Topology and Geometry of 2 and 3 dimensional manifolds

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1 Introduction

In this project I started with studying the classification of Surface and then I started studying some preliminary topics in 3 dimensional manifolds.

2 Surfaces

Definition 2.1. *A simple closed curve c in a connected surface S is separating if $S \setminus c$ has two components. It is non-separating otherwise. If c is separating and a component of $S \setminus c$ is an annulus or a disk, then it is called inessential. A curve is essential otherwise.*

An observation that we can make here is that a curve is non-separating if and only if its complement is connected. For manifolds, connectedness implies path connectedness and hence c is non-separating if and only if there is some other simple closed curve intersecting c exactly once.

2.1 Decomposition of Surfaces

Lemma 2.2. *A closed surface S is prime if and only if S contains no essential separating simple closed curve.*

Proof. Consider a closed surface S. Let us assume that $S = S_1 \# S_2$ is a non trivial connected sum. The curve along which $S_1 \setminus (disc)$ and $S_2 \setminus (disc)$ where identified is an essential separating curve in S. Hence, if S contains no essential separating curve, then S is prime. Now assume that there exists a essential separating curve c in S. This means neither of the components of $S \setminus c$ are disks i.e. $S_1 \neq S^2$ and $S_2 \neq S^2$. □

Lemma 2.3. *The torus, the projective plane and the sphere are all prime surfaces.*

Lemma 2.4. *The only closed prime surfaces are the torus, the projective plane and the sphere.*

Theorem 2.5. *(The classification of closed compact surfaces) Every closed compact connected surface is homeomorphic to a sphere or a connected sum of tori or a connected sum of projective planes.*

Example The figure below gives the prime decomposition of a 2-torus.

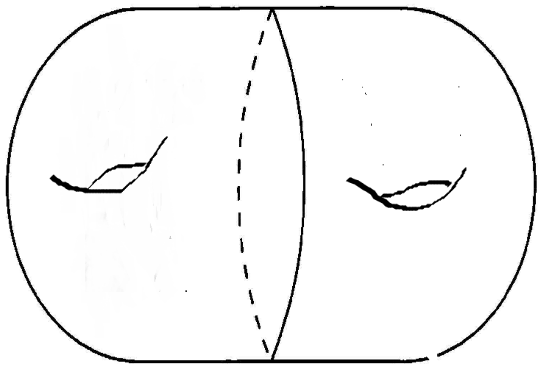


Figure 1: Prime decomposition of a 2-torus

Definition 2.6. *The number of tori in the connected sum is called the genus of a orientable surface. For non orientable surfaces, the number of projective planes in the connected sum is called the genus.*

Corollary 2.7. *For a surface S of genus g, $\chi(S) = 2 - 2g$. For a non orientable surface S of genus g, $\chi(S) = 1 - g$. Here $\chi(S)$ is the Euler characteristic of the surface S.*

There is one another way of decomposition of surfaces, the pants decomposition.

Definition 2.8. *A pants decomposition of a compact surface S is a collection of pairwise disjoint simple closed curves $\{c_1, ..., c_n\}$ such that each component of*

$$S \setminus (c_1 \cup c_2 \cup ... \cup c_n)$$

is a pair of pants. Two pants decompositions are equivalent if they are isotopic.

Pants decomposition of surfaces are not unique. The following figure contains different decomposition of 2-torus.

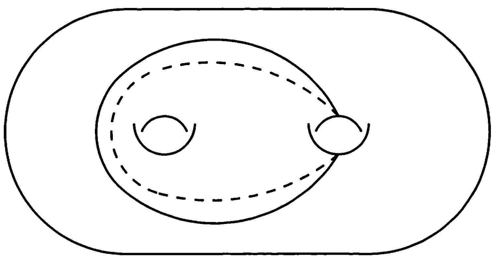


Figure 2: Pants decomposition of a 2-torus

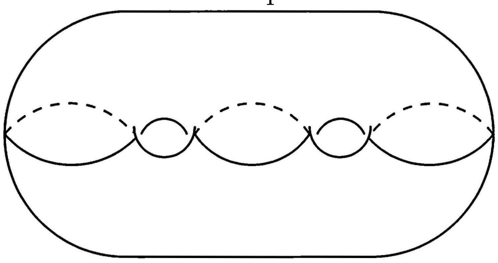


Figure 3: Pants decomposition of a 2-torus

2.2 Covering spaces and branched covering spaces

Definition 2.9. *Let $p : E \to B$ be a continuous map. The open set $U \subset B$ is said to be evenly covered if $p^{-1}(U)$ is a disjoint union of open sets $\{U_\alpha\}$ such that $\forall \alpha,\ p\mid_{U_\alpha}: U_\alpha \to U$ is a homeomorphism.*

Definition 2.10. *Let E be a manifold, B a connected manifold, and $p : E \to B$ a continuous map. The triple (E, B, p) is a covering if $\forall b \in B$ there is an open set $U \subset B$ with $b \in U$ that is evenly covered. The map p is called the covering map.*

Example The triple $(\mathbb{R}, \mathbb{S}^1, p)$, where $p : \mathbb{R} \to \mathbb{S}^1$ is given by $p(x) = e^{2\pi ix}$ is a covering.

Example The triple $(\mathbb{R}^2, \mathbb{T}^2, p)$, where $p : \mathbb{R}^2 \to \mathbb{T}^2$ is given by $p(x) = (e^{2\pi ix},\ e^{2\pi iy})$, is a covering.

If suppose $p : E \to B$ is a covering map, then p is a local homeomorphism, but p a local isomorphism may not be a covering map. For example, consider $p : \mathbb{R}^+ \to \mathbb{S}^1$ given by $p(x) = e^{2\pi ix}$.

Definition 2.11. *Let (E, B, p) be a covering space. A homeomorphism $t : E \to E$ is a covering transformation if $p \circ t = p$.*

Example For $n \in Z$ the map $t : \mathbb{R} \to \mathbb{R}$ given by $t(x) = x+n$ is a covering transformation for the covering $p(x) = e^{2\pi ix}$ where $p : \mathbb{R} \to \mathbb{S}^1$.

The covering transformation for a given space forms a group which is called the group of covering transformations. Example, for the triple $(\mathbb{R}, \mathbb{S}^1, p)$ where $p : \mathbb{R} \to \mathbb{S}^1$ given by $p(x) = e^{2\pi ix}$ is a covering, the group covering transformation is $\mathbb{Z}$.

Definition 2.12. *Let $E,\ B$ be manifolds, E' a submanifold of $E,\ B'$ a submanifold of B, and $p : E \to B$ a continuous map. The quintet (E, E', B, B', p) is a branched covering if*
(i) $P\mid_{E\setminus E'}: E \setminus E' \to B \setminus B'$ is a covering map;
(ii)$P\mid_{E'}: E' \to B'$ is a covering map.
Here B' is called the branch locus and E' is called the ramification locus.

Example The quintet $(\mathbb{C}, 0, \mathbb{C}, 0, f)$, where $f(z) = z^n$, is a branched covering.

2.3 Homotopy and Isotopy on Surfaces

Isotpies are always homotopies. But in case of surfaces, we have more. Here, there is a the homotopy classes of closed simple curves correspond to the isotopy classes of simple closed curves.

Definition 2.13. *Let F be a surface and let $C = c_1, ..., c_n$ be a collection of simple closed curves in F that have been isotoped to intersect in a minimal number of points. We say that C is filling if $F \setminus (c_1 \cup ... \cup c_n)$ is a union of disks.*

Lemma 2.14 (Alexander Trick)**.** *. Suppose that $f : \mathbb{D}^n \to \mathbb{D}^n$ is a homeomorphism such that $f\mid_{\partial\mathbb{D}^n}$ is the identity. Then f is isotopic to the identity.*

Proof. Define $H : \mathbb{D}^n \times I \to \mathbb{D}^n$ as follows

$$H(x,t) = \begin{cases} tf(x/t) \; if \; 0 \leq \|x\| \; < t, \\ x \qquad\quad if \; t \leq \|x\| \leq 1 \end{cases} .$$

The schematic for the Alexander Trick is shown in the figure below. This is the required isotopy.

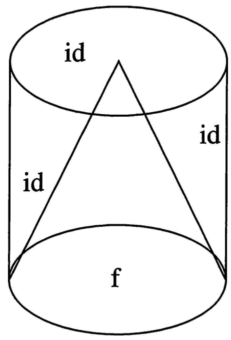


Figure 4: Schematic for the above proof

□

Theorem 2.15. *Let F be a closed orientable surface and $h : F \to F$ a homeomorphism. Now, if h is homotopic to the identity, then h is isotopic to the identity.*

2.4 The Mapping Class Group

The set of self-diffeomorphisms forms a group which is denoted by Diffeo(S) if S is oriented surface. Also, the set of orientation preserving self-diffeomorphism $\text{Diffeo}_+(S)$ and the set of self-diffeomorphisms homotopic to identity $\text{Diffeo}_0(S)$ form normal subgroups. Homeomorphisms are isotopic to diffeomorphisms in case of surfaces, hence we will be considering only diffeomorphisms.

Definition 2.16. *Let S be a compact orientable surface. The mapping class group of S, denoted by $\mathcal{MCG}(S)$ is $\text{Diffeo}_+(S)/\ \text{Diffeo}_0(S)$, the group of orientation-preserving self-diffeomorphisms of S modulo the subgroup consisting of self-diffeomorphisms of S homotopic to the identity.*

Lemma 2.17. $\mathcal{MCG}(\mathbb{T}^2) \cong SL(2, \mathbb{Z})$.

We will denote the embedding of a closed regular neighborhood of a closed simple orientation preserving curve c as $N(c)$ and an open regular neighborhood as $\eta(c)$.

Definition 2.18. *Let c be an orientation-preserving simple closed curve in a compact surface S and let $N(c)$ be a regular neighborhood of c that is oriented via the parametrization $i : \mathbb{S}^1 \times [O, 1] \to \eta(c)$. A map from $f : S \to S$ is called a left Dehn twist around c if*

(i) $f\ |_{S \setminus N(c)}$ is the identity map; and
(ii) $f\ |_{N(c)}$ is the map of the annulus given by

$$f(e^{2i\pi\theta},\ t) = (e^{2i\pi(\theta+t)}) \quad \forall (e^{2i\pi\theta}, t) \in \mathbb{S}^1 \times [0,1].$$

Likewise, a map $f : S \to S$ is called a right Dehn twist around c if

(i) $f\,|_{S\setminus N(c)}$ *is the identity map; and*
(ii) $f\,|_{N(c)}$ *is the map of the annulus given by*

$$f(e^{2i\pi\theta},\ t) = (e^{2i\pi(\theta-t)}) \quad \forall (e^{2i\pi\theta}, t) \in \mathbb{S}^1 \times [0,1].$$

Theorem 2.19 (Dehn, 1938; Lickorish, 1962)**.** *Every surface isomorphism is isotopic to a composition of Dehn twists i.e. the mapping class group is generated by Dehn twists.*

We will state two lemmas and use them to prove the above theorem.

Lemma 2.20. *If the simple closed curves α, β in the compact surface S intersect exactly once, then there is a pair f_1, f_2 of Dehn twists such that $f_2 \cdot f_1(\alpha)$ is isotopic to β.*

Lemma 2.21. *If α, β are oriented simple closed curves in the orientable surface S, then there is a series of Dehn twists $f_1, f_2, ..., f_k$ such that $f_k \cdot ... \cdot f_2 \cdot f_1(\alpha)$ is either disjoint from β or intersects β in exactly two oppositely oriented points.*

Sketch of the proof. Let S be a closed orientable surface $a_1, b_1..., a_g, b_g$ be collection of curves which cut S into a disk. Self-homeomorphisms take this collection of curves to another such collection of curves. Now, using a series of Dehn twists we reverse the effect of the surface diffeomorphism on the new collection of curves. This can be done because of the above stated lemmas. The collection of Dehn twists can be seen as "undoing" the effect of the surface diffeomorphism on the specified collection of curves. Using the Alexander Trick, we get that the map on the complementary disk is isotopic to identity.

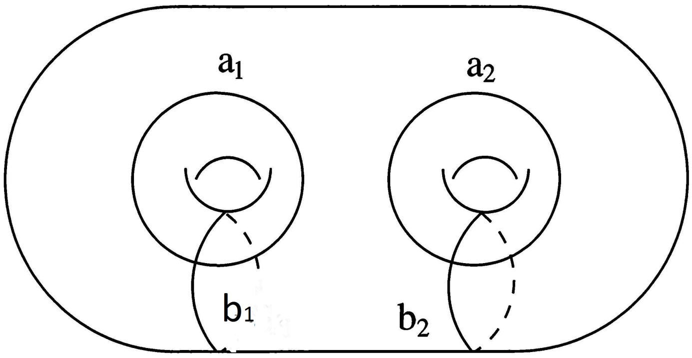


Figure 5: Standard collection of curves a_1, b_1, a_2, b_2 in a genus 2-surface

Definition 2.22. *A diffeomorphism $h : S \to S$ is called periodic if $h^n = id_S$ for some $n \in \mathbb{N}$. It is called reducible if $\exists$ an essential simple closed curve $c \subset S$ such that $\phi(C) = C$ (setwise).*

Example Let $h_1 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $h_1(e^{i\theta}, e^{i\phi}) = (e^{-i\phi}, e^{i\theta})$. Then h_1 is periodic; in fact, $h_1^4 = id$.
Let $h_2 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $h_2(e^{i\theta},\ e^{i\phi}) = (e^{i\theta},\ e^{i(\theta+\phi)})$. Then h_2 is reducible, and it preserves the essential simple closed curve $\{(1, e^{i\phi}) \mid 0 \leq \phi < 2\pi\}$.

Let $h_3 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $h_2(e^{i\theta},\ e^{i\phi}) = (e^{i(2\theta+\phi)},\ e^{i(\theta+\phi)})$. This is neither periodic nor reducible.

Definition 2.23. *An element of the mapping class group of S is periodic if it has a periodic representative. An element of the mapping class group of S is reducible if it has a reducible representative.*

3 3-dimensional Manifolds

3.1 Bundle

Definition 3.1. *A bundle is a quartet (M, F, B, π) where $M, F, andB$ are manifolds and $\pi : M \to B$ is a continuous map such that the following hold:*

(i) for every $b \in B, \pi^{-1}(b)$ is homeomorphic to F;
(ii) there is an atlas $\{U_\alpha\}$ for B such that $\forall\alpha\exists$ a homeomorphism

$$h_\alpha : \pi{-}1(U_\alpha) \to U_\alpha \times F;$$

(iii) the following diagram commutes (the right arrow is projection onto the first factor) :

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times F \\ \pi\downarrow & & \downarrow \\ U_\alpha & \xrightarrow{\text{id}} & U_\alpha \end{array}$$

.

Here M is called the total space, F is called the fiber, B is called the base space, and π is called the projection. A pair (U_α, h_α) is called a bundle chart. The family $\{(U_\alpha, h_\alpha)\}$ is called a bundle atlas. We sometimes refer to (M, F, B, π) simply as an $F-$bundle over B.

Example Any product manifold $X \times Y$ is a bundle. The total space is $X \times Y$, the fiber is Y (or X, respectively) , the base space is X (or Y, respectively) , and the projection is projection onto the first (or second, respectively) factor.

Example The Mobius band is a bundle with fiber I and base space $\mathbb{S}^1$. The projection can be described by considering the rectangle in Figure below. The projection consists in vertical projection onto the core curve, that is, the marked curve.

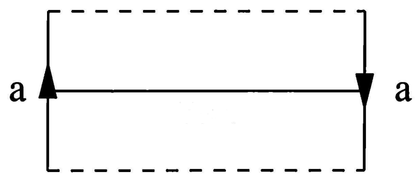
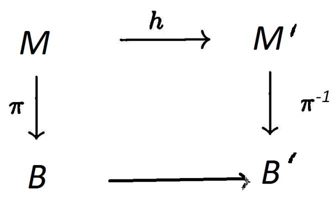


Figure 6: The Mobius strip .

Example Let S be a closed connected n-manifold and $f : S \to S$ a homeomorphism. The mapping torus of f is the $(n + 1)$-manifold obtained from $S \times [-1, 1]$ by identifying the points $(x, -1)$ and $(f(x), 1)$, $\forall x \in S$. The mapping torus of f is a bundle with fiber S and base space $\mathbb{S}^1$. It is denoted by $(S \times I)/\sim f$. The Mobius band is an example of a mapping torus, with $S = [-1, 1]$ and $f : [-1, 1] \to [-1, 1]$ given by $f(x) = -x$.

Definition 3.2. *Suppose that* (M, F, B, π) *and* (M', F, B', π) *are bundles. An isomorphism between the bundles is a pair of homeomorphisms* $h : M \to M'$ *and* $f : B \to B'$ *such that the following diagram commutes:*



Two bundles are said to be isomorphic or equivalent if there is an isomorphism between them.

Example The annulus and the Mobius band are inequivalent bundles.

Definition 3.3. *A bundle that is isomorphic to a product bundle is called a trivial bundle. A bundle that is not a trivial bundle is called a non-trivial bundle.*

Example The annulus is a trivial bundle. The Mobius band is a non-trivial bundle.

Definition 3.4. *Given a bundle* $E = (M, F, B, \pi)$ *and a submanifold* B' *of* B*, the restriction of* E *to* B' *is a bundle with total space* $M' = \pi^{-1}(B')$ *, fiber* F*, base space* B'*, and projection* $\pi\mid_{M'}$ *We denote this restriction by* $E\mid_{B'}$*.*

Example Construction of a non-trivial I-bundle over Mobius band. In the Figure **??**, the points lie in $[-1, 1] \times [-1, 1]$. To form the Mobius band, we identify $(-1, x)$ with $(1, -x)$. Now consider $[-1, 1] \times [-1, 1] \times [-1, 1]$ and identify $(-1, x, y)$ with $(1, -x, -y)$. The result is a non-trivial bundle over the Mobius band. We see that the total space of this bundle is a torus as shown in Figure 7.

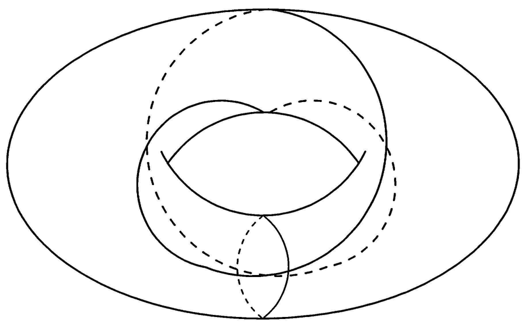


Figure 7: Solid torus containing the mobius band

Definition 3.5. *A section of a bundle is a continuous map* $\sigma : B \to E$ *such that* $\pi \circ \sigma = id_B$*(where* id_B *is the identity map on* B*) .*

Definition 3.6. *Let* M *be an n-manifold. Let* S *be a submanifold of* M *of dimension* m*. A regular neighborhood of* S *is a submanifold* $N(S)$ *of* M *of dimension* n *that is the total space of a bundle over* S *with fiber* $\mathbb{B}^{n-m}$*. A regular neighborhood of a 1-manifold in a 3-manifold is also called a tubular neighborhood. A regular neighborhood of an* $(n-1)-$*dimensional submanifold of an n-manifold that is a trivial bundle is also called a collar (or, if* $n = 2$*, a bicollar) . An open regular neighborhood of* S *is a submanifold* $\eta(S)$ *of* M *of dimension* n *that is the total space of a bundle over* S *with fiber interior* $(\mathbb{B}^{n-m})$*.*

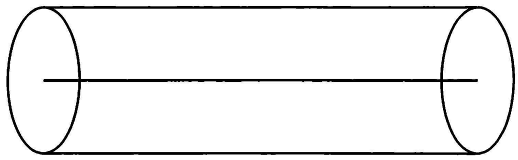


Figure 8: Regular neighborhood of a 1- manifold in a 3-manifold.

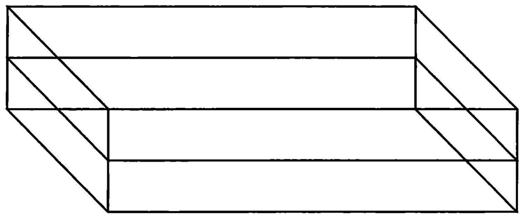


Figure 9: Regular neighborhood of a 2- manifold in a 3-manifold.

Theorem 3.7. *Let M be an n−manifold and let S be a k−manifold in M . There exists a regular neighborhood N(S) for S in M. Furthermore, any two regular neighborhoods of S in M are isotopic.*

Theorem 3.8. *If S is a surface in the 3-manifold M and both M and S are orientable, then N(S) is a trivial I-bundle.*

Definition 3.9. *Let S be a surface in the 3-manifold M. We say that S is a 2-sided surface if a regular neighborhood of S in M is a trivial I-bundle. We say that S is a I -sided surface if a regular neighborhood of S in M is a non-trivial, or twisted, I-bundle.*

3.2 Schönflies Theorem

Definition 3.10. *A 3-manifold M is irreducible if every 2-sphere in M bounds a 3-ball. A 3-manifold is reducible if it contains a 2-sphere that does not bound a 3-ball.*

Example The 3-manifold $\mathbb{S}^2 \times \mathbb{S}^1$ is reducible.

We will prove the following lemma which will be used to prove of Schönflies theorem.

Lemma 3.11. *Let S_1, S_2 be surfaces in a 3-manifold M . Suppose that $S_1 \cap S_2$ contains a simple closed curve c such that, for i = 1, 2, one component of $Si \setminus c$ is a disk D_i such that D_2 is disjoint from S_1 . If $c \cup D_1 \cup D_2$ bounds a 3-ball in M, then*

- *$(S_1 \setminus D_1) \cup D_2$ is isotopic to S_1;*
- *there is an isotopy of S_1 that eliminates the curve c from $S_1 \cap S_2$ and introduces no new components of intersection.*

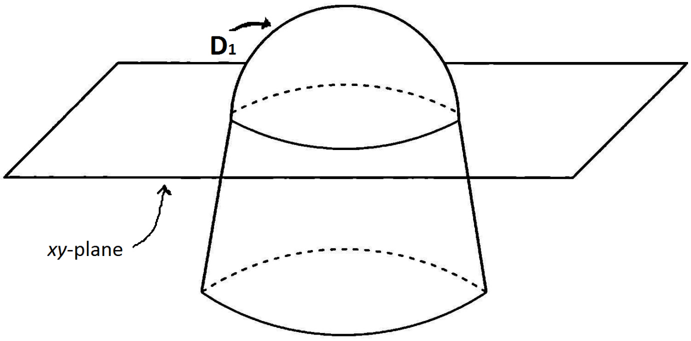


Figure 10: .

Proof. Since $D_1 \cup D_2$ is homeomorphic to $\mathbb{S}^2$ any neighborhood of the 3-ball in M bounded $D_1 \cup D_2$ is homeomorphic to neighborhood of upper hemisphere of the standard 3-ball in $\mathbb{R}^3$. From the Figure 10 we see that we can define a straight line projection of, say D_1 onto the xy-plane which is an isotopy that replaces S_1 with $(S_1 \setminus D_1) \cup D_2$. The straight line isotopy of the portion of S_1 above $y = -\epsilon$ to the plane $y = -\epsilon$ gives the isotopy removing the curve c from $S_1 \cap S_2$. Figure 11 shows the $S_1 \cup S_2$ after the second isotopy. □

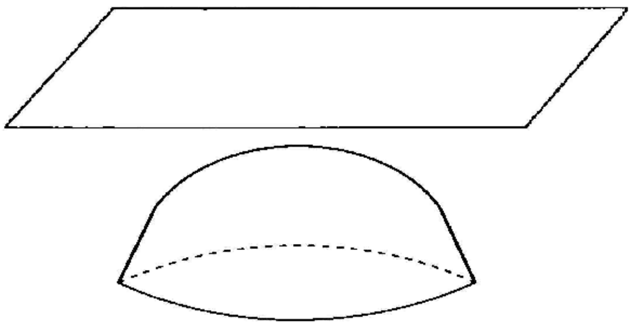


Figure 11: .

Definition 3.12. *Let X be an n-manifold and x a subset of X (typically a sub manifold of dimension $n-1$). To cut X along x means to consider $X \setminus \eta(x)$. Likewise, x cuts X into a given set of manifolds means that $X \setminus \eta(x)$ consists of this set of manifolds.*

Theorem 3.13 (Schönflies Theorem)**.** *Any 2-sphere in $\mathbb{R}^3$ bounds a 3-ball.*

Proof. For the following proof of the theorem we will work in DIFF category. Firstly, let us define a *height function.* A Morse function with at most two critical points, a maximum and a minimum, is called a height function h.
Let $S \subset \mathbb{R}^3$ be a 2-sphere submanifold. We isotope S so that the height function $h : \mathbb{R}^3 \to \mathbb{R}$ given by projection onto the third coordinate restricts to a Morse function $h\mid_S$ By the Poincare-Hopf Index Theorem,

$$\#maxima + \#minima - \#saddles = \chi(S) = 2$$

Since S is compact, $h\mid_S$ has at least one maximum and one minimum. Let n be the number of saddle points. If $n = 0$, then S has one maximum, y_2, and one minimum, y_1 We may assume, by multiplying h with an appropriate constant, that $y_2 - y_1 = 2$. We may further assume, by translating h upwards or downwards, that $y_2 = 1$ and hence $y - 1 = -1$. Each r such that $-1 < r < 1$ corresponds to a level curve in S that is a simple closed curve in the plane $z = r$. It follows from the Jordan Curve Theorem that each such curve can be isotoped within the plane $z = r$ into the circle such that $x_2 + y_2 = 1 - r_2$. The isotopy is vary smoothly with r. Thus S is isotopic to the standard 2-sphere in $\mathbb{R}^3$ and hence bounds a 3-ball. When n=1, either there are two maxima and a minima or vice-versa but both contain a saddle. Now there are two possibilities, one a non-nested saddle or a nested saddle as shown in the following figures.

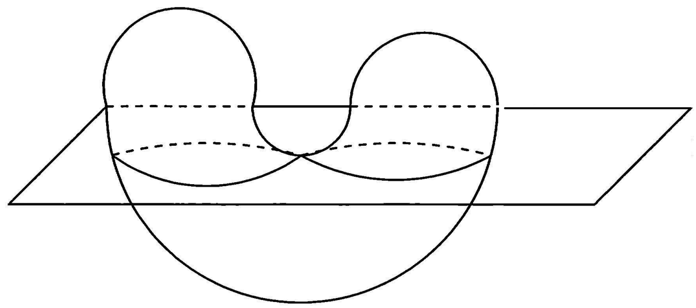


Figure 12: A sphere with one non-nested saddle .

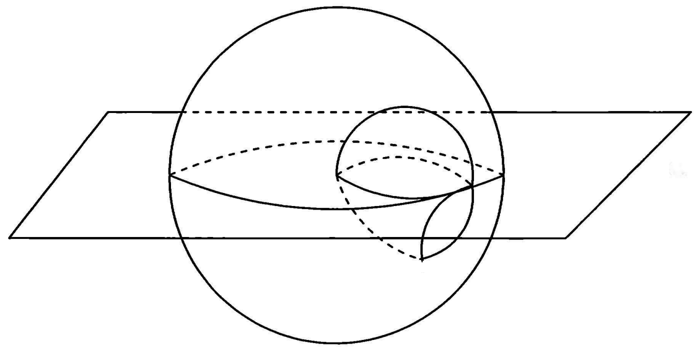


Figure 13: Sphere with one nested saddle .

In both the cases there is a plane $z = c$ (containing the saddle) that meet S in a figure 8. Let the plane be H. Then $H \cap S$ is a figure 8 that cuts S into three disks, D_1, D_2, D_3 . Each of these disks contains exactly one critical point (either a maximum or a minimum) . Furthermore, the figure 8 cuts H into two disks and one unbounded component. At least one of these two disks, call it D, shares its boundary with one of D_1, D_2, D_3 . Attach ∂D to the boundary of the appropriate D_1, D_2, D_3 , say D_1 , to obtain a piecewise smooth 2-sphere $D \cup D_1$. After a small isotopy (that introduces only one new critical point) , $D \cup D_i$ has exactly two critical points (a maximum and a minimum). Since it has no saddles, the case above applies and it bounds a 3-ball. By previous lemma, $(S \setminus D_1) \cup D$ is isotopic to S. After a small isotopy, $(S \setminus D_1) \cup D$ is smooth and has no saddles. Hence the case reduces to $n = 0$.

For $n \geq 2$ we assume that the hypothesis is true for every 2-sphere with less than n saddles. Then there is a plane H', given by $z = r$, for a regular value r, such that there are saddle points both above and below H' and such that the number of components of $H' \cap S$ is minimal. Denote the number of components of $H' \cap S$ by $\#| H' \cap S |$. We will now use induction on $(n, \# \mid H' \cap S \mid)$ to prove the result. Since $H' \cap S$ is a compact 1-manifold (without boundary) , so $H' \cap S$ is a finite union of disjoint simple closed curves in H' and a be the innermost componentof the above curve in H'. Then there is a a disk $D \subset H'$ that meets S only in a. Thus a separates S into two disks, D_1, D_2. Set $S_1 = D \subset D_1$ and $S_2 = D \subset D_2$. Then both S_1 and S_2 are piecewise smooth 2-spheres in $\mathbb{R}^3$. We isotope S_1 and S_2 slightly (in a way that introduces only one new critical point) to be smooth. See the Figures 14 and **??**.

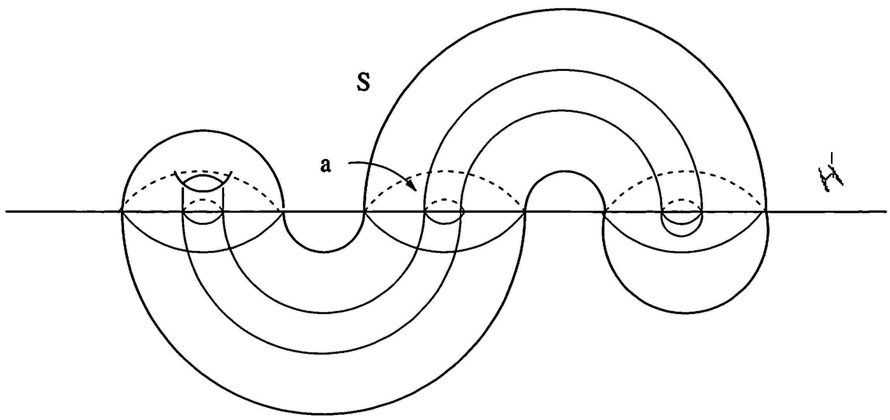


Figure 14: A choice of a .

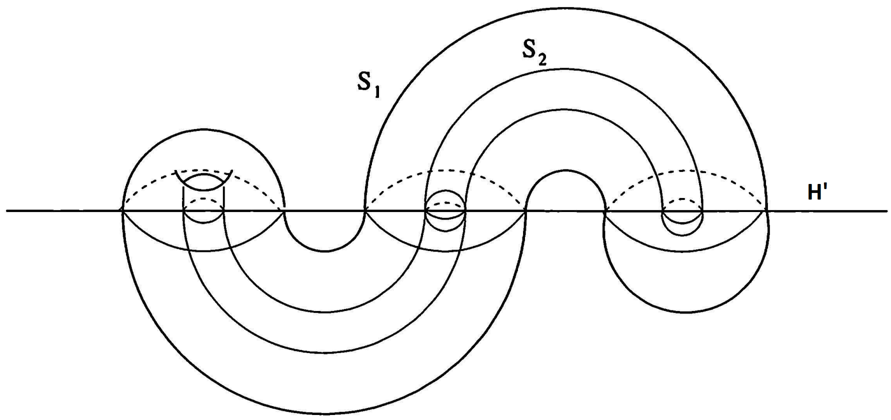


Figure 15: The two spheres $S_1,\ S_2$.

Now, at least one of S_1 and S_2 has lesser number of saddles than S. Let it be S_2, then S is isotopic to S_1 by the previous lemma. Since S_1 cannot have more saddles than S, another isotopy as the types in the lemma reduces the number of intersections with H'. Thus by induction we have proved the theorem. □

Theorem 3.14 (Alexander)**.** *Let $\mathbb{T}^2$ be a torus submanifold in $\mathbb{S}^3$, then one of the components $\mathbb{S}^3 \setminus \mathbb{T}^2$ has closure homeomorphic to a solid torus $\mathbb{S}^1 \times \mathbb{B}^2$.*

Definition 3.15. *A 3-manifold M is boundary irreducible if every simple closed curve c in ∂M that bounds a disk in M cuts M into two 3-manifolds one of which is a 3-ball.*

Example A boundaryless 3-manifold is necessarily boundary irreducible.

Example The 3-ball is boundary irreducible.

Example The solid torus $V = \mathbb{S}^1 \times \mathbb{D}^2$ is not boundary irreducible. Indeed, consider the curve $m = p \times \partial\mathbb{D}^2 \subset \partial V$. Then m is non-separating in ∂V; hence the disk $D = px\mathbb{D}^2$ is also non-separating.

Definition 3.16. *For V, m, D in the above, m is called a meridian and D is called a meridian disk. A simple closed curve in ∂V that intersects m in one point is called a longitude.*

3.3 Prime but Reducible Manifolds

There are two compact connected 3-manifolds that deserve attention in the context of prime and reducible 3-manifolds. These two 3-manifolds are both $\mathbb{S}^2$-bundles over $\mathbb{S}^1$. In fact, they are both mapping tori with S = $\mathbb{S}^2$. One of these 3-manifolds is the product manifold $\mathbb{S}^2 \times \mathbb{S}^1$ $(= (\mathbb{S}^2 \times I)/\ id_{\mathbb{S}^2})$. The other is also a mapping torus, but twisted, over $\mathbb{S}^2$.

Definition 3.17. *We denote by $\mathbb{S}^2 \times \mathbb{S}^2$ the 3-manifold $(\mathbb{S}^2 \times I)/\ I$ where $f : \mathbb{S}^2 \to \mathbb{S}^2$ is the antipodal map. We also call $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ the twisted product of $\mathbb{S}^2$ over $\mathbb{S}^2$.*

Definition 3.18. *A submanifold A of a connected manifold X is separating if $X \setminus A$ has at least two components; otherwise it is non-separating.*

Theorem 3.19. *An irreducible closed connected 3−manifold is prime. An orientable closed connected prime 3−manifold is either irreducible or $\mathbb{S}^2 \times \mathbb{S}^1$.*

Proof. A non-trivial connected sum of 3-manifolds contains a sphere that does not bound a 3-ball. Hence an irreducible 3-manifold is prime. Assume M is prime and let S be a 2-sphere in M. If S is separating, then $M - S$ has two components, N_1 , N_2. If neither N_1 nor N_2 is a 3-ball, then $M = N_1 \# N_2$ is not prime. Thus either N_1 or N_2 is a 3-ball; i.e. , S bounds a 3-ball. If S is non-separating, let a be a simple closed curve in M that intersects S once transversely. Furthermore, let $N(S)$ be a regular neighborhood of S in M and let $N(a)$ be a regular neighborhood of a. Both $N(a)$ and $N(S)$ are trivial I-bundles, so, in particular, $\partial N(S)$ consists of two copies of S. Thus by transversal isotopy of S and a, $N(a)$ intersect $N(S)$ in a solid cylinder. Let $\bar{M}$ be the submanifold of M obtained by removing the interior of $N(S) \cup N(a)$. Then $\partial\bar{M}$ is a 2-sphere $\bar{S}$ in M and $\bar{S}$ is a separating 2-sphere. To one side of $\bar{S}$ is $N(S) \cup N(a)$. this means there is a 3-ball to the other side of $\bar{S}$. Therefore $M = N(S) \cup N(a) \cup (3 - ball) = \mathbb{S}^2 \times \mathbb{S}^1$. □

3.4 Incompressible Surfaces

The essential simple curves are studied a lot because using them we can properly define cut and paste techniques which helps to classify surfaces. Like wise, we would like to do for 3-manifolds and for other higher dimensions if possible. Here, for 3-manifolds, instead of essential curves we study incompressible surfaces.

Definition 3.20. *A submanifold S in a compact n-manifold M is proper if $\partial S = S \cap \partial M$.*

Definition 3.21. *A simple arc α in a surface F is essential if there is no simple arc β in ∂F such that $\alpha \cup \beta$ is a closed 1-manifold that bounds a disk in F.*

Definition 3.22. *Let M be a 3-manifold. A surface S in M is compressible if it is either a 2-sphere bounding a 3-ball in M or there is a simple closed curve c in S that bounds a disk D, called the compressing disc, with interior in $M \setminus S$ but that bounds no disk such that its interior is a component of $S \setminus c$. A surface which is not compressible is called incompressible.*

The following Figures illustrate examples of compressible and incompressible surfaces.

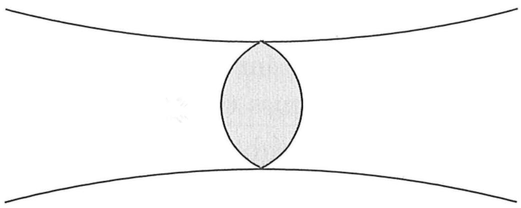


Figure 16: A compressing disk for a surface .

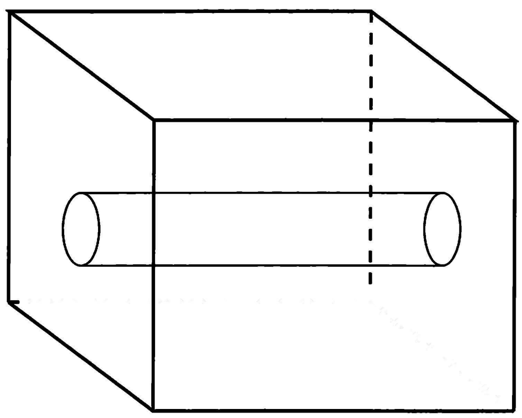


Figure 17: A compressible 2-torus in the 3-torus .

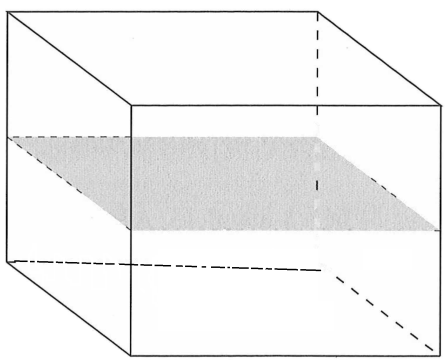


Figure 18: An incompressible 2-torus in the 3-torus .

If a surfaces is incompressible, then it is a union of disks and each component of ∂S is a simple closed curve c. If we consider $S \subset \mathbb{B}^3$, then $\partial \mathbb{B}^3 \setminus c$ is the union of two disjoint disks and hence bounds a disk in $\mathbb{B}$. Thus each component of ∂S must bound a disk in S as well.

Definition 3.23. *Let M be a 3-manifold. A surface $S \subset M$ is boundary compressible, or ∂-compressible, if there is an essential simple arc α in S and an essential simple arc β in ∂M such that $\alpha \cup \beta$ is a closed 1-manifold that bounds a disk D in M with interior disjoint from S. A surface that is not boundary compressible is boundary incompressible.*

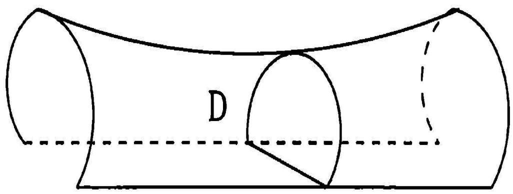


Figure 19: A boundary compressing disk .

Definition 3.24. *Let M be a connected 3-manifold. A 2-sphere S subset M is essential if it does not bound a 3-ball. A surface $F \subset M$ is boundary parallel if it is separating and a component of $M \setminus F$ is homeomorphic to $F \times I$. A surface F in a 3-manifold M is essential if it is incompressible, boundary incompressible, and not boundary parallel.*

Definition 3.25. *An orientable irreducible 3-manifold that contains a proper essential surface is called a Haken 3-manifold.*

3.5 Dehn's lemma

Theorem 3.26. *Suppose that M is a 3-manifold and $f : D \to M$ is a continuous map from the disk into M such that $f(\partial D) \subset \partial M$. If for some neighborhood U of ∂D, $f\mid_U$ is an embedding, then $f\mid_{\partial D}$ extends to an embedding.*

Theorem 3.27 (The Loop Theorem)**.** *Let M be a 3-manifold and F a connected surface in ∂M . If N is a normal subgroup of $\pi_1(F)$ and if $ker(\pi(F) \to \pi((M))/N \neq 0$, then there is a proper embedding $g : (D, \partial D) \to (M, F)$ such that $[g\mid_{\partial D}]$ is not in N .*

Theorem 3.28 (The Sphere Theorem)**.** *Let M be an orientable 3-manifold and N a $\pi_1(M)-$ invariant subgroup of $\pi_2(M)$. If $\pi_2(M)/N \neq 0$, then there is an embedding $g : \mathbb{S}^2 \to M$ such that [g] is not in N.*

References

1. Jennifer Schultens; Introduction to 3-manifolds, Graduate Studies in Mathematics;(2014) vol. 151; American Mathematical Society