H_2(M)$ with appropriate coefficients, and denote the appropriate class by $[F]$. Let $i : F \hookrightarrow M$ be the inclusion map. Then the class $i_*[F]$ in $H_2(M, \partial M)$ or $H_2(M)$ with \mathbb{Z} or \mathbb{Z}_2 coefficients (as appropriate) is said to be represented by F . Say F' is obtained by performing surgery on F along C using a disc D . Then $i_*([F'] - [F])$ is represented by the 2-cycle $F' - F$. F' is $F \cup B - A$, so $F' - F = B - A$. This is the boundary of N and hence is a null-homologous 2-cycle in M . Hence, $i_*([F'] - [F]) = 0$, ie, $i_*([F']) = i_*([F])$. We can argue similarly for the case when F' is obtained from F by surgery along an arc. \square

Replacing any compact properly embedded surface by a normal surface

We fix a triangulation of M . We will perform a sequence of moves to obtain the required normal surface, preserving the transversality of F to the triangulation (which is obtained in move 0 of the process).

We define w , the weight of F , to be the number of points of intersection of F with $M^{(1)}$, the 1-skeleton of M . For each 2-simplex σ , u_σ is defined to be the number of components of $F \cap \sigma$, and $u = \sum u_\sigma$. For each 3-simplex τ , v_τ is the number of components of $F \cap \partial\tau$ which do not bound a disc component of $F \cap \tau$, and $v = \sum v_\tau$. n is defined to be the number of components of F .

The complexity $c(F)$ of a compact surface F embedded in M transversely to the given triangulation is defined to be the quadruple (w, u, v, n) , and the ordering on these quadruples is lexicographical. Our aim will be to reduce the complexity of F as far as possible to get our required normal surface.

Move 0: We isotop F to be transverse to the triangulation of M .

Move 1: Say there is a 2-simplex σ such that $F \cap \sigma$ contains a circle. We choose a circle C in $F \cap \sigma$ which is innermost in σ , and let D be the disc bound by C in the interior of σ . Performing surgery on F along C using D removed C from $F \cap \sigma$ and reduces u , while keeping w unchanged. If σ lies in ∂M , this move removes a circle from ∂F .

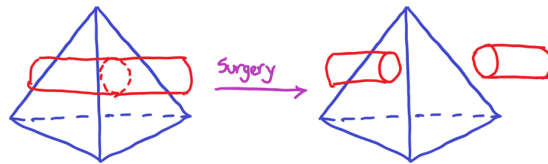


Figure 1.16:

Moves 2 and 2b: σ is a 2-simplex such that $F \cap \sigma$ contains an arc λ such that $\partial\lambda$ lies on one edge e of σ and $F \cap \sigma$ contains no circles (all these are removed by repeatedly performing move 1). We assume that the arc λ is innermost in σ , so that the disc D in σ bound by λ and a sub-arc μ of e meets no other component of $F \cap \sigma$.

Move 2: We isotop λ across D just beyond μ , and extend this to an isotopy of F supported on a small neighbourhood λ . This reduces w by at least 2.

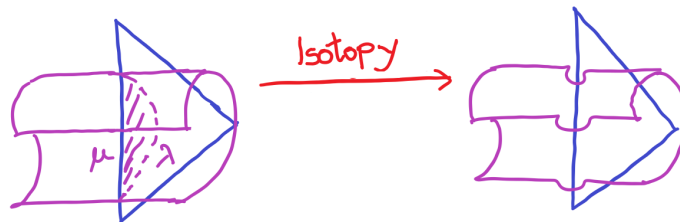


Figure 1.17:

Move 2b: If e lies in ∂M but σ does not, then perform surgery on F along λ using D . This too reduces w by at least 2.

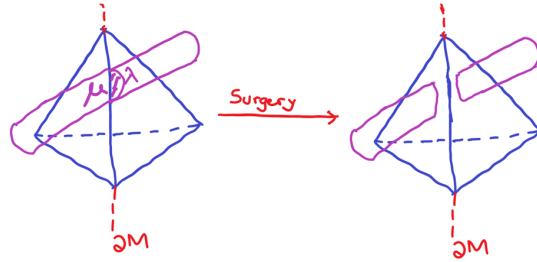


Figure 1.18:

Move 3: Let τ be a 3-simplex such that $F \cap \tau$ is not a disc. Let C be a boundary component of X , ie, C is a simple closed curve in $\partial\tau$ and bounds a disc D in $\partial\tau$. If D is the innermost of all such discs, any component of $F \cap \partial\tau$ which lies in D bounds a disc component of $F \cap \tau$. Let C' be a circle in the interior of X parallel to C , and D'' be the disc made up by the union of D and the collar in X between C and C' . Let D' be a disc obtained in τ from D'' by pushing D'' off $X \cup \partial\tau$ while fixing $\partial D' = C'$. Thus C' bounds D' , which is embedded in τ and meets X only in C' . The only other components of $F \cap \tau$ which meet D' are the discs with boundary in D . So we can isotop D' till it meets F only in C' . We then perform surgery on F along C' using D' . This reduces v by at least one, while leaving w and u unchanged.

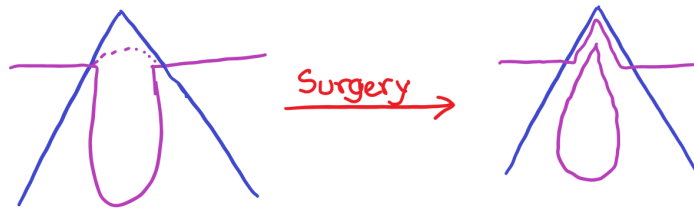


Figure 1.19:

Move 4: Any component of F in the interior of a 3-simplex of M is discarded. This reduces n , while leaving w , u and v unaltered.

Moves 5 and 5b: These moves are used only after we have used the previous moves for as long as possible. Then F meets each 2-simplex in normal arcs and each 3-simplex in a disjoint union of discs. Let τ be a 3-simplex such that a component D of $F \cap \tau$ is not a normal disc; ie, ∂D meets the 1-skeleton of τ in more than four points, by lemma 1.1.24. Thus, $\tau^{(1)}$ cuts ∂D into more than four arcs, so that ∂D meets some 2-simplex of $\partial\tau$ in at least 2 arcs and some 1-simplex e of $\partial\tau$ in at least two points. We choose two of these points to be adjacent on e so that there are no other intersection points on the sub-arc μ of e between them. Let λ be an arc in D such that $\partial\lambda = \partial\mu$ and let B be a disc embedded in τ with boundary $\lambda \cup \mu$ such that $B \cap F = \lambda$.

Move 5: If e does not lie in ∂M then we isotop λ across B just beyond μ , and extend this to an isotopy of F supported on a small neighbourhood of λ . This reduces w by at least 2.

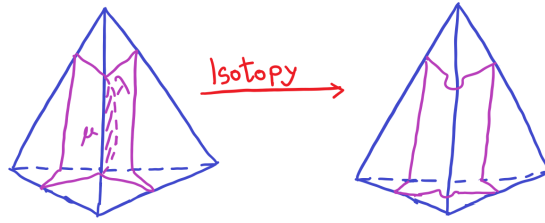


Figure 1.20:

Move 5b: If e lies on ∂M then we perform surgery on F along λ using the disc B . This too reduces w by at least 2.

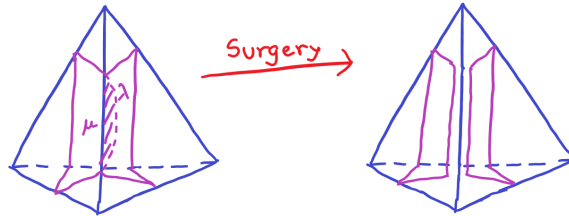


Figure 1.21:

Corollary 1.3.4. *Let M be a 3-manifold. Then any element of $H_2(M)$ is represented by a closed embedded normal surface.*

Proof. We know from lemma 1.1.26 that any element of $H_2(M)$ can be represented by a closed embedded surface F . On this surface, we perform the moves 1-5b. These moves involve isotopies and surgeries, hence they leave the homology class of F unchanged. If a component of F is deleted in move 4, it lies inside a 3-simplex and hence represents the zero element of homology; so its deletion does not change the homology class of F . Thus finally we obtain a normal surface Σ which has the same homology class as F . \square

Definition 1.3.5. A connected surface F properly embedded in a 3-manifold M is called incompressible in M if the following conditions hold:

1. Whenever a simple closed curve C on F bounds a 2-disc D in M such that $D \cap F = C$, then C also bounds a 2-disc in F .
2. If F is S^2 or D^2 , then in addition to (1), which is automatic, F does not separate M into two pieces, one of which is a 3-ball.

If F is disconnected then it is incompressible iff each of its components is incompressible.

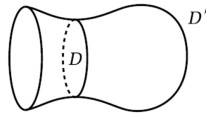


Figure 1.22:

Definition 1.3.6. Let F be a properly embedded surface in a 3-manifold M . A compressing disc for F is a 2-disc D in M such that $D \cap F = \partial D$ is an essential curve on F .



Figure 1.23:

Lemma 1.3.7. *Let F be a properly embedded surface in a 3-manifold M and let F_1 be a component of F which admits a compressing disc. Then F also admits a compressing disc.*

Proof. Let D be the compressing disc for F_1 . We isotop D to be transverse to F such that $D \cap F$ consists of ∂D and some discs in the interior of D .

If $D \cap F = \partial D$, then D is the required compressing disc for F .

If the above is not true, let C denote an innermost circle such that C bounds a disc D' in D with $\text{int}(D') \cap F = \emptyset$. Then either D' is the required compressing disc for F , or C bounds a disc D'' in F . In the latter case, we replace D' by D'' and isotop D'' off F , reducing the number of components of $D \cap F$ by one.

On repeating this argument, we ultimately obtain a compressing disc for F . \square

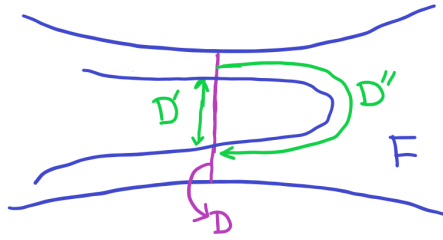


Figure 1.24:

Changing moves 1 and 3 to isotopies

Here we assume that M is a triangulated irreducible 3-manifold, and that F is an incompressible surface embedded in M transverse to the triangulation. Let D be a 2-disc that meets F only in ∂D . As F is incompressible, so ∂D bounds a 2-disc D' in F . D and D' together form a 2-sphere, which must bound a 3-ball B in M as M is irreducible. Thus any surgery on F using D replaces F by a copy of F and a 2-sphere disjoint from it.

Moves 1' and 3': Each of these is an isotopy of F supported on a small neighbourhood of D' , which isotopes D' across B and just past D . This is the same as performing surgery on F using D and then discarding the 2-sphere produced.



Figure 1.25: Move 1'

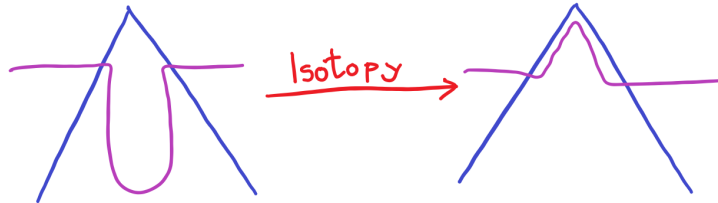


Figure 1.26: Move 3'

Lemma 1.3.8. *Let M be an irreducible 3-manifold with a fixed triangulation and let F be a closed incompressible surface embedded in the interior of M such that no component of F lies in a 3-ball. Then F is isotopic to a normal surface Σ in M .*

Proof. We apply the moves 1', 2, 3' and 5 to F . The complexity $c(F)$ is a finite number, so the process must end after a finite number of steps. Further, since we are not performing any surgeries, so move 4 is unnecessary. \square

Lemma 1.3.9. *Let M be an irreducible 3-manifold with a fixed triangulation and let F be a compact incompressible surface properly embedded in M such that no component of F lies in the interior of a 3-ball. In addition, ∂F is normal with respect to the given triangulation of ∂M and ∂F meets each edge of ∂M in at most one point. Then F is isotopic rel ∂F to a normal surface Σ in M .*

Proof. As in the previous lemma, we apply moves 1', 2, 3' and 5. F meets

each edge of the triangulation of ∂M in at most one point, so we do not need moves 2b and 5b since they need F to meet some edge of ∂M in two or more points. \square

Remark 1.3.10. *The condition that ∂F meets each edge of ∂M in at most one point is not usually restrictive as we can subdivide the triangulation of M to arrange for this, while still keeping ∂F normal to the new triangulation.*

Definition 1.3.11. *A connected surface F properly embedded in a 3-manifold M is arc-incompressible in M if, whenever there is a disc D in M with boundary consisting of an arc λ in F and an arc μ in ∂M such that $D \cap F = \lambda$, then there is an arc μ' in ∂F such that $\mu' \cup \lambda$ bounds a disc in F , ie, λ is inessential in F .*

If F is disconnected, then F is arc-incompressible iff each component of F is arc-incompressible.

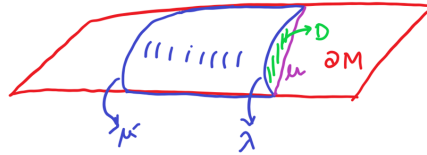


Figure 1.27:

Definition 1.3.12. *Let F be a properly embedded surface in a 3-manifold M . Then an arc-compressing disc for F is a 2-disc D in M with boundary consisting of an arc λ in F and an arc μ in ∂M such that $D \cap F = \lambda$ and λ is an essential arc on F .*

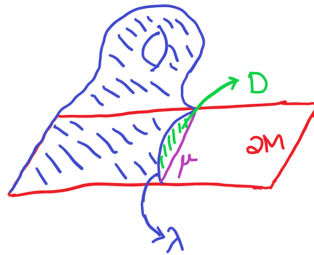


Figure 1.28:

Lemma 1.3.13. *Let F be a properly embedded surface in a 3-manifold M and let F_1 be a component of F that admits an arc-compressing disc. Then F also admits an arc compressing disc.*

Proof. Similar to the proof of lemma 1.3.7. □

Changing moves 2b and 5b to isotopies

Here we assume that M is a triangulated irreducible 3-manifold and that F is an incompressible surface embedded in M transverse to the triangulation. In addition, F is arc-incompressible and ∂M is incompressible in M .

Moves 2b and 5b are surgeries on F along an arc λ using a disc D . Let the part of ∂D in ∂M be μ . As F is arc-incompressible, there is an arc μ' in ∂F such that $\mu' \cup \lambda$ bounds a disc D' in F . But then $\mu \cup \mu'$ is a circle in ∂M which bounds $D \cup D'$ in M . But we assumed that ∂M is incompressible in M , so $\mu \cup \mu'$ bounds the disc D'' in ∂M . $D \cup D' \cup D''$ is a 2-sphere embedded in M , but M is irreducible, so it bounds a ball B in M .

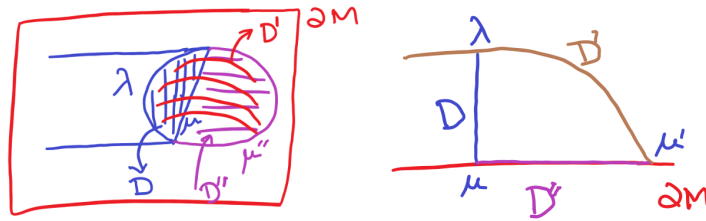


Figure 1.29:

Moves 2b' and 5b': Each of these moves is an isotopy of F supported on a small neighbourhood of D' , which isotopes D' across B and just past D . This is the same as performing the surgery moves as described in moves 2b and 5b, and then discarding the 2-disc produced.

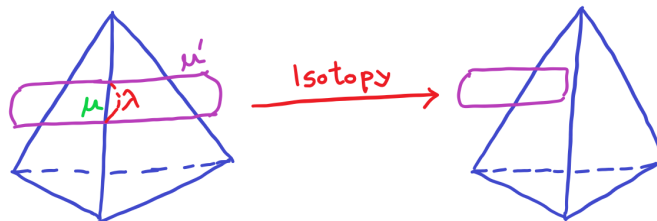


Figure 1.30: Move 2b'

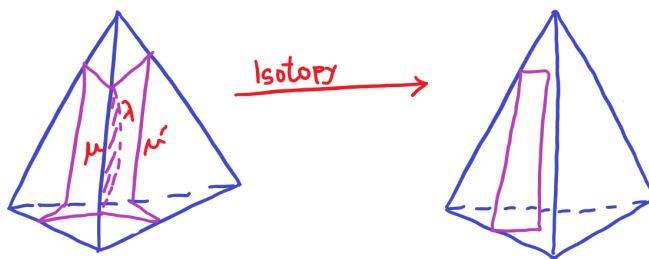


Figure 1.31: Move 5b'

Lemma 1.3.14. *Let M be an irreducible compact 3-manifold with a fixed triangulation and let F be a compact incompressible surface properly embedded in M such that no component of F lies in the interior of a 3-ball. If, in addition, F is arc-incompressible and the boundary of M is incompressible in M , then F is isotopic to a normal surface Σ in M .*

Proof. The given conditions in the lemma are sufficient for applying moves 0, 1', 2, 2b', 3', 5 and 5b', all of which are isotopies. \square

1.4 Barriers

Definition 1.4.1. *A barrier is a normal surface Φ such that given some surface F , we want to make the normalising moves for F in the complement of Φ .*

Remark 1.4.2. *A barrier does not always exist.*

Lemma 1.4.3. *Let Φ be a normal surface in a compact triangulated 3-manifold M , and let F be a surface embedded in $M - \Phi$. Then the normalising moves 0 to 5b can be applied to F , preserving disjointness from Φ at every stage, so that the normal surface Σ obtained from F is disjoint from Φ .*

Proof. We apply the normalising moves 0-5b to the disjoint union of F and Φ . Φ is already given to be normal, hence it is preserved by the moves. \square

Corollary 1.4.4. *Let Φ be a normal surface in a compact triangulated 3-manifold M such that Φ does not bound a 3-ball in M . Let F be a sphere embedded in $M - \Phi$ which also does not bound a 3-ball in M . Then the normalising moves 1, 2, 3, 4 and 5 can be applied to F , preserving disjointness from Φ at every stage, so that the normal surface Σ obtained from F is disjoint from Φ , and at least one component of Σ is a sphere which does not bound a 3-ball in M .*

Proof. F is a sphere, hence closed and boundary-less, so we do not need the moves 2b and 5b.

After applying each normalising move, we see that the surface obtained is a disjoint union of spheres, at least one of which does not bound a 3-ball. \square

Lemma 1.4.5. *Let Φ be a normal surface in a compact triangulated and irreducible 3-manifold M . Suppose no component of Φ lies in a ball in M . Let F be a closed incompressible surface embedded in $M - \Phi$, and not lying in a ball in M . Then F can be isotoped in the complement of Φ to a normal surface Σ .*

Proof. We cannot apply lemma 1.3.8 directly to $F \cup \Phi$ as Φ may not be incompressible. However F satisfies the requirements to apply lemma 1.3.8, so we will proceed by applying each move to F and checking if disjointness from Φ is preserved at each step.

Move 0: This move simply makes F transverse to the triangulation of M ; hence it preserves disjointness from Φ .

Move 1': This move is applied if there is a 2-simplex σ such that $F \cap \sigma$ contains a circle C innermost in σ . Say C bounds a disc D in the interior of σ . As F is incompressible, so ∂D bounds a 2-disc D' in F . Then $D \cup D'$ form a 2-sphere in M , which bounds a ball B as M is irreducible. This means move 1' is an isotopy of F supported on a small neighbourhood of D' which isotopes D' across B past D . Now, as F and Φ are disjoint, so Φ does not meet D' . Further, it is normal, so it does not meet D on σ . Lastly, no component of Φ lies in a ball, hence Φ cannot intersect B . Thus the isotopy is completely disjoint from Φ .



Figure 1.32:

Move 2: This move is applied when there is a 2-simplex σ such that $F \cap \sigma$ contains an arc λ such that $\partial\lambda$ lies on an edge e of σ . The disc D bounded in σ by λ and a sub-arc μ of e meets no other component of $F \cap \sigma$. Then move 2 is an isotopy of F supported on a small neighbourhood of λ which isotopes λ across D just past μ . Φ is normal and disjoint from F , so it cannot intersect D . We can perform move 2 in a way that the disjointness from Φ is preserved.

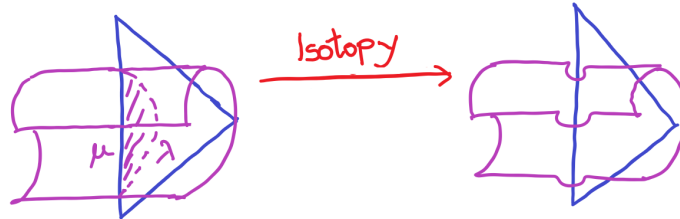


Figure 1.33:

Move 3: This move is applied when there is a component X of $F \cap \tau$ which is not a disc. We assume there is a boundary component C of X which bounds a disc component of $F \cap \tau$. If D meets a component of $\Phi \cap \partial\tau$, this component must lie in the interior of D as F is disjoint from Φ . Now Φ , hence the aforementioned component bounds a disc component of $\Phi \cap \tau$. Let C' be a circle in the interior of X which is parallel to C . Further, let D'' be the disc consisting of the union of D and the collar in X between C and C' . Let D' be the disc in the τ obtained from D'' by pushing D'' off $X \cup \partial\tau$ while fixing $\partial D'' = C'$. Thus C' bounds D' , and D' is embedded in τ and meets X only in C' . The only other components

of $F \cap \tau$ and $\Phi \cap \tau$ which meet D' are discs with boundary in D . Hence, we can isotop D' till it meets F only in C' and becomes disjoint from Φ . Now F is incompressible in M , hence C' bounds a disc E in F . Further, M is irreducible, so $E \cup D'$ bounds a ball B in M . Then move 3' is an isotopy of F supported in a small neighbourhood of E , which isotopes E across B and just past D' . As Φ and F are disjoint, Φ cannot meet E . No component of Φ lies in a ball, so $\Phi \cap B = \phi$. Thus we can perform the isotopy in a way that preserves disjointness from Φ .

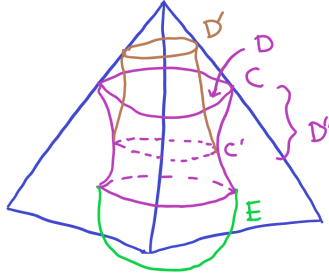


Figure 1.34:

Move 5: This move is used if there is a 3-simplex τ and a component D of $F \cap \tau$ which is not a normal disc. That is, ∂D intersects some 1-simplex e of $\partial\tau$ in two points adjacent on e ; the sub-arc determined by these points is called μ . Further there is an arc λ in D such that $\partial\lambda = \partial\mu$. Let B be the disc embedded in τ with boundary $\lambda \cup \mu$ such that $B \cap F = \lambda$. Then move 5 is an isotopy of F supported on a small neighbourhood of λ which isotopes λ across B and just past μ . Φ cannot meet λ as it is disjoint from F . It intersects B , if at all, in circles or arcs with both endpoints on μ . But as it is normal, so it cannot have such arcs. So each component of $\Phi \cap \tau$ is a disc. We can also isotop τ to get a B such that $\Phi \cap B = \phi$. Thus, we can perform move 5 to preserve disjointness from Φ . \square

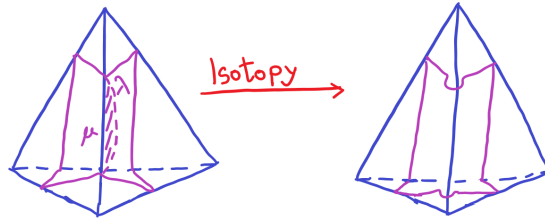


Figure 1.35:

Lemma 1.4.6 (Fixed boundary case). *Let Φ be a normal surface in a compact triangulated and irreducible 3-manifold M . Suppose no component of Φ lies in a ball in M . Let F be a compact incompressible surface properly embedded in $M - \Phi$ such that no component of F lies in the interior of a 3-ball. Further, ∂F is normal with respect to the given triangulation of ∂M and ∂F meets each edge of ∂M in at most one point. Then F can be isotoped in the complement of Φ and rel ∂F to a normal surface Σ .*

Proof. The conditions in the lemma are sufficient to apply moves 0, 1', 2, 3' and 5 to F . We can use lemma 1.4.5 to argue disjointness of F from Φ . \square

Lemma 1.4.7 (Free Boundary Case). *Let Φ be a normal surface in a compact triangulated and irreducible 3-manifold M . Suppose no component of Φ lies in a ball in M . Let F be a compact incompressible surface properly embedded in M and disjoint from Φ such that no component of F lies in the interior of a 3-ball. If, in addition, F is arc-incompressible and ∂M is incompressible in M , then F can be isotoped in the complement of Φ rel ∂F to a normal surface Σ .*

Proof. The conditions in the lemma are sufficient to apply moves 0, 1', 2, 2b', 3', 5 and 5b' to F . From the proof of lemma 1.4.5, we know that 0, 1', 2, 3' and 5 preserve disjointness of F from Φ . So we only need to prove the same for moves 2b' and 5b'.

Move 2b': The arcs λ on F and μ on ∂M bound a disc. As F is incompressible, so there is an arc μ' on ∂F such that $\lambda \cup \mu'$ bounds a disc D . $\mu \cup \mu'$ also bounds a disc D' on σ . $D \cup D'$ is a 2-sphere, and as M is irreducible, so it bounds a ball, across which we can perform isotopy. Arguing similar to move 1' in the proof of lemma 1.4.5, we see that disjointness from Φ is preserved.

Move 5b': The proof for this case is similar to that of move 3' in the proof of lemma 1.4.5. \square

1.5 More Finiteness Results

Theorem 1.5.1. *1. Let M be an irreducible compact 3-manifold. Then there is a number $k(M)$ such that if Σ is an incompressible closed surface embedded in the interior of M with more than $k(M)$ components such that no component lies in a 3-ball, then two components of Σ are parallel.*

2. Let M be an irreducible compact 3-manifold, Then there is a number $k(M)$ such that if Σ is an incompressible and arc-incompressible surface properly embedded in M with more than $k(M)$ components such that no component lies in the interior of a 3-ball, then two components of Σ are parallel.

Proof. This theorem is a combination of lemma 1.2.5, lemma 1.3.8 and lemma 1.3.14. \square

Definition 1.5.2. Let M be a 3-manifold and let X be a connected 3-dimensional submanifold of M . Then X is called bad if it is disjoint from ∂M and is homeomorphic to S^3 with the interiors of some disjoint 3-balls removed; X is called good otherwise.

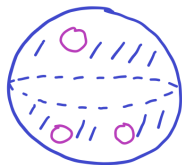


Figure 1.36: 2D analogue of a bad submanifold

Definition 1.5.3. Let S be a compact surface embedded in the interior of a 3-manifold M such that each component of S is a 2-sphere. Let N denote the interior of a regular neighbourhood of S . S is called good if all components of $M - N$ are good; it is called bad otherwise.

Lemma 1.5.4. Let P and Q be compact connected 3-dimensional submanifolds of a 3-manifold M such that $P \cap Q$ is a 2-disc or a 2-sphere. Then $P \cup Q$ is bad iff both P and Q are bad.

Proof. First, suppose that both P and Q are bad. Then both lie inside M and are homeomorphic to S^3 with the interiors of some disjoint 3-balls removed. So $P \cup Q$ lies in the interior of M . Further, we are given that $P \cap Q$ is a 2-disc or a 2-sphere. Hence, each of P and Q must have at least one boundary component, otherwise we will end up with non-manifold points. So each of P and Q is a 3-ball with the interiors of some disjoint 3-balls removed. The union of two 3-balls that intersect in a sphere is S^3 . The union of two 3-balls that intersect in a 3-ball is a 3-ball. Hence, $P \cup Q$ is homeomorphic to S^3 with the interiors of some disjoint 3-balls removed; ie, it is bad.

Conversely, say $P \cup Q$ is bad. Then $P \cup Q$ lies inside M and is homeomorphic to S^3 with the interiors of some disjoint 3-balls removed. So P and Q must lie in the interior of M . Now a sphere in S^3 cuts S^3 into two 3-balls, and a disc properly embedded in a 3-ball cuts it into two 3-balls. So P and Q must be homeomorphic to S^3 with the interiors of some disjoint 3-balls removed; ie, they are both bad. \square

Theorem 1.5.5. Let M be a compact 3-manifold and let S denote a compact surface embedded in M , each component of which is a 2-sphere. Then there is

a number $l(M)$ such that if S has more than $l(M)$ components, then it must be bad.

Proof. Let S be a compact surface embedded in $\text{int}(M)$, each component of which is a 2-sphere. We fix a triangulation of M .

If S is normal with respect to the triangulation, then by lemma 1.2.5, there is a number $k(M)$ such that if S has greater than $k(M)$ components, then two components are parallel. Such an S is bad as $S^2 \times I \cong S^3 - (B_1 \cup B_2)$, ie, it is bad.

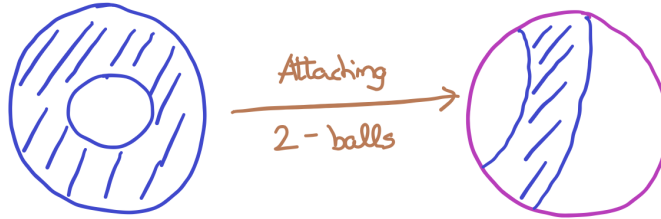
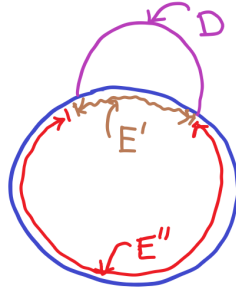


Figure 1.37: 2D analogue

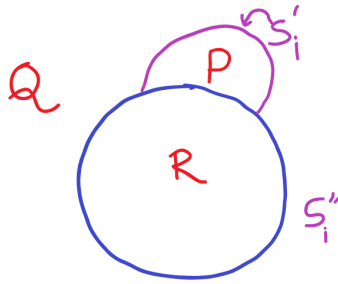
Next, say S is not normal to the triangulation. We will prove the theorem by contradiction. Suppose S is good. We can apply moves 0-5 on S to obtain a normal surface Σ ; all the moves except 1 and 3 are isotopies and hence preserve ‘goodness’. We will replace moves 1 and 3 by new moves 1” and 3” such that each yields a new good surface S' which has the same number of components as S . No component of S' can lie in the interior of a ball in M , as an innermost such sphere will bound a 3-ball in M and hence be a bad component. So we will not need to use move 4. We will use moves 1”, 2, 3” and 5 to obtain a surface Σ that is normal to the triangulation of M and is good, contradicting the previous case.

Let D be a 2-disc in M which intersects S only in ∂D . Let S_i be the component of S containing ∂D . ∂D cuts S_i into two 2-discs, E' and E'' . We call $D \cup E'$ as S'_i and $D \cup E''$ as S''_i . We claim that replacing S_i by S'_i or S''_i yields a good family.

We define X to be the component of $\overline{M - N(S - S_i)}$ which contains S_i . Then we have four possible cases.



Case I: S_i separates X and D separates the component of $X - S_i$ on which it lies.



$S_i \cup D$ separates X into three pieces - P , Q and R . The pieces are labelled as: S_i cuts X into $P \cup Q$ and R ; S_i' cuts X into $Q \cup R$ and P ; and S_i'' cuts X into $P \cup R$ and Q . We can replace S_i by either of S_i' and S_i'' unless one of P and $Q \cup R$ is bad, or one of Q and $P \cup R$ is bad.

S_i is good, so $P \cup Q$ and R must be good. R is good, so by lemma 1.5.4, $P \cup R$ and $Q \cup R$ are good. At least one of P and Q must be good, otherwise $P \cup Q$ will be bad, which is a contradiction. So the replacement is possible.

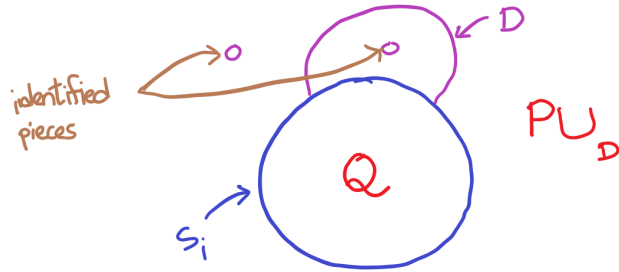
Case II: S_i separates X , but D does not separate the component of $X - S_i$ it lies on.

We define the following notation:

$P \cup_D$:= the manifold obtained from P by gluing two disjoint copies of D in the boundary of P .

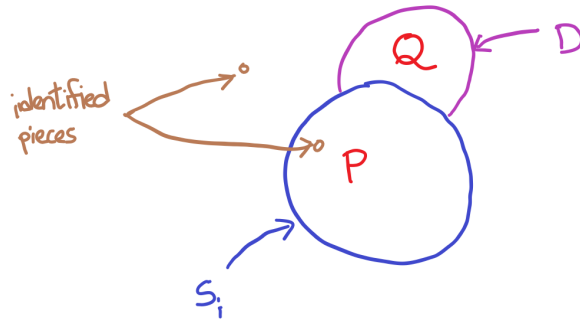
$P \cup_A Q$:= the manifold obtained by gluing P and Q along A .

Then we declare that $S_i \cup D$ cuts X into P and Q labelled such that S_i cuts X into $P \cup_D$ and Q . Then S_i' cuts X into one piece, $P \cup_{E''} Q$; S_i'' cuts X into one piece, $P \cup_{E'} Q$. Since S_i is good, so both $P \cup_D$ and Q are good. Then $P \cup_{E''} Q$ and $P \cup_{E'} Q$ are automatically good as Q is good. And we can replace S_i by



either of S'_i or S''_i .

Case 3: S_i does not separate X , but D separates the component of $X - S_i$ on which it lies.



$S_i \cup D$ cuts X into P and Q , labelled such that S_i cuts X into $P \cup_D Q$. Since S_i is good, so $P \cup_D Q$ is good; hence at least one of P and Q is good. As S''_i cuts X into $P \cup_{E'} Q$, so it is good. Similarly for S'_i .

Case 4: Both S_i and D are non-separating.

In this case, $S_i \cup D$ is a connected manifold P , with S_i cutting X into $P \cup_D$. Hence, S''_i cuts X into $P \cup_{E'}$, which is good as it has non-trivial fundamental group. Similarly for S'_i .

Moves 1" and 3" consist of replacing S_i by S'_i and S''_i , as is appropriate. \square

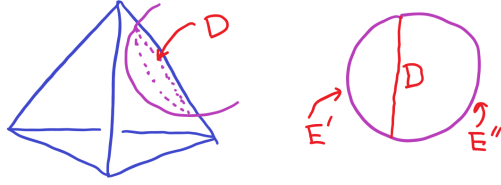


Figure 1.38: Representative situation for move 1”

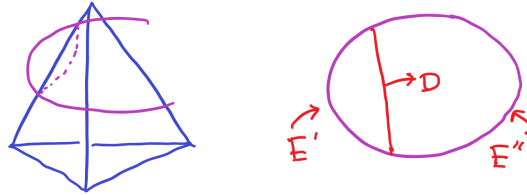


Figure 1.39: Representative situation for move 3”

Theorem 1.5.6 (Kneser). *Let M be a compact 3-manifold. Then there is a number $k(M)$ such that if M is expressed as the connected sum of manifolds M_1, \dots, M_r and $r > k(M)$, then $M_i \cong S^3$ for some i . Further, the decomposition is unique upto homeomorphism.*

Proof. We fix some triangulation of M . Let $M = M_1 \# M_2 \# \dots \# M_r$. Then there are spheres S_1, \dots, S_r in the interior of M such that cutting M along the S_i 's gives manifolds X_1, \dots, X_r with X_i homeomorphic to M_i with the interiors of some 3-balls removed. We claim that taking $k(M) = l(M) + 1$ works, where $l(M)$ is the same as given by theorem 1.5.5.

Let $r > l(M) + 1$, let S be the collection of $(r - 1)$ 2-spheres in M determined by the connected sum decomposition and let N be the interior of a regular neighbourhood of S . Then by theorem 1.5.5, some component of $M - N$ is bad, and the corresponding summand M_i of M is homeomorphic to S^3 .

For uniqueness, let us consider two cases:

Case I: First, say we have a compact orientable 3-manifold M with connected sum decompositions given by:

$$\begin{aligned} M &= M_1 \# M_2 \# \dots \# M_k \# (S^1 \times S^2)_h \\ &= M'_1 \# M'_2 \# \dots \# M'_k \# (S^1 \times S^2)_{h'} \end{aligned}$$

where $(S^1 \times S^2)_d$ denotes a connected sum of d copies of $S^1 \times S^2$.

We say $S \subset M$ of disjoint spheres is a reducing set of spheres for the decomposition $M = M_1 \# M_2 \# \dots \# M_k \# (S^1 \times S^2)_h$ if $M - S$ consists precisely of one M_i with some holes, and some balls with holes. In general, S is constructed by taking the spheres of the prime decomposition, plus one non-separating sphere inside each $S^1 \times S^2$ summand. Let S' be the prime decomposition for the other prime decomposition.

If we add any sphere Σ disjoint from S to S , we still get a reducing set of spheres for the same decomposition, as this sphere lies in some irreducible N , so Σ bounds a ball B . If $S^* = S \cup \Sigma$, then $M - S^*$ contains the same holed N , and possibly a new holed ball B (if $B \cap S = \emptyset$).

We assume that S and S' intersect transversely in circles (this can be arranged by an isotopy) and pick an innermost circle in a component of S such that it bounds a disc $D \subset S$. Surgery of S' along D replaces $S'_0 \in S'$ with $S'_1 \sqcup S'_2$. We isotope S'_0 , S'_1 and S'_2 so that they become disjoint but together bound a ball B' having holes. Further, $S' \cup S'_1 \sqcup S'_2$ is still a reducing set of spheres.

We continue this process till $S \cap S' = \emptyset$. Then $S \cup S'$ is a reducing set of spheres for both systems. This is only possible if M_i and M'_j are pairwise diffeomorphic. So we only need to show that $h = h'$. We have

$$M = N \# (S^2 \times S^1)_h = N \# (S^2 \times S^1)_{h'}$$

where the prime summands are the same in both decompositions by the above arguments. Taking first homology gives:

$$H_1(M) = H_1(N) \oplus \mathbb{Z}^h = H_1(N) \oplus \mathbb{Z}^{h'}$$

which gives $h = h'$.

Case II: From lemma 1.1.6, we know that there are two reducible 3-manifolds - $S^2 \times S^1$ (which is orientable) and $S^2 \tilde{\times} S^1$ (which is non-orientable). If N is non-orientable, we have in the spirit of a similar result for surfaces (lemma 3.0.3):

$$N \# S^2 \times S^1 \cong N \# S^2 \tilde{\times} S^1$$

So by replacing all the $S^2 \times S^1$ terms with $S^2 \tilde{\times} S^1$ we can repeat the arguments above (as $H_1(S^2 \times S^1) = H_1(S^2 \tilde{\times} S^1) = \mathbb{Z}$) to get the required uniqueness result. \square

Definition 1.5.7. *Theorem 1.5.6 says that every 3-manifold $M = M_1 \# \dots \# M_k$, where k is finite and each M_i is prime. This is called the prime decomposition of M .*

Chapter 2

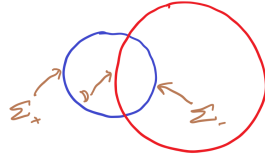
Normal Immersions and PL Area

2.1 Smooth Area

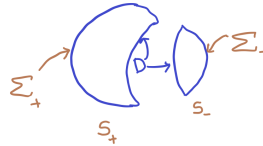
We consider 3-manifolds with a Riemannian metric, and maps of surfaces into such manifolds which have least area in their homotopy class. As the area of a map is an integral, so we only consider those maps for which the integral is convergent. Then the area of the map is not the area of the image, but rather the metric of the 3-manifold is pulled back to obtain a metric on the surface whose area is to be measured.

Lemma 2.1.1. *Let Σ and Σ' be smooth embedded spheres in a 3-manifold M which is equipped with a Riemannian metric. Suppose that each sphere has least area among all smoothly embedded spheres in M which do not bound a 3-ball. Further, if Σ and Σ' intersect, they must do so transversely or coincide. Then either Σ and Σ' do not coincide or they are disjoint.*

Proof. Let $A(\Sigma)$ denote the area of Σ , and $A(\Sigma')$ denote the area of Σ' . By hypothesis, $A(\Sigma) = A(\Sigma')$. If possible, say they intersect transversely. Then they will intersect in finitely many circles. Each such circle bounds two discs on each sphere. Let D be the disc of least area among all these discs. Then no circle of $\Sigma \cap \Sigma'$ lies in the interior of D . Let us assume that D lies in Σ' , and let $C := \partial D$. Then C cuts Σ into two discs, Σ_+ and Σ_- .



As no double curve meets the interior of D , so $D \cup \Sigma_+$ and $D \cup \Sigma_-$ are embedded spheres. We call $D \cup \Sigma_+$ as S_+ and $D \cup \Sigma_-$ as S_- .



$A(D)$ is minimum, so

$$A(D) \leq A(\Sigma_+), A(\Sigma_-) \implies A(S_+) = A(D) + A(\Sigma_+) \leq A(\Sigma_-) + A(\Sigma_+) = A(\Sigma)$$

Similarly, $A(S_-) \leq A(\Sigma)$.

S_+ and S_- have corners along C , so we can round these corners to obtain spheres S'_+ and S'_- with areas strictly less than S_+ and S_- respectively. So $A(S'_+), A(S'_-) < A(\Sigma)$. So S'_+ and S'_- must bound balls as Σ and Σ' are the spheres of least area not bounding balls. But as S'_+ and S'_- are obtained by slightly modifying S_+ and S_- , so S_+ and S_- must themselves also bound 3-balls.

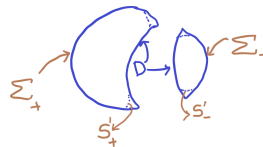
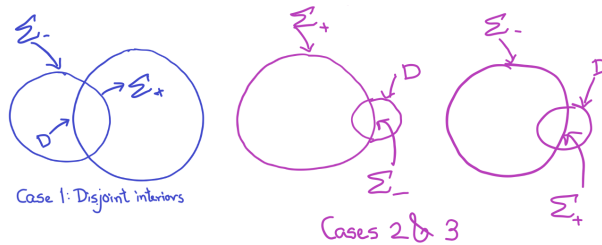


Figure 2.1: S'_+ and S'_- are obtained from S_+ and S_- by rounding the corners

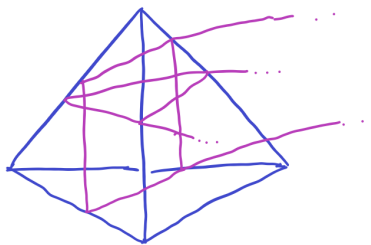
There are three possible configurations for the 3-balls B_+ and B_- bounded by S_+ and S_- - they may lie inside one another, or may have disjoint interiors. In all three cases, Σ must bound the 3-ball $B_+ \cup B_-$, which is a contradiction. \square



2.2 Normal Immersions

Definition 2.2.1. Let M be a 3-manifold with a fixed triangulation and let F be a compact surface. An immersion $f : F \rightarrow M$ is called normal if it satisfies the following conditions:

- f is transverse to the triangulation of M , ie, f is transverse to each open simplex of M .
- For each 2-simplex σ of M , $f^{-1}(\sigma)$ is a disjoint union of arcs each mapped by an embedding to a normal arc in σ .
- For each 3-simplex τ of M , $f^{-1}(\tau)$ is a disjoint union of 2-discs each mapped by an embedding to a normal disc in τ .



Definition 2.2.2. Let M be a 3-manifold with a fixed triangulation and let F be a compact surface. Two maps $f, f' : F \rightarrow M$ which are transverse to the triangulation of M are called normally homotopic if they are properly homotopic through maps which are all transverse to the triangulation of M .

Replacing any Proper Map of a Compact Surface by a Normal Immersion

Let M be a 3-manifold with a fixed triangulation, F be a compact surface and $f : F \rightarrow M$ be a proper map.

Move 0: We homotop f to be triangulation of M . If $f|_{\partial F}$ is already transverse to the triangulation, the homotopy can be done by fixing. Further, the pre-image $f^{-1}(\sigma)$ of each 2-simplex σ of M is mapped into σ in general position; the pre-image $f^{-1}(e)$ of each 1-simplex e is mapped injectively into e .

Now we define the following terms:

w is the weight of f , ie, the number of points of $f^{-1}(M^{(1)})$.

u_σ is the number of components of $f^{-1}(\sigma)$, where σ is a 2-simplex.

$u := \sum_\sigma u_\sigma$.

z_σ is the number of double points of f in σ .

$z := \sum_\sigma z_\sigma$.

v_τ is the number of components of $f^{-1}(\partial\tau)$ which do not bound a disc component of $f^{-1}(\tau)$, where τ is a 3-simplex.

$v := \sum_\tau v_\tau$.

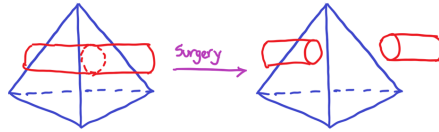
n is the number of components of F .

The complexity, $c(f) := (w, z, u, v, n)$ with lexicographic ordering.

Move 1: Let σ be a 2-simplex such that $f^{-1}(\sigma)$ contains a circle C . As f is transverse to the triangulation, so C must be two-sided.

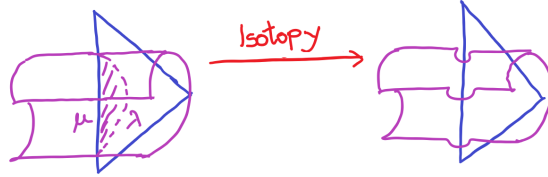
Let D be a 2-disc. σ is contractible, so we can extend $f|_C$ to a map $\bar{f} : D \rightarrow \text{int}(\sigma)$. A 3-complex X is constructed from F and $D \times I$ by gluing $\partial D \times I$ to an annulus neighbourhood of C in F . If $\sigma \in \partial M$, we identify D with $D \times \{0\}$; else, it is identified with $D \times \{\frac{1}{2}\}$. f and \bar{f} are then extended to a map $\tilde{f} : X \rightarrow M$ such that if $f_1 = \tilde{f}|_{D \times I}$, then $f_1^{-1}(\sigma) = D$ and f_1 maps $D \times I$ to a small collar neighbourhood of σ in M .

Let F' be the surface in X obtained by performing surgery on F along C using D . Move 1 consists of replacing F by F' and f by $f|_{F'}$. This is also called surgery on the map f along the circle C . It removes C from $f^{-1}(\sigma)$. This reduces u , while keeping w unchanged and not increasing z , thus reducing $c(f)$.

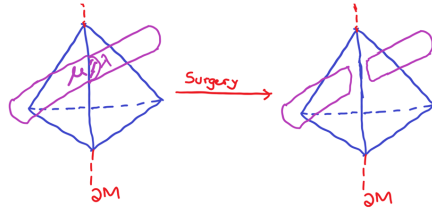


Moves 2 and 2b: Say there is a 2-simplex σ such that $f^{-1}(\sigma)$ contains an arc λ such that $f(\partial\lambda)$ lies in one edge e of σ . If σ lies in ∂M or e does not lie in

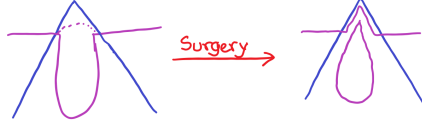
∂M , then we can homotop $f|_\lambda$ into e and just beyond, and extend this to a homotopy of f supported on a small neighbourhood of λ . This reduces w by at least 2, thus decreasing $c(f)$.



Now suppose e lies in ∂M but σ does not. Let D be a 2-disc whose boundary is divided into two arcs λ and μ which intersect only on their boundaries. σ deformation retracts to e , so we can extend $f|_\lambda$ to a map $\bar{f} : D \rightarrow \sigma$ with $\bar{f}^{-1}(e) = \mu$. We construct a 3-complex X from F and $D \times I$ by gluing $\partial D \times I$ to a collar neighbourhood of λ in F . Further, we identify D with $D \times \{\frac{1}{2}\}$. We extend f and \bar{f} to $\tilde{f} : X \rightarrow M$ such that if $f_1 = \tilde{f}|_{D \times I}$, then $f_1^{-1}(\sigma) = D$ and f_1 maps $D \times I$ to a small collar neighbourhood of σ in M . Let F' denote a surface in X obtained by performing surgery on F along λ using D . We replace F by F' and f by $\tilde{f}|_{F'}$. This too reduces w by 2.



Move 3: Let τ be a 3-complex and let a non-disc component Y of $f^{-1}(\tau)$ meet $\partial\tau$. We choose a boundary component C of Y and let C' be a circle in $\text{int}(Y)$ parallel to C . Let D be a 2-disc bounded by C' . As τ is simply connected so $f|_{C'}$ extends to $\bar{f} : D \rightarrow \tau$. Move 3 consists of performing surgery on F along C' using D to obtain F' , and replacing f by $\tilde{f}|_{F'}$. This move reduces v by at least 1 as C bounds a disc component of $f^{-1}(\tau)$ after surgery, while leaving w , z and u unaltered.



Move 4: We discard any component of F mapped into the interior of a 3-simplex of M . This reduces n , while leaving w, z, u and v unaltered.

Moves 5 and 5b:

Moves 1' and 3': Let M be a triangulated irreducible 3-manifold with $\pi_2(M) = 0$ and let $f : F \rightarrow M$ be a π_1 -injective map of a surface F into M which is transverse to the triangulation. Moves 1 and 3 comprise performing surgery on the map f along a circle C embedded in F using an extension of $f|_C$ to a map to the 2-disc D into M . As f is π_1 -injective, so C must bound a 2-disc D' in F . Further, as $\pi_2(M) = 0$, so the map of the 2-sphere formed by the union of D and D' is nullhomotopic. So surgery on f replaces f by a new map of F and a nullhomotopic map of a 2-sphere.

We define moves 1' and 3' to be a homotopy of f supported on a small neighbourhood of D' which homotopes D' to D and a little beyond.



Figure 2.2: Move 1'



Figure 2.3: Move 3'

Lemma 2.2.3. *Let M be a compact 3-manifold with a fixed triangulation. Let F be a closed connected surface and let $f : F \rightarrow M$ be a map. In addition,*

$\pi_2(M) = 0$ and $f_* : \pi_1(F) \rightarrow \pi_1(M)$ is injective. Then f is homotopic to a normal immersion of F into M .

Proof. Case 1: F is homeomorphic to S^2 .

$\pi_2(M) = 0$ implies that all maps of F into M are homotopic. So f is trivially homotopic to a normal immersion - an embedding whose image is the link of a vertex of the triangulation of M .

Case 2: F is not homeomorphic to S^2 .

In this case, we can use moves 1', 2, 3' and 5. Move 4 will never be used as F cannot be mapped into a 3-ball even after a homotopy. The compactness of F ensures that $c(f)$ is finite, so the process cannot continue indefinitely. \square

Definition 2.2.4. A map f of a 1-manifold into a triangulated surface is called normal if it is transverse to the given triangulation and, for each 2-simplex σ of the triangulation, the pre-image $f^{-1}(\sigma)$ consists of arcs which are each embedded in σ as normal arcs.

Lemma 2.2.5 (Fixed Boundary Case). Let M be a compact 3-manifold with a fixed triangulation, let F be a compact surface and let $f : F \rightarrow M$ be a proper map which embeds ∂F . Suppose $\pi_2(M) = 0$ and $f_* : \pi_1(F) \rightarrow \pi_1(M)$ is injective. In addition, suppose the restriction of f to ∂F is normal with respect to the given triangulation of ∂M and that for each edge e of ∂M , $f^{-1}(e)$ consists of at most 1 point. Then f is homotopic rel ∂F to a normal immersion of F into M .

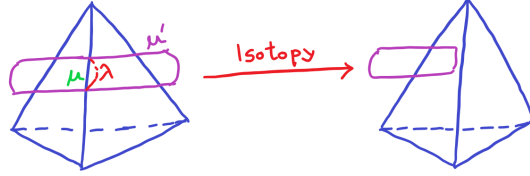
Proof. We can clearly apply moves 1', 2, 3' and 5. As $f^{-1}(e)$ has at most only one point, so it cannot enclose any arcs in ∂M ; hence we do not need moves 2b and 5b. \square

Definition 2.2.6. The boundary ∂M of a 3-manifold M is called π_1 -injective if the natural map $\pi_1(S) \rightarrow \pi_1(M)$ is injective for each component S of ∂M .

Move 2b': Let M be a triangulated 3-manifold with a π_1 -injective boundary, and let $f : F \rightarrow M$ be a π_1 -injective proper map of a surface into M such that f is transverse to the triangulation. Further, let $\pi_2(M) = 0$ and let $f_* : \pi_1(F, \partial F) \rightarrow \pi_1(M, \partial M)$ be injective. In move 2b, we perform surgery on f along an arc λ embedded in F , using a disc D (with ∂D divided into the arcs λ and μ) and a map of D into M which sends μ into ∂M . $f : \pi_1(F, \partial F) \rightarrow \pi_1(M, \partial M)$ is injective and $f \circ \lambda$ is homotopic to μ in ∂M , so there is an arc μ' in ∂F such that $\mu' \cup \lambda$ bounds the disc D' in F . Hence, under the given conditions, move 2b replaces F by a copy of F and a disc, $D \cup D'$. In M , $\mu \cup \mu'$ bounds $D \cup D'$ and $\pi_1(S) \rightarrow \pi_1(M)$ is injective for any boundary component of M , so $\mu \cup \mu'$ bounds a disc D'' in ∂D . $\pi_2(M) = 0$, so $D \cup D' \cup D''$ bounds a 3-ball B .

Move 2b' is then defined to be a proper homotopy of f supported on a small neighbourhood of D' which homotops D' across B and a little beyond. This is

the same as performing the surgery in move 2b and then discarding the 2-disc produced.



Lemma 2.2.7 (The free boundary case). *Let M be a compact 3-manifold with a π_1 -injective boundary and a fixed triangulation. Let F be a compact connected surface, and let $f : F \rightarrow M$ be a proper map. Suppose that, in addition, $\pi_2(M) = 0$, that $f_* : \pi_1(M) \rightarrow \pi_1(M)$ and that $f_* : \pi_1(F, \partial F) \rightarrow \pi_1(M, \partial M)$ is injective. Then f is properly homotopic to a normal immersion of F into M*

Proof. Case 1: F is S^2 or D^2 .

We homotope F so that it becomes an embedding of the link of a vertex of the triangulation of M .

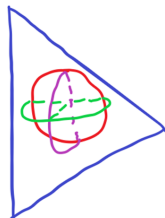
Case 2: F is any other surface.

Then we can use moves 0, 1', 2b', 3' and 5b'. Move 4 is not needed as F cannot be mapped into a 3-ball. Compactness of F implies that $c(f)$ is finite, so the process stops after a finite number of steps. \square

2.3 Barriers

Let M be a triangulated 3-manifold, g be a normal immersion of a surface Σ into M and $f : F \rightarrow M - g(\Sigma)$ be a map. In many cases, the normalising moves can be performed in the complement of $g(\Sigma)$ so that $g(\Sigma)$ acts as a barrier.

Definition 2.3.1. *A normal immersion g of a surface Σ into M is in good position if for each 3-simplex τ of M , any two normal discs of $g(\Sigma)$ which lie in τ intersect in a disjoint union of trees, each of which meets the boundaries of both discs.*



Lemma 2.3.2. *Let X be the union of a finite family of 2-discs D_1, \dots, D_k properly embedded in 3-ball B , such that the intersection of any two of the discs is a disjoint union of trees, each of which meets the boundaries of both discs. Then each component of $B - X$ is contractible.*

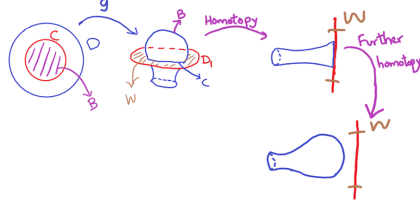
Proof. We will show that each component of $B - X$ has trivial fundamental group and hence is contractible.

Let $f : S^1 \rightarrow B - X$ be a map. B is contractible, so we can extend f to a map $g : D \rightarrow B$, where D is a 2-disc. We will homotope $g \text{ rel } \partial D = S^1$ until $g(D) \subset B - X$.

Let $g(D)$ intersect D_1 . First, we homotope g to be transverse to D_1 . Then $g^{-1}(D_1)$ consists of disjoint circles in D . Let C be one of these circles, and let the subdisc of D bounded by C be denoted by Z . D_1 is contractible, so there is a map of Z into D_1 which agrees with g on $\partial Z = C$. In place of g we use this map and then further homotope it further (supported on a small neighbourhood of Z) to remove Z from $g^{-1}(D_1)$. Repeating this process, we do away with all intersections of $g(D)$ with D_1 .

Say when we started, $g(D)$ did not meet the disc D_j (for some j). We claim that even after the homotopy, $g(D)$ will not intersect D_j . Suppose otherwise. The image of C in D_1 separates D_1 into two components - we name the one containing ∂D_1 as W . The new map is chosen to take Z to the closure of $D_1 - W$. If, after homotopy, we get a new intersection with D_j , then D_j has to meet $D_1 - W$. This is a contradiction, as $D_1 \cap D_j$ must meet ∂D_1 , which means D_j must have already intersected $g(D)$ when we started, we are wrong by assumption.

So we see that the above described homotopy reduces the number of D_i 's that $g(D)$ intersects. We repeat the process till $g(D)$ does not intersect any of the D_i 's, and lies in $B - X$. This completes the proof. □



Lemma 2.3.3. *Let M be a compact triangulated 3-manifold such that $\pi_2(M) = 0$, g be a good position normal immersion of a surface Σ into M and $\pi_2(M - g(\Sigma)) = 0$. Let F be a closed connected surface and let $f : F \rightarrow M - g(\Sigma)$ be a map such that the induced image $\pi_1(F) \rightarrow \pi_1(M)$ is injective. Then f can be homotoped in the complement of $g(\Sigma)$ to a normal immersion.*

Proof. We will check that the moves 0, 1', 2, 3' and 5 can be applied to f so that $f(F)$ stays in the complement of $g(\Sigma)$.

Move 0 makes f transverse to the triangulation of M ; if we choose a small enough homotopy, f stays disjoint from $g(\Sigma)$.

Move 1' is applied to f when there is a 2-simplex σ such that $f^{-1}(\sigma)$ contains a circle C . g is a normal immersion, so $g(\Sigma) \cap \sigma$ consists of a finite family of embedded normal arcs, with each component of the complement of these arcs in σ is simply connected. Then there is an extension of $f|_C$ to a map of the 2-disc D into σ which is disjoint from $g(\Sigma)$. f is π_1 -injective, so C must bound a disc D' in F . Then the 2-sphere $D \cup D'$ is mapped into $M - g(\Sigma)$; this map is nullhomotopic as $\pi_1(M - g(\Sigma)) = 0$. Then move 1' is a homotopy of f supported on a small neighbourhood of D' which homotopes D' to D and just beyond σ . By taking a small enough homotopy we can perform move 1' to be disjoint from $g(\Sigma)$.

Move 2 is applied when there is a 2-simplex σ such that $f^{-1}(\sigma)$ contains an arc λ such that $f(\partial\lambda)$ lies in a single edge e of σ . $f(\lambda)$ lies in the complement of $g(\Sigma)$ and each component of this complement is simply connected, so the move is performed in the complement of $g(\Sigma)$.

Move 3' is used when there is a 3-simplex τ such that $f^{-1}(\tau)$ contains a non-disc component Y that intersects $\partial\tau$. Let C be a boundary component of Y and C' be a circle in Y parallel to it. g is a good position normal immersion, so $g(\Sigma) \cap \tau$ consists of a finite family of embedded normal arcs, such that each pair meet in arcs only. As the complement of these discs in σ is simply connected, so $f|_C$ extends to a map of the disc D into τ which lies in the complement of $g(\Sigma)$. f is π_1 -injective, so C' must bound a 2-disc D' in F . Both these discs are mapped into $M - g(\Sigma)$ and are homotopic in $M - g(\Sigma)$. So move 3' can be performed in the complement of $g(\Sigma)$. \square

Lemma 2.3.4 (Fixed Boundary Case). *Let M be a compact triangulated 3-*

manifold such that $\pi_2(M) = 0$. Let g be a good position normal immersion of the surface Σ into M and let $\pi_2(M - g(\Sigma)) = 0$. Let F be a compact surface and let $f : F \rightarrow M$ be a proper map which embeds ∂F such that the induced map $\pi_1(F) \rightarrow \pi_1(M)$ is injective. In addition, let the restriction of f to ∂F be normal with respect to the given triangulation of ∂M and that for each edge e of ∂M , $f^{-1}(e)$ consists of at most one point. If the images of f and g are disjoint, then f can be homotoped rel ∂F in the complement of $g(\Sigma)$ to a normal immersion.

Proof. Only moves 0, 1', 2, 3' and 5 are needed, so we can apply the proof of lemma 2.3.3. \square

Lemma 2.3.5 (Free boundary case). *Let M be a compact triangulated 3-manifold with π_1 -injective boundary, such that $\pi_2(M) = 0$. Let g be a good position normal immersion of the surface Σ into M and let $\pi_2(M - g(\Sigma)) = 0$. Let F be a compact surface and let $f : F \rightarrow M$ be a proper map such that $f_* : \pi_1(F) \rightarrow \pi_1(M)$ is injective and $f_* : \pi_1(F, \partial F) \rightarrow \pi_1(M, \partial M)$ is injective. If the images of f and g are disjoint, then f can be homotoped in the complement of $g(\Sigma)$ to a normal immersion.*

Proof. We apply moves 0, 1', 2, 2b', 3', 5 and 5b' to f . For moves 0, 1', 2, 3' and 5, we can use the proof of lemma 2.3.3. The proofs for moves 2b' and 5b' are similar to 1' and 3' respectively. \square

2.4 PL-Area

Definition 2.4.1. *Let w be the weight, ie, the number of points in $f^{-1}(M^{(1)})$. Let us choose some metric on each 2-simplex of M . Then there is an induced length on $f^{-1}(M^{(2)})$. We define $L(f)$ to be the total length of $f^{-1}(M^{(2)})$. Then the pair (w, L) with lexicographic ordering is called the PL-area of f , denoted as $A(f)$.*

Definition 2.4.2. *Let \mathcal{F} be a family of maps of surfaces into M . An element f of \mathcal{F} is said to be least area in \mathcal{F} if $A(f) \leq A(g)$ for any map g in \mathcal{F} for which $A(g)$ is defined. If \mathcal{F} consists of all maps of a given surface into M which are homotopic to a given map, then an element of \mathcal{F} of least area in \mathcal{F} is called least area.*

Definition 2.4.3. *We identify each 3-simplex of M with the regular ideal tetrahedron in hyperbolic 3-space \mathbb{H}^3 . This gives a metric on each 3-simple with the vertices removed. As all the faces of this hyperbolic simplex are congruent, we can glue these hyperbolic tetrahedra by isometries to obtain a metric on the complement of the vertices of M . We refer to this choice of metric as a*

hyperbolic metric on the 3-simplices of M . The advantage of this metric is that if a group acts simplicially on M , the action is given by isometries.

Definition 2.4.4. *A normal immersion $g : F \rightarrow M$ which has least area in its normal homotopy class is called a geodesic normal immersion. Its image in the 2-skeleton of M consists of geodesic arcs.*

Lemma 2.4.5. *Let M be a 3-manifold with a fixed triangulation and with a hyperbolic metric on its 3-simplices. Let F be a compact connected surface and let $f : F \rightarrow M$ be a normal immersion. Then the following hold:*

1. *Suppose that f is not normally homotopic arbitrarily close to a vertex of M . Then there is a standard normal immersion $g : F \rightarrow M$ normally homotopic to f and such that $A(g) \leq A(h)$ for all normal immersions h which are normally homotopic to f , ie, g has least area in the normal homotopy class of f .*
2. *If ∂F is non-empty, then there is a standard normal immersion $g : F \rightarrow M$ normally homotopic to f rel ∂F and such that $A(g) \leq A(h)$ for all normal immersions h which are normally homotopic to f rel ∂F , ie, g has least area in the normal homotopy class of f fixing ∂F .*

Proof. 1. Let X_i denote $f^{-1}(M^{(i)})$, $i = 1, 2$. Given any normal immersion $f : F \rightarrow M$, there is a standard normal immersion f' of F into M which agrees with f on X_1 . The image of f' is unique due to our choice of metric on the 2-simplices of M . f' maps X_2 to geodesic arcs, so $A(f') \leq A(f)$. Hence, we will only consider f to be a geodesic normal immersion.

The area of f is determined by $f|_{X_1}$. The choice of metric induces a metric on each open edge of M , and we can choose an isometry of each open edge to \mathbb{R} . We identify the set of all geodesic normal immersions normally homotopic to f with the points of \mathbb{R}^w , where w is the weight of f .

Let $\{f_n\}$ be a sequence of geodesic normal immersions, all normally homotopic to f such that $L(f_n) \rightarrow I = \inf L(h) : h$ is normally homotopic to f . $\{f_n\}$ is a sequence of points in \mathbb{R}^w . If it is bounded, it has a convergent subsequence with limit g and $L(g) = I$. Say this is not true. Then $\{f_n\}$ is not bounded, ie, there is a point x of X_1 such that $\{f_n(x)\}$ is not bounded. On taking a subsequence and changing the orientation of an edge of M if needed, we can assume that $f_n(x) \rightarrow +\infty$ as $n \rightarrow \infty$. Then for the closed edge of M that contains $f_n(x)$, $f_n(x) \rightarrow v$ for some vertex v of the edge e on which $f_n(x)$ lies. $\{L(f_n)\}$ is convergent implies that it is bounded; let K be its upper bound. Let x' be some point of X_1 joined to x by an edge l of X_2 . $L(f_n(l)) \leq K$, so $f_n(x') \rightarrow \infty$ in some 2-simplex of M , ie, it also goes to v in M . Inductively, we can repeat this process and use the connectedness of X_2 to argue that $f_n(y) \rightarrow v$ in M for all points y in X_1 . This contradicts the assumption that f is not normally homotopic arbitrarily close to a vertex of M . So $\{f_n\}$ must be bounded.

2. We can adopt the proof for (1) by using the fact that f cannot be normally homotopic arbitrarily close to a vertex of M by any homotopy fixing ∂F . \square

Chapter 3

Appendix - Previously Covered Theorems

Lemma 3.0.1. *Any homeomorphism of the sphere S^2 is isotopic to the identity or to a reflection.*

Theorem 3.0.2. *Let M be a closed connected orientable surface. Let Σ be a closed 1-submanifold of M with r components, such that no component of Σ is nullhomotopic in M and no two components of Σ are parallel. Then an upper bound for r is given by the following formulae:*

*0 if M is S^2
1 if M is the torus T
 $-3\chi(M)/2$ otherwise.*

Lemma 3.0.3. *If N is a non-orientable surface, $N\#T$ and $N\#K$ are homeomorphic, where T is the torus and K is the Klein bottle.*

References

1. *An Introduction to 3-Manifolds* by G Peter Scott (primary reference)
2. *Introduction to 3-Manifolds* by Jennifer Schultens
3. *An Introduction to Geometric Topology* by Bruno Martelli
4. *Notes on Basic 3-Manifold Topology* by Allen Hatcher