Semester Project Report

Geometry of Surfaces

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Contents

1 Introduction

In this report, we study the Sphere and Hyperbolic space, with the help of isometries. To do this, we identify the lines of these spaces, via reflections and use that to generate all possible isometries. This method of studying the space using isometries works well in these two cases because they are manifolds of constant curvature. We in fact further exploit this constant curvature to help classify some kinds of Spherical and Hyperbolic surfaces. None of the below material is original work by the author. It has been taken from the several books and notes mentioned in the references. The main reference followed for the material was [1] with the others providing relevant information and context.

2 Preliminaries

2.1 Euclidean Plane

The Euclidean Plane is the set $R^2 = \{(x,y)|x,y \in \mathbf{R}\}$ together defined with the distance function:

$$d(P_1,P_2) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Where $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$

Our goal in this section is introduce isometries and some ideas related to it.

Definition 2.1. An isometry is a function f on a metric space (S,d) such that:

$$d(f(P_1),f(P_2)) = d(P_1,P_2) \text{ for all } P_1,P_2 \in S$$

Some examples of Isometries of the plane are as follows:

1. *Translations:* This function $\tau_{a,b}$ moves the entire plane. It is defined as:

$$\tau_{a,b}(P) = (x+a,y+b) \text{ , P = (x,y)}$$

2. *Reflections:* This function reflects the entire plane about a line. We shall define it about only the x-axis, but it can be defined over any line by a change of coordinates.

$$\Phi(P) = (x,-y) \text{ , P = (x, y)}$$

3. *Rotations:* This function rotates the plane about a point by any angle θ. We shall define it about the origin, but can be shifted to any point via a change of coordinates.

$$r_\theta(P) = (xcos\theta - ysin\theta, xsin\theta + ycos\theta) \text{ , P = (x, y)}$$

From the definition we can see that

$$r_\theta r_\phi(P) = (x(cos\theta cos\phi - sin\theta sin\phi) - y(sin\theta cos\phi + sin\phi cos\theta), x(sin\theta cos\phi + sin\phi cos\theta) + y(cos\theta cos\phi - sin\theta sin\phi)) =$$

$$(xcos(\theta+\phi) - ysin(\theta+\phi), xsin(\theta+\phi) + ycos(\theta+\phi)) = r_{\theta+\phi}(P)$$

Hence, since $r_\theta r_\phi(P) = r_{\theta+\phi}(P)$ we can conclude that $r_\theta^{-1} = r_{-\theta}$

It can be easily checked that these examples are isometries, by just routine calculations.
It is also clear that the set of isometries is closed under composition, as:

$$d(f \circ g(P_1), f \circ g(P_2)) = d(g(P_1),g(P_2)) = d(P_1,P_2)$$

Where f and g $\in Iso(\mathbf{R^2})$

Hence if we take the set of isometries generated by the rotations, reflections and translations via the composition map, we see that since each of them are invertible, with inverses $r_{-\theta}$, Φ and $\tau_{-a,-b}$ respectively, they form a subgroup within the set of isometries.

2.1.1 Change of Coordinates

As described earlier, a lot of the reflections and rotations in the plane can be obtained by a change of coordinates. The corresponding functions obtained after a change of coordinates can be described algebraically using conjugation. A few relevant examples are described below:

Example 2.2. *Rotation about any point* P by an angle θ (denoted by $r_{P,\theta}$) can be done by first translating P to the origin, performing the rotation about the origin and then translating back to P. Hence, the rotation is described as a conjugation of r_θ with $\tau_{a,b}$ as:

$$r_{P,\theta} = \tau_{a,b} r_\theta \tau_{-a,-b} = \tau_{a,b} r_\theta \tau_{a,b}^{-1}$$

Where $P = (a, b)$ is any point in R^2.

Example 2.3. *Reflection about any line* L passing through the origin (denoted by Φ_L) can be done by first rotating L to the x axis, performing the reflection about the x-axis and then rotating the x-axis back to L. Hence, it is described as a conjugation of Φ with r_θ as:

$$\Phi_L = r_\theta \Phi r_{-\theta} = r_\theta \Phi r_\theta^{-1}$$

Since $r_\theta r_\phi(P) = r_{\theta+\phi}(P)$, it is also clear that $r_\theta \Phi_L r_{-\theta}$ is also a reflection about some line L' passing through the origin.

Reflection about any possible line can also be done in a similar fashion by conjugating with the composition of an appropriate rotation and translation.

This idea of conjugation with other isometries to obtain new isometries can be extended to any general metric space. In particular, the idea of conjugation to obtain reflections about different lines and points can be done in Spherical Space and Hyperbolic space as well.

3 Spherical Geometry

In this section, we shall develop some basic theory about spherical surfaces. We shall study a little bit about isometries about these surfaces and the form they take. Note that like the Euclidean plane, the Sphere has a constant curvature of $\frac{1}{R}$, so studying it's isometries does give us a lot of information about the space. Since the sphere is also not locally isometric to the plane, this is the first example where we shall study a non Euclidean Geometry.

3.1 The Sphere

Definition 3.1. The Sphere in defined as the set $S^2 = \{(x, y, z \mid x^2 + y^2 + z^2 = 1)\}$ in R^3.

We know the distance in R^3 is given by the function:

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \text{Where } P_1 = (x_1, y_1, z_1) \text{ and } P_2 = (x_2, y_2, z_2).$$

We consider the distance between points on the Sphere to be the distance along the surface of the sphere. Hence it looks as follows:

Hence from the figure it is clear that the distance metric on the sphere is:

$$d_{S^2}(P, Q) = 2sin^{-1}(\frac{1}{2}d(P, Q))$$

Clearly, since the distance on the sphere is a function of the distance in R^3, two points on S^2 which are equidistant in R^3 are equidistant in S^2.

Definition 3.2. A plane in R^3 is defined to be the set of all (x, y, z) such that $ax + by + cz = 0$, for given a, b, c.

A line in R^3 passing through (x_0, y_0, z_0) is defined as the set of (x, y, z) such that, $x = x_0 + v_1 t$, $y = y_0 + v_2 t$ and $z = z_0 + v_3 t$

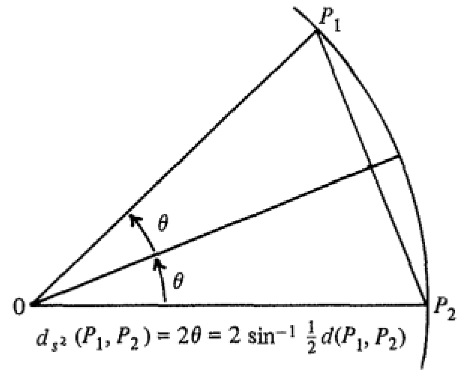


Figure 1: Distance along the Sphere

3.2 Isometries of the Sphere

We start out with some basic examples of isometries on the sphere.

Example 3.3. An example of an isometry on S^2 is rotation in the z-axis, denoted by $r_{z,\theta}$. It is given by:

$$r_{z,\theta}(x) = (x_1 cos\theta - x_2 sin\theta, x_1 sin\theta + x_2 cos\theta, z)$$

Similarly we can define reflection in a plane as follows:

Example 3.4. Reflection in the plane $z = 0$ is given by:

$$\bar{r_E} = (x_1, x_2, -x_3)$$

In fact, via rotations about the x and y-axis, we can bring any plane to the position of $z = 0$. Hence, we can define reflection in arbitrary plane as conjugation of $\bar{r_E}$ with an appropriate rotation.

Lemma 3.5. *The set of points equidistant from two points P and Q on S^2 is a plane through the origin O. Reflection in this plane exchanges P and Q.*

Proof. Let $P = (x, y, z)$ and let $Q = (x', y', z')$. If (a, b, c) is equidistant from P and Q, we have:

$$(a - x)^2 + (b - y)^2(c - z)^2 = (a - x')^2(b - y')^2(c - z')^2 \implies (x - x')a + (y - y')b + (z - z')c = 0$$

Which is the equation of a plane passing through the origin. As described before, with appropriate rotations, we can move P and Q and the plane such that, $(x - x')a + (y - y')b + (z - z')c = 0$ goes to $z = 0$ and P,Q go to $(a, b, 0)$ and $(a, -b, 0)$ respectively. Hence, the proposition follows from Example 3.4 above. □

Definition 3.6. We define a $line in S^2$ to be the equidistant set of two points P and Q on S^2. Hence via the above proposition, the lines in S^2 correspond to the intersection of a plane through the origin and the sphere, which turn out to be the set of *great circles* on S^2.

Theorem 3.7. *Any isometry in S^2 is a product of 1,2 or 3 reflections.*

Proof.

Lemma 3.8. *Any isometry of S^2 is determined by it's image of 3 points not on a line.*

Proof. This can be seen in the same way as done for the Euclidean space. Consider the set of points A,B,C. We shall show that any point is determined by it's distances from A,B and C. This implies that the image of any point is determined by it's distances from $f(A), f(B)$ and $f(C)$ Suppose there are two points P and P' with the same set of distances from A,B and C. Then, clearly A,B and C are all on the plane between P and P' via the previous proposition. Hence, since A,B and C are all on the sphere, this means that they are on a great circle and hence, they are in a line, which is a contradiction. □

Now, the rest of the proof follows the following argument:

Choose 3 points not in a line, and consider their images via an isometry f.

- **Case 1:** All images coincide with their original point. In this case the only possible isometry is the identity map.
- **Case 2:** $f(A)$ and $f(B)$ coincide with A and B respectively, but not C and $f(C)$. In this case, Since f is an isometry, $A = f(A)$ and $B = f(B)$ are on the line L equidistant from C and $f(C)$. Hence, reflection in this line exchanges these two points and therefore agrees with f on 3 points. Hence $f = \bar{r_L}$
- **Case 3:** Suppose only $A = f(A)$ and B and C don't coincide with their f images. Here, Let the line of equidistant points between B and $f(B)$ be L_B. Therefore, the isometry $\bar{r_{L_B}}$ exchanges B and $f(B)$, via the previous proposition. Consider the line of equidistant points between $C' = \bar{r}_{L_B}(C)$ and $f(C)$. Clearly, $A = f(A)$ is on the line, as $d(f(A), f(C)) = d(A, C) = d(A, \bar{r}_{L_B}(C))$. Similarly, $f(B) = \bar{r}_{L_B}(B)$ is also on the line, as $d(f(C), f(B)) = d(C, B) = d(\bar{r}_{L_B}(C), \bar{r}_{L_B}(B)) = d(\bar{r}_{L_B}(C), f(B))$. Therefore, reflection in $L_{C'}$ sends $\bar{r}_{L_B}(C)$ to $f(C)$. Hence consider the isometry $\bar{r}_{L_{C'}}\bar{r}_{L_B}$. Clearly it agrees with f on the points A, B and C. Hence, $f = Barr_{L_{C'}}\bar{r}_{L_B}$
- **Case 4:** Suppose no $f(A), f(B)$ $f(C)$ coincides with A, B or C. In this case, we can follow the same argument as Case 2, with this time starting with the line between A and $f(A)$ instead and then moving to B and C.

□

Corollary 3.9. *The isometries of S^2 form a group.*

Proof. This follows from the fact, that all reflections have an inverse. The properties of associativity and identity are obvious. □

We shall now take a look at properties of rotations in the Sphere.

Lemma 3.10. *A rotation in S^2 about P of angle θ is given by a product of reflections in any lines L and L' through P, with angle $\frac{\theta}{2}$ between them.*

Proof. This can be seen as follows. Consider line passing through P and the antipodal point P'. Now this line can be rotated to the z-axis. We know that rotations along the z-axis are just rotations along the xy plane, along with the z-coordinate. Since in R^2 rotations can be written as a product of reflections in line with angle $\frac{\theta}{2}$ about the origin, this means that Rotations in S^2 about the z-axis can be written as a product of reflections in great circles about the origin. Hence by conjugating the line through P and P' with appropriate rotations to take it to the z-axis, we see that the proposition follows. □

Theorem 3.11. *The product of two rotations is again a rotation. The rotations of S^2 form a group.*

Proof. Given a rotation about the point P and another rotation about the point Q choose the lines of reflections as described in the diagram below:

□

Therefore, we have the lines of reflection as M, N and L, where $r_{P,\theta} = \bar{r_M}\bar{r_L}$ and $r_{Q,\phi} = \bar{r_N}\bar{r_M}$. This gives us:

$$r_{Q,\phi}r_{P,\theta} = \bar{r_N}\bar{r_L} = r_{R\chi} \quad \text{Where } \chi \text{ and } R \text{ are the angle and intersection point of } N \text{ and } L.$$

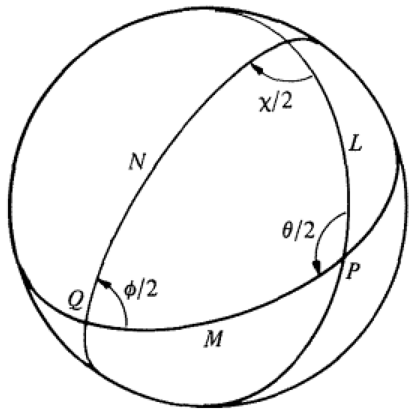
Hence, the product of any two rotations is a rotation. It is easy to see that this subgroup has the identity element and that every rotation has an inverse (obtained as $r_{P,-\theta}$).

Hence, the proposition follows.

Corollary 3.12. *The group of Rotations makes up the orientation preserving isometries. The set of orientation reversing isometries are given by the coset $\bar{r}Iso^+(S^2)$, where $\bar{r}$ is any reflection.*

Proof. Since any two great circles meet at some point, their product will always give a rotation $r_{P,\theta}$. Hence, the set of all rotations forms a group, this means that all isometries generated by even number of reflections is just the group generated by the rotations. Hence, the first part follows.

Given any orientation reversing isometry r, we can write r as $\bar{r_L}\bar{r_L}r$. Since both $\bar{r_L}$ and r are orientation reversing, their product has an even number of reflections and hence is a rotation $r_{X,\rho}$. Therefore, we have $r = \bar{r_L}r_{X,\rho}$. Hence, $Iso^-(S^2) \subseteq \bar{r_L}Iso^+(S^2)$. The reverse inclusion is obvious from the definition of orientation reversing isometry. Hence we get $Iso^-(S^2) = \bar{r_L}Iso^+(S^2)$ □



3.3 Forms of Isometries

An important map to give a nice form to the set of all isometries is the stereographic projection. This maps all points of S^2 except for the North pole, onto the plane. It can be easily shown to be a homeomorphism of $S^2 - N$ and R^2. The function at P is obtained by drawing a line from N to P and returning the point where this line cuts the plane. The final form of this function is:

$$f(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z})$$

The inverse of the function is given by:

$$f^{-1}(u, v) = (\frac{2u}{1+v^2+u^2}, \frac{2v}{1+v^2+u^2}, \frac{u^2+v^2-1}{1+v^2+u^2})$$

Unfortunately, the stereographic projection is not an isometry, but it plays a role in converting isometries of S^2 into a form over the complex plane.

Definition 3.13. We define an *inversion* about the circle with center P and radius ρ as $i(S) = S'$, where P' is a point on the line between P and S such that $|OP||OP'| = \rho^2$.

As the radius and center of the circle tends to ∞ the circle itself turns into a straight line. Hence, we also consider the limiting case of reflection about lines to be inversions. Inversions look as given in the diagram below:

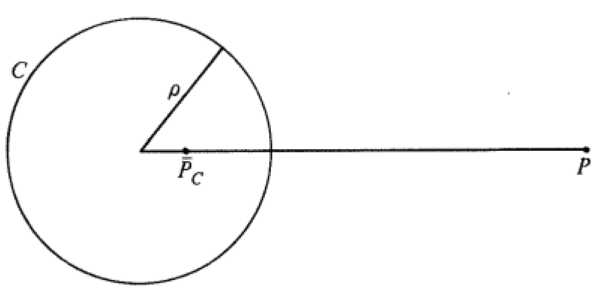


Figure 2: Inversion

Theorem 3.14. *If $\bar{r}$ is a reflection in S^2, then the induced map via the stereographic projection onto the plane is an inversion.*

Proof. Let $P = (a, b, c)$ and $Q = (a', b', c')$ be two points of S^2 exchanged by the reflection. Hence it is given by:

$$(a - a')x + (b - b')y + (c - c')z = 0$$

Let u and v denote the coordinates of the stereographic projection of a point P on S^2. Therefore, the corresponding points on S^2 are:

$$(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$$

Hence the points on the plane and S^2 satisfy:

$$(a-a')\frac{2u}{u^2+v^2+1} + (b-b')\frac{2v}{u^2+v^2+1} + (c-c')\frac{u^2+v^2-1}{u^2+v^2+1} = 0$$

This simplifies to:

$$(u+\frac{a-a'}{c-c'})^2 + (v+\frac{b-b'}{c-c'})^2 = 1 + (\frac{a-a'}{c-c'})^2 + (\frac{b-b'}{c-c'})^2$$

Hence, it is a circle with center $(-\frac{a-a'}{c-c'}, -\frac{b-b'}{c-c'})$ and radius $\frac{\sqrt{2(1-aa'-bb'-cc')}}{c-c'}$

It can further be checked that the map corresponding to the reflection turns into an inversion about this circle. In the limiting case that $c = c'$, this map turns into a reflection in the plane R^2. The case where $c = 1$ is also not defined, as this corresponds to the point N on S^2. ☐

Instead of looking at the stereographic projection onto R^2 we can look at it on C. Here, the inversion in the unit circle looks as follows:

If $w = pe^{i\theta}$, then the inversion about the unit circle gives us $I(w) = p^{-1}e^{1\theta}$. This is the same as $\frac{1}{pe^{-i\theta}}$. Hence it is clear that $I(w) = \frac{1}{\bar{w}}$

We now state some basic geometric properties of inversions without proof. These are properties are easy to check and verify.

Theorem 3.15. *Inversions are maps which:*

- *Map circles to circles*
- *Preserve magnitude of angles, but reverse sign (They are conformal maps).*

Hence, in C we can now give a form to the inversion described by the theorem above. It goes as follows:

Let $r_1 = a - a'$, $r_2 = b - b'$ and $r_3 = c - c'$. Hence, the map of the reflection about the plane $r_1x + r_2y + r_3z = 0$ induces an inversion about a circle with center $-(\frac{r_1}{r_3} + i\frac{r_2}{r_3})$ and radius $\frac{\sqrt{r_1^2+r_2^2+r_3^2}}{r_3}$, which if we normalise r_1, r_2 and r_3 becomes $\frac{1}{r_3}$. Hence we get the following theorem.

Theorem 3.16. *The map induced by reflection about the plane $r_1x + r_2y + r_3z = 0$, where $r_1^2 + r_2^2 + r_3^2 = 1$ is:*

$$g(z) = \frac{(r_1 + ir_2)\bar{z} - r_3}{-r_3\bar{z} - (r_1 - ir_2)}$$

Proof. To generalise function of an inversion from a unit circle, we can compose it with dilation and translations. It is easy to see that the inversions are preserved with dilation and translations. Hence, in order to find the inversion about any circle, with center d and radius p, we look at the function $t_d d_p I d^{-1} p t_{-d}$. Since $I(z) = \frac{1}{\bar{z}}$, we see that:

$$g(z) = d + p(\frac{1}{\frac{(\bar{z-d})}{p}}) = \frac{d\bar{z} + p^2 - d\bar{d}}{\bar{z} - \bar{d}}$$

Now, we can substitute the values for d and p for the center and radius respectively to obtain the final expression. It turns out to be:

$$g(z) = \frac{(r_1 + ir_2)\bar{z} - r_3}{-r_3\bar{z} - (r_1 - ir_2)}$$

Hence the proposition follows. ☐

In order to make this form a little nicer, we can call $l = r_1 + ir_2$ and impose the condition $|l|^2 + r_3^2 = 1$. So, the final form would be:

$$g(z) = \frac{l\bar{z} - r_3}{-r_3\bar{z} - \bar{l}}$$

Therefore, we have successfully given a form to all possible reflections in S^2, as inversions in C. Since rotations are products of two reflections, the following corollary also pops out.

Corollary 3.17. *The maps on C induced by the rotations of S^2 are precisely of the form:*

$$f(z) = \frac{az + b}{-\bar{b}z + \bar{a}}$$

With $|a|^2 + |b|^2 = 1$

Proof. Since a rotation is a product of reflections, consider the reflections given as:

$$g_1(z) = \frac{l_1\bar{z} - r_1}{-r_1\bar{z} - \bar{l_1}} \qquad g_2(z) = \frac{l_2\bar{z} - r_2}{-r_2\bar{z} - \bar{l_2}}$$

Now by looking at their forms, we can see that these are fractional linear transformations. In order to find their product, it can be done by routinely substituting one equation in the other. Once done it will turn out that the following is true.

The coefficient of each term in the numerator and denominator is given by the product of the following matrices:

$$\begin{bmatrix} l_1 & -r_1 \\ -r_1 & -l_1 \end{bmatrix} \begin{bmatrix} l_2 & -r_2 \\ -r_2 & -l_2 \end{bmatrix}$$

The condition that $|l|^2 + r_3^2 = 1$ translates into the determinant of the above matrices. Therefore, we can check that the new coefficients give a rotation, by looking at the determinant of the product which is the product of determinants which is 1. Hence, the corollary follows. ☐

From the above forms, the form of the orientation reversing isometries can also be calculated. Although this has been omitted, the calculation follows the same route as that of the orientation preserving isometries.

3.4 Spherical Surfaces

Definition 3.18. Spherical surfaces are defined analogous to Euclidean surfaces. It is defined as a set S with a metric d such that, about every point $P \in S$, $\ni \epsilon$ ball around P which is isometric to a disc in S^2.

Theorem 3.19 (Killing Hopf)**.** *Any connected complete Spherical surface, can be expressed as $\frac{S^2}{\Gamma}$ where Γ is a subgroup of isometries of S^2 which are discontinuous and fixed point free.*

We shall not prove this theorem. The argument follows similarly to that of the Euclidean case, albeit with more detail. Here, **discontinuous** means that the orbit of any points don't have limit points and **fixed point free** means that $g(p) \neq p \forall g$ and p

Thus, the problem reduces to finding fixed point free and discontinuous isometries. Hence, this rules out all possible rotations, since they have fixed points. Since the square of any orientation reversing isometry is a rotation, clearly, it must be such that $f^2 = Id$. This leads to the following proposition:

Theorem 3.20. *The only fixed point free, orientation reversing isometry with $f^2 = Id$ is the antipodal map defined by:*

$$m(x, y, z) = (-x, -y, -z)$$

Proof. Clearly m is an orientation reversing isometry such that $m^2 = 1$. Since $x \neq -x$ in S^2 for any x, this also implies it is fixed point free. Hence it satisfies the necessary condition. Since $m(rotations) =$ orientation reversing isometries, any orientation reversing isometry is of the form mr for some rotation r. wlog, choose the z-axis as the axis of rotation of r.

Consider a point P on $S^2 \cap z = 0$. Clearly $mr(P)$ lies again on $S^2 \cap z = 0$, since it is a rotation about the z-axis. if r is a rotation by the angle θ, that means that $mr(P)$ is at an angle of $\pi + \theta$. Now, since $(mr)^2 = 1$, this means $2\pi + 2\theta = 2n\pi \implies \theta = 0$ or π. If $\theta = 0$, then it has a fixed point. Hence, $\theta = \pi$. Therefore, the proposition follows and m is the only such isometry. □

Corollary 3.21. *The only connected complete spherical surface is the Projective Plane denoted by* $\frac{S^2}{\{1,m\}}$.

Hence, with that we have classified all isometries of S^2 and all connected and complete Spherical surfaces.

4 Hyperbolic Geometry

Until now we have studied Spherical surfaces by computing the it's isometry group and studying it's properties. This works for these manifolds as it has a constant curvature $\frac{1}{R}$. This methodology can also be extended to surfaces of constant negative curvature. This takes us into the realm of Hyperbolic Geometry.

4.1 Motivation

Hyperbolic geometry first arose in the study of space-time via the Minkowski model. We shall describe hyperbolic space from this perspective.

Space time in this model, is viewed as the space R^{n+1} with the indefinite norm:

$$x * x = |x|^2 = \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 \qquad \text{Where } x = (x_1, x_2, ..., x_{n+1})$$

The set of all points with norm 0 is called the light cone. In the same way we define a Sphere in the Euclidean space, we define the following:

Definition 4.1. The n-dimensional hyperbolic space is defined to be the set:

$$H^n = \{x \in R^{n+1} | x * x = -1, x_{n+1} > 0\}$$

Clearly, in R^3 the light cone is the set $S = \{x \in R^3 | x_1^2 + x_2^2 = x_3^2\}$, which is a cone. H^2 here, is of the form $\{x \in R^3 | x_1^2 + x_2^2 = x_3^2 - 1\}$. This can be visualised in R^3 as follows:

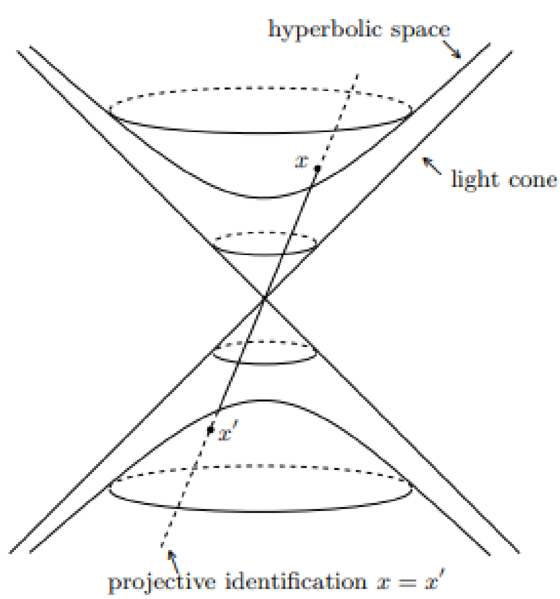


Figure 3: H^2 in R^3 [1]

Now, we wish to understand H^n using the metric given by $*$. When we think about distance of 2 points p_1 and p_2 on H^n, we want the distance to be along the surface on the manifold. Hence it does not make sense to assign $|p_1 - p_2|$ as distance between p_1 and p_2. Since H^n can be viewed as a sub manifold of R^{n+1}, in the style of Riemannian manifolds, we can instead look at the inner product restricted to the Tangent spaces of H^n.

Theorem 4.2. *The inner product $*$ restricted to the tangent space of H^n is positive definite.*

Proof.

Lemma 4.3. *The tangent vectors to $p \in H^n$, are hyperbolically orthogonal to the p.*

Proof. Let $x : (-\infty, \infty) \to H^n$ be a path in H^n. Let $x(t) = (x_1(t), x_2(t), ..., x_n(t), x_{n+1}(t))$ and $x(t_1) = p$. By the definition of H^n,

$$\sum_i^n x_i(t)^2 - x^2(t)_{n+1} = 1$$

On differentiating, we get $\sum_i^n 2x_i(t)x_i'(t) - 2x_{n+1}(t)x_{n+1}'(t) = 0$.
Hence we get:

$$\sum_i^n x_i(t)x_i'(t) - x_{n+1}(t)x_{n+1}'(t) = 0 \implies x(t) * x'(t) = 0 \implies x(t_1) * x'(t_1) = 0$$

For each tangent vector v, there exists a path x through p such that $x'(t_1) = v$. Hence the Lemma follows. ☐

Now consider $p = (\bar{p}, p_{n+1}) \in R^{n+1}$. Let $x = (\bar{x}, x_{n+1}) \in T_P(H^n)$ and . denote the inner product in R^n.

If $x_{n+1} = 0$, then $x * x = x.x > 0$. Hence it is positive definite. If $x_{n+1} \neq 0$, then:

$$x * p = \bar{x}.\bar{p} - x_{n+1}p_n + 1 = 0. \text{ and } p * p = \bar{p}.\bar{p} - p_{n+1}^2 = -1.$$

Therefore, by Cauchy-Schwarz inequality for R^n we have:

$$\bar{x}^2.\bar{p}^2 \geq (x.p)^2 = x_{n+1}^2(\bar{p}.\bar{p} + 1) \implies (\bar{x}^2 - x_{n+1}^2)(\bar{p}.\bar{p}) \geq x_{n+1}^2 \implies (x * x)(\bar{p}.\bar{p}) > 0 \ \text{, if } x \neq 0$$

Hence it has been proved for all x_{n+1}.

☐

Therefore, now using the above, we can now define a Riemannian metric on H^n.

Definition 4.4. A **Riemannian metric** of a manifold M (denoted by ds^2) is the assignment of a positive definite inner product $<,>$ to each point p (denoted by $<,>_p$) on M such that, if X and Y are vector fields on the manifold, the function $p \to < X_p, Y_p >_p$ is smooth.

It turns out that the assignment $< x, y >_p = \sum_{i=0}^n x_i y_i - x_{n+1}y_{n+1}$, induced by the inner product on R^{n+1} in the Minkowski model gives a Riemannian metric on H^n. Hence we can define the following in order to have a well defined notion of distance on H^n.

if p_1 and p_2 are points on H^n, and $f : [a, b] \to H^n$ is a path connecting them, then we define:

$$length(f) = \int_a^b |f'(t)|dt$$

Since this gives us a distance of a path between p_1 and p_2, if $\Gamma(p_1, p_2)$ is the set of all piece-wise smooth paths between p_1 and p_2, we can define:

$$d_{H^n}(p_1, p_2) = inf\{length(f) : f \in \Gamma(p_1, p_2)\}$$

This distance, coming from the Riemannian metric on H^n can be shown to satisfy the properties of a normal metric, making H^n into a metric space.

4.2 Models of Hyperbolic Space

There are several ways in which one studies the hyperbolic space. Conventionally, Hyperbolic space can be described via several different models, each giving a different perspective and intuition for the space. All the models are isometrically equivalent, which makes sure they all describe the same space. Here, we shall briefly mention a few models, with particular focus on two models; H, the Half space model and D, the Disk model.

- The Loid Model (Hyperboloid model). This is the space we used to originally define the Hyperbolic space.

$$L = \{(x_1, x_2, ..., x_{n+1} : \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}$$

- The Jemisphere Model.

$$J = \{(x_1, x_2, ..., x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = 1, x_{n+1} > 0\}$$

- The Klein Model.

$$K = \{(x_1, ..., x_n, 1) : \sum_{i=1}^{n} x_i^2 < 1\}$$

- The Half Space Model.

$$H = \{(1, x_2, ..., x_{n+1}) : x_{n+1} > 0\}$$

- The Disk Model.

$$D = \{(x_1, ..., x_n, 0) : \sum_{i=1}^{n} x_i^2 < 1\}$$

The Disk model can be obtained from the Loid Model as follows:
Consider the map $f : L \to D$ such that

$$(x_1, ..., x_{n+1}) \to \frac{(x_1, ..., x_n)}{1 + x_{n+1}}$$

This is the map obtained via stereographic projection. from the point $(0, ..., 0, -1)$ Clearly, the inverse is given by:

$$(y_1, ..., y_n) \to \frac{(-2y_1, ..., -2y_n, y_1^2 + ... + y_n^2 + 1)}{(y_1^2 + ... + y_n^2 - 1)}$$

In order to use this as a model for hyperbolic space, we need to give an appropriate metric for this space for it to be isometric to the Loid model. Hence we define it using the pullback metric, with respect to the inverse function. Hence, the Riemannian metric on these spaces is given by:

$$ds^2 = \frac{4(dx_1^2 + dx_2^2 + ... + dx_n^2)}{(1 - x^2 - ... - x_n^2)^2}$$

Hence, the metric on the space is:

$$d_{D^n}(p_1, p_2) = inf\{\int_a^b \frac{2}{(1 - x^2 - ... - x_n^2)} |f'(t)| dt : f \in \Gamma_D(p_1, p_2)\}$$

Similarly, from the Disk Model, we can obtain the Half Space model as well via the following map $f : D^n \to H^n$:

$$f(x) = \frac{2(x + e_n)}{|x + e_n|^2}$$

Where $|.|$ is the Euclidean Norm and $e_n = (0, ..., 0, 1)$.
Therefore, one can calculate and see that the Riemannian metric in the Half Space model is

$$ds^2 = \frac{(dx_2^2 + dx_3^2 + ... + dx_{n+1}^2)}{x_{n+1}^2}$$

Which gives:

$$d_{H^n}(p_1, p_2) = inf\{\int_a^b \frac{1}{|(f_{n+1}(t)|)}|f'(t)|dt : f \in \Gamma_D(p_1, p_2)\}$$

4.3 Hyperbolic Plane

In this section, we spend time describing the Hyperbolic Plane H^2 and describe some of it's properties.

As described before, H^2 is the model of hyperbolic space described as $H^2 = \{(x, y) \in R^2 : y > 0\}$ with the Riemannian metric given by $ds^2 = \frac{dx^2+dy^2}{y^2}$. Instead of viewing this space in R^2, we can equivalently look at it in C as $H^2 = \{x + iy \in C : y > 0\}$.

Hence, if $\Gamma(x, y)$ is the set of piecewise smooth paths from x to y, then we get a metric from this as described prior as,

$$d_{H^2}(x, y) = inf\{\int_x^y \frac{1}{Im(f(t))}|f'(t)|dt : f \in \Gamma(x, y)\}$$

4.3.1 Isometries

In order to find out all the isometries of H^2, we shall first look at some basic examples:

- *Limit Rotation:* It is given by the function $\tau_a(z) = z + a$, $a \in R$. This can be seen to be an isometry as, for any path $p \in \Gamma(x, y)$, the length of $\tau_a \circ p$ is given by:

$$\int_x^y \frac{1}{Im(\tau_a \circ p(t))}|(\tau_a \circ p)'(t)|dt = \int_x^y \frac{1}{Im(p(t) + a)}|(p + a)'(t)|dt = \int_x^y \frac{1}{Im(p(t))}|p'(t)|dt = len_{H^2}(p)$$

  Hence the length of any path remains the same. So the infimum and hence the distance between any two points will be the same.

- *Translation:* This is given by the function $d_\rho(z) = \rho z$, $\rho \in R^+$. This is an isometry as, for any path $p \in \Gamma(x, y)$, the length of $d_\rho \circ p$ is given by:

$$\int_x^y \frac{1}{Im(d_\rho \circ p(t))}|(d_\rho \circ p)'(t)|dt = \int_x^y \frac{1}{Im(\rho p(t))}|(\rho p)'(t)|dt = \int_x^y \frac{\rho}{\rho Im(p(t))}|p'(t)|dt = len_{H^2}(p)$$

  Hence the length of any path remains the same. So the distance between any two points doesn't change.

- *Reflection:* This is given by the function $\bar{r_y}(z) = -\bar{z}$. This is an isometry as for any path $p \in \Gamma(x, y)$, the length of $\bar{r_y} \circ p$ is given by:

$$\int_x^y \frac{1}{Im(\bar{r_y}(p(t)))}|(\bar{r_y} \circ p)'(t)|dt = \int_x^y \frac{1}{Im(p(t))}|(\bar{p})'(t)|dt = \int_x^y \frac{1}{Im(p(t))}|p'(t)|dt = len_{H^2}(p)$$

  Hence the length of any path remains the same. So the distance between any two points doesn't change.

Unfortunately, these are not enough to generate all possible isometries. We can construct more isometries by looking at H^2 in another model. We shall now look at it in D^2.

The map we described in the above section reduces to the following in H^2 and D^2:
We have $J : H^2 \to D^2$ defined as:

$$J(z) = \frac{iz+1}{z+i}$$

The inverse function $J^{-1} : D^2 \to H^2$ is given by:

$$J^{-1}(z) = \frac{-iw+1}{w-i}$$

The distance function in D^2 is obtained as above as:

$$d_D(p_1, p_2) = inf\{\int_a^b \frac{2}{1-|f(t)|^2}|f'(t)|dt : f \in \Gamma_D(p_1, p_2)\}$$

From the form of the distance function it's possible to see that the following are also isometries.

- The function $r_\theta(w) = e^{i\theta}w$ is an isometry. This can be seen as:

$$\int_x^y \frac{2}{1-|r_\theta \circ f|^2}|(r_\theta \circ f)'(t)| = \int_x^y \frac{2}{1-|f|^2}|e^{i\theta}f'(t)| = \int_x^y \frac{2}{1-|f|^2}|f'(t)| = len_{H^2}(f)$$

- Another function which turns out to be an isometry is $\bar{r}(w) = \bar{w}$. It is seen as:

$$\int_x^y \frac{2}{1-|\bar{r} \circ f|^2}|(\bar{r} \circ f)'(t)| = \int_x^y \frac{2}{1-|f|^2}|\bar{f}'(t)| = \int_x^y \frac{2}{1-|f|^2}|f'(t)| = len_{H^2}(f)$$

Therefore, we have now found more isometries in D^2 that we didn't get from H^2. In order to convert these to isometries of H^2, since the distance in D^2 is obtained via the pullback of the bijection between them, if f is an isometry of D^2, $J^{-1}hJ$ is an isometry of H^2.

For $h = \bar{r}$,

$$r_{\bar{H}^2} = J^{-1}hJ(z) = J^{-1}(\frac{1-i\bar{z}}{\bar{z}-i}) = \frac{-i-\bar{z}+\bar{z}-i}{1-i\bar{z}-i\bar{z}-1} = \frac{1}{\bar{z}}$$

For $h = r_\theta$,

$$J^{-1}hJ(z) = \frac{-ie^{i\theta}(iz+1)+(z+i)}{e^{i\theta}(iz+1)-iz+1} = \frac{(e^{i\theta}+1)z+i(1-e^{i\theta})}{i(e^{i\theta}-1)z+(e^{i\theta}+1)} = \frac{(e^{i\frac{\theta}{2}}+e^{-i\frac{\theta}{2}})z+i(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}})}{i(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}})z+(e^{i\frac{\theta}{2}}+e^{-i\frac{\theta}{2}})} = \frac{az+b}{-bz+a}$$

Where $a, b \in R$. Notice that conjugation of isometries with other isometries is also an isometry. Hence given $h = \bar{r}$, clearly $\tau_a d_\rho r_{\bar{H}^2} d_\rho^{-1} \tau_a^{-1}$ is also an isometry. Hence, since h is the inversion about the unit circle in H^2, this implies that the inversion about the circle $C_{\rho,a}$ where a is the center on the x axis and $\rho > 0$, is also an isometry. These inversions make up the reflections of H^2. With that in mind, we make the following definition:

Definition 4.5. We define the **lines** of H^2 to be the fixed point sets of the reflections of H^2.

Hence, clearly all $x = \alpha$ and all semicircles with centers on the x-axis are the lines in H^2.

Theorem 4.6. *Between any two points z_1 and z_2 in H^2, there is a unique line between them.*

Proof. If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ then consider the following.

If $y_1 = y_2 = \alpha$, then the line $x = \alpha$ passes through them both. Since no semicircle with center on the x-axis passes through both, it is unique.

if $y_1 \neq y_2$, then consider the line:

$$y = \frac{x_1 - x_2}{y_2 - y_1}x + (\frac{(x_2^2+y_2^2)-(x_1^2+y_1^2)}{2(y_2-y_1)}$$

This is the line representing the perpendicular bisector of z_1 and z_2. It crosses the x-axis at $x = \frac{(x_1^2+y_1^2)-(x_2^2+y_2^2)}{2(x_2-x_1)}$. Hence, the semi circle with center at $(\frac{(x_1^2+y_1^2)-(x_2^2+y_2^2)}{2(x_2-x_1)}, 0)$ and radius, the length of the line segment between the center and z_1 or z_2. This is a H^2 line passing through z_1 and z_2. Since the perpendicular bisector cuts the x-axis in only 1 place, the uniqueness of the line follows directly. ∎

Lemma 4.7. *Conjugation of $\bar{r}$ with r_θ defines an isometry which is an inversion in H^2.*

Proof. We can see this by looking first in the D^2 model. Clearly, this conjugation is equivalent to reflection in D^2 about the line $y = \alpha x$, where $\alpha = \frac{sin\theta}{cos\theta} = tan\theta$. Therefore, we have that the isometry $r_\theta \bar{r} r_\theta^{-1}$ gives us:

$$z \to e^{-i\theta} z \to \bar{z} e^{i\theta} \to e^{2i\theta} \bar{z}$$

Now converting it into an isometry in H^2 via $J^{-1}HJ$,

$$J^{-1}HJ(z) = J^{-1}(\frac{e^{2i\theta}(\overline{iz+1})}{z+i}) = J^{-1}(\frac{e^{2i\theta}(1-i\bar{z})}{\bar{z}-i}) = \frac{(1-e^{2i\theta})\bar{z} - i(1+e^{2i\theta})}{-i(1+e^{2i\theta})\bar{z} + (e^{2i\theta}-1)}$$

From this we get:

$$\frac{(1-e^{2i\theta})\bar{z} - i(1+e^{2i\theta})}{-i(1+e^{2i\theta})\bar{z} + (e^{2i\theta}-1)} = \frac{(e^{-i\theta}-e^{i\theta})\bar{z} - i(e^{-i\theta}+e^{i\theta})}{-i(e^{i\theta}+e^{-i\theta})\bar{z} + (e^{i\theta}-e^{-i\theta})} = \frac{\alpha\bar{z}+1}{\bar{z}-\alpha}$$

This is the inversion of at the circle with center $(\alpha, 0)$ and radius $\sqrt{\alpha^2+1}$. Hence proved. □

Theorem 4.8. *The distance between any two points is the length of the H^2 line segment between them.*

Proof. We know that isometries preserve distances. Hence, Let l be the H^2 line segment between z_1 and z_2. If $z_1 = (a, b)$, then consider the transformation given by $d_b^{-1}\tau_{-a}(l)$. Clearly, z_1 is mapped to $(0, 1)$ and z_2 is mapped to some other point in H^2.

Now looking at the images $z_1' = d_b^{-1}\tau_{-a}(z_1)$ and $z_2' = d_{-b}\tau_{-a}(z_2)$ in the D^2 model, via J, we can see that $J(z_1') = (0, 0)$ and $J(z_2') = ke^{i\theta}$ for some θ. Therefore the isometry $f(w) = we^{i(\frac{\pi}{/2}-\theta)}$ takes $J(z_2')$ to the y-axis. Now J^{-1} maps the y-axis in D^2 to the y-axis in H^2. Since τ_a, d_p, the map J, J^{-1} and r_θ preserve circles (r_θ, J and J^{-1} are given by inversions or products of inversions in H^2, which preserves circles), this means that the image of the l is the circular arc/line passing through the images of the points, with center on the x-axis.

Hence, the line is the segment of the y-axis passing through the points. If l' is any other line passing through z_1 and z_2, clearly it's image after all these isometries is another curve between these points on the y-axis, call it C. Hence the length of C is given by :

$$\int_C \frac{1}{y}\sqrt{dx^2+dy^2} \geq \int_P^Q \frac{1}{y} dy = \quad H^2 \text{ length of } z_1 \text{ and } z_2.$$

Where P and Q are the images of z_1 and z_2 after all the isometries described. Hence, this implies that $d_{H^2}(z_1, z_2) =$ length of H^2 line segment between them.

□

Corollary 4.9. *If P, Q and R are points in H^2, then*

$$H^2 \textit{ length of } PQ + H^2 \textit{ length of } QR \geq H^2 \textit{ length of } PR$$

With strict inequality when R is not on the line PQ.

Proof. The first part of the statement follows from the previous proposition and the triangle inequality for d_{H^2}. For the strict inequality, note that as before, via isometries it is possible to take PQ to the y-axis. If $R = (a, b)$, Since $R \notin PQ$, we have:

$$\int_{PR} ds > \int_P^{(0,b)} dy \quad \text{and} \quad \int_{RS} ds > \int_{(0,b)}^Q dy$$

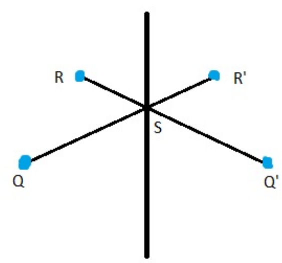
Hence, the proposition follows directly. □

Theorem 4.10. *The set of points equidistant from two points in H^2 is a H^2 line and a H^2 reflection in this line exchanges the two points.*

Proof. Consider the points P_1 and P_2. Let them be denoted by (x_0, y_0) and (x_1, y_1). Let $P_1' = d_{y_0}^{-1} \circ \tau_{-x_0}(P_1)$ and $P_2' = d_{y_0}^{-1} \circ \tau_{-x_0}(P_2)$. Clearly now, $P_1' = (0, 1)$.

Now again, looking at it in the D^2 model, $P_1' = (0, 0)$ and $P_2' = ke^{i\theta}$ Now, we can rotate by some $r_{\theta'}$. Denote $R(\theta, z) = r_\theta(z)$. It can be seen that R is a continuous function. Therefore, via intermediate value theorem, we can choose ϕ such that the image of $r_\phi(ke^{i\theta})$ has the same y coordinate as P_1' in H^2. Therefore, after the series of isometries we have the points $P_1'' = (0, 1)$ and $P_2'' = (x, 1)$. Therefore, applying $\tau_{\frac{x}{2}}$, we get the final points $Q = (-\frac{x}{2}, 1)$ and $Q' = (\frac{x}{2}, 1)$.

Since via a series of isometries, we get two points in H^2 which are mirror images on the y axis, we can figure out the set of equidistant points for these two points and then map it back to P_1 and P_2. $\bar{r_y}$ is an isometry and exchanges Q and Q', so it clearly shows that, since the y-axis is the set of fixed points, the y-axis is contained in the set of fixed points.



Consider a points R outside the y-axis. Let R' be it's mirror image along the y-axis. Let S be the point of intersection of QR' and $Q'R$ on the y-axis. Clearly $|Q'R'| = |QR|$ and $|SR'| = |SR|$ since they are images via reflections. By choice of R, we also have $|Q'R| = |QR|$. Clearly, we have:

$$|Q'R'| = |Q'R| = |Q'S| + |SR| = |Q'S| + |SR'|$$

This is a contradiction to the triangle inequality, since S is not on the line $Q'R'$. Hence R cannot be equidistant from Q and Q'. Hence, the set of equidistant points is only the y-axis. Since, isometries of the form r_θ, τ_a and d_p, all preserve H^2 lines, it is clear that the set of equidistant points between P_1 and P_2 is a H^2 line between them. Since $\bar{r_y}$ exchanged Q_1 and Q_2, the corresponding function between P_1 and P_2 exchanges them about the given H^2 line. Hence the proposition follows.

□

Theorem 4.11. *Each H^2 isometry is a product of at most 3 reflections.*

Proof.

Lemma 4.12. *Each H^2 isometry is determined by it's effect on 3 points of H^2.*

Proof. Given three points A, B and C, not on the same line if there were two points P and P' that were equidistant from all 3 individually, by the previous lemma, A, B and C would all lie on the line equidistant from P and P' which is a contradiction to the choice of A, B and C. Hence the lemma follows. □

Now, with the given lemma and the above proposition, the proof is the exact same as the one for the analogous proposition described in the section on the isometries of the sphere. Hence the same proposition follows. □

Therefore, we have established the 3 reflection theorem for Isometries in the hyperbolic plane.

Definition 4.13. The set of isometries which are given as products of even number of reflections are called *orientation preserving* isometries (denoted as $Iso^+(H^2)$).

The set of isometries which are given as products of odd number of reflections are called *orientation reversing* isometries.

Corollary 4.14. *The H^2 isometries form a group. The set $Iso+(H^2)$ forms a subgroup.*

Remark 4.15. As before, it can be seen that the orientation reversing isometries are given by the coset $\bar{r_y}.Iso^+(H^2)$. Since $\bar{r_y}$ has a line of fixed points and orientation preserving isometries have at max 2 (except for the identity), the result follows.

4.3.2 Form of Isometries

Now that we have the 3 reflection theorem, we can establish an expression corresponding to the isometries of H^2.

Theorem 4.16. *The isometries of H^2 of the form:*

$$f(z) = \frac{az+b}{cz+d} \quad \textit{where} \quad ad-bc=1 \quad \textit{are the orientation preserving isometries}$$

$$f(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \quad \textit{where} \quad ad-bc=-1 \quad \textit{are the orientation reversing isometries}$$

Proof.

Lemma 4.17. *τ_a and d_p are both orientation preserving isometries.*

Proof. This can be easily seen as, $\tau_a = \tau_{\frac{a}{2}}\bar{r_y}\tau_{\frac{-a}{2}} \circ \bar{r_y}$. Since both are reflections, it is a product of two reflections and hence is an orientation preserving isometry.

Similarly, $d_p = d_{\frac{p}{2}} r_{\bar{H}^2} d_{\frac{2}{p}} \circ r_{\bar{H}^2}$. Hence, since both are reflections it becomes an orientation preserving isometry. ☐

We shall prove the proposition for orientation preserving isometries. The proposition for orientation reversing isometries follows from that and the remark above.

$\Longrightarrow$ From the 3- reflection theorem, we know that every isometry can be written as a product of 3 reflections. The reflections are of the form $\tau_a d_\rho \bar{r} d_\rho^{-1} \tau_{-a}$ and $\tau_a \bar{r_y} \tau_{-a}$ Hence they are all inversions in circles with center on the origin or reflections about the vertical lines. Hence, the composition of two reflections, gives a an isometry of the form $f(z) = \frac{az+b}{cz+d}$, as described in the section on orientation preserving isometries in S^2. Hence, by normalising the numerator and denominator, we can make $ad-bc=1$ and hence satisfying the appropriate form.

$\Longleftarrow$ We need to show that every inversion of the given form is an isometry of H^2. Given a function $f(z) = \frac{az+b}{cz+d}$, we can see that:

$$\frac{az+b}{cz+d} = \frac{az+b+\frac{ad}{c}-\frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{b-\frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} = \frac{a}{c} - \frac{1}{c(cz+d)}$$

From this we can see that $f(z) = \tau_{\frac{a}{c}} \circ \bar{r_y} \circ J^{-1} r_{\bar{H}^2} J \circ d_c \circ \tau_d \circ d_c(z)$. Since all of these are isometries, their composition is also an isometry. It is orientation preserving as τ_a and d_p are both orientation preserving and it is a product of an even number of reflections otherwise. Hence the proposition follows. ☐

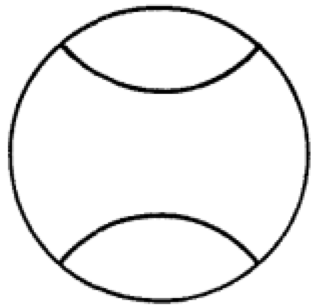
Theorem 4.18. *Every orientation preserving isometry can be written as a Rotation, Translation or Limit rotation.*

Proof. We know all orientation preserving isometries can be written as a product of 2 reflections. Let us now look at all the possibilities of lines about which reflections happen in the D^2 model.

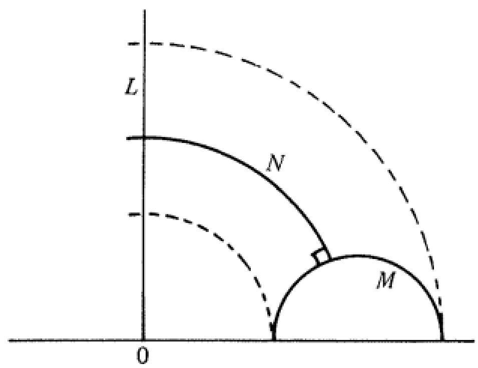
- **Case 1:** The lines are ultra parallel. The lines in this case look as such:

  Clearly the images of these lines in D^2 are just disjoint semicircles with centers on the x-axis. Call them l_1 and l_2. Therefore, just as in Theorem 3.10, we can map l_1 to the y-axis via a series of isometries, call it's composition i. Since i is also an isometry, this means that $i(l_2)$ is a line in H^2 disjoint from the y-axis.

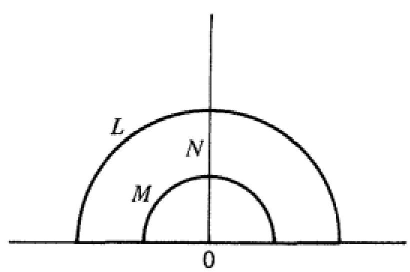
  Now, by intermediate value theorem we know there exists an l_3 that cuts the y-axis and $i(l_2)$ such that it is perpendicular to both. It will look as such:



ultra-parallel



Therefore, we know that there exist a series of isometries who take l_3 to the y-axis. Call their composition j. Since j is made up of τ_a, d_p and $J^{-1}r_\theta J$, which are all inversions and hence preserve angles, This means that $j(i(l_1))$, $j(i(l_2))$ and $j(l_3)$ will look as such:



Hence, since in the end $j \circ i(l_1)$ and $j \circ i(l_2)$ end up parallel, it means their product gives d_p for some p. Therefore, this implies that:

$$r_1 \circ r_2 = ji \circ d_p \circ i^{-1}j^{-2}$$

Therefore, every orientation preserving isometry obtained from the product of reflections about ultra parallel lines is a translation.

- **Case 2:** The lines l_1 and l_2 are asymptotic. In this case, they look as below:

  Now clearly as can be seen, there exists r_θ such that $r_\theta(l_1)$ and $r_\theta(l_2)$ are asymptotic with the y-axis at $+\infty$. Now consider the new lines in the H^2 model. Since the y-axis maps to the y-axis and $J^{-1}r_\theta(l_1)$, $J^{-1}r_\theta(l_2)$ are asymptotic with the y-axis, this means that $J^{-1}r_\theta(l_1)$, $J^{-1}r_\theta(l_2)$ are of the form $x = \alpha_i$ respectively.

  Therefore, clearly their product is τ_a for some a. Hence, we have:

$$r_1 \circ r_2 = l\tau_a l^{-1} \qquad \text{Where } l \text{ is an isometry of appropriate form.}$$

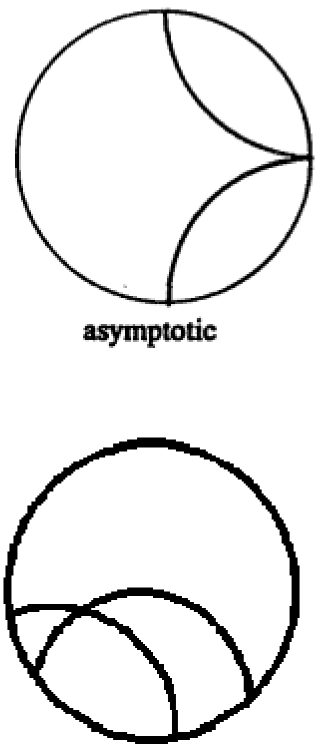


Figure 4: Lines that intersect

Therefore, product of two reflections about asymptotic lines gives us a Limit Rotation.

- **Case 3:** The lines l_1 and l_2 intersect. In this case, Consider the picture in H^2. Clearly it is that of two semicircles intersecting, with centers on the x-axis. Consider the isometry, i taking l_1 to the y-axis. Therefore, $i(l_2)$ is a semicircle which intersects the y-axis at one point. Let that point be $(0, y)$. Consider the isometry $d\frac{1}{y}$. Now the images of l_1 and l_2 respectively are the y-axis, and a semi circle cutting the y-axis at the point $(0, 1)$. From the section on the isometry $e^{i\alpha}$, we know that this corresponds to a reflection about the line $y = \alpha x$.

  Now the image of the reflections r_1 and r_2 in D^2 are the reflections about the y-axis and the line $y = \alpha x$. Therefore, it corresponds to the isometry r_θ for some θ. Hence if we denote $J^{-1} r_\theta J$ as h, we have:

  $$r_1 r_2 = d_{\frac{1}{y}}^{-1} i^{-1} h i d_{\frac{1}{y}}$$

  Hence, this means that the product of reflections about lines that intersect is a rotation in D^2 or h.

□

Remark 4.19. The following proposition extends to orientation reversing isometries in the following way:

"*All orientation reversing isometries are glide reflections.*"

We shall not prove this proposition, but it follows by considering reflections about 3 intersecting lines and moving them accordingly via isometries to produce the product of a reflection and translation, which are glide reflections.

Hence, with that all isometries of H^2 have been classified.

4.4 Hyperbolic Surfaces

Definition 4.20. A hyperbolic surface is defined to be a set S with a distance function d_S such that $\forall P \in S$ there is an ϵ such that $\{x \in S \mid d_S(x, P) < \epsilon\}$ is isometric to a disk of H^2.

Theorem 4.21. *Any complete connected Hyperbolic surface is of the form* $\frac{H^2}{\Gamma}$*, where* Γ *is a discontinuous, fixed point free subgroup of isometries of* H^2 *isometries.*

We shall not prove the above theorem. The proof is similar to the case of Spherical surfaces with a few more details.

As with the Spherical case, fixed point free means that the subgroup cannot contain any rotations or reflections. Hence, any subgroup will be generated by translations and glide reflections. Unfortunately, unlike the Euclidean space, the group of isometries can be generated by more than 2 elements.

4.4.1 Example

We shall now describe an example of a hyperbolic surface.

Consider the subgroup of isometries generated by $< \tau_{2\pi} >$. Therefore, the surface $\frac{H^2}{\Gamma}$ has a fundamental domain given by vertical strips with width 2π. Hence, if we want to visualise this surface, we can think of the analogous surface in the Euclidean space, the cylinder. Since the metric in H^2 is given by $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$, the points further away from the x-axis are further apart and the points closer to the x-axis are close. Hence, in R^3, it can be imagined as the diagram below.

The lines in this space are defined as the Γ images of the lines. It can be visualised by superimposing images of lines in H^2 in each domain.

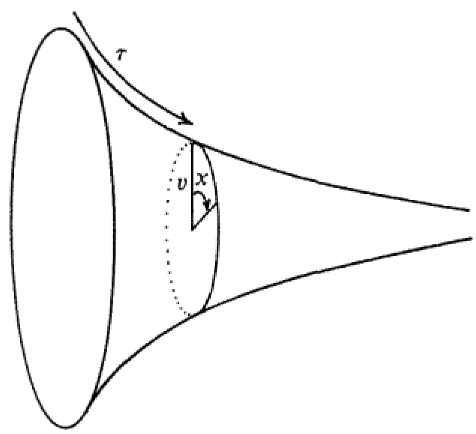


Figure 5: Pseudosphere

4.4.2 Hyperbolic Surfaces from Polygons

Definition 4.22. A *hyperbolic polygon* (Π) is defined to be a region in H^2 bounded by finitely many simple polygonal paths in H^2 and δH^2 called proper and improper edges of Π. The endpoints of these edges are called vertices.

An example of the same is polygonal fundemental domain we saw in the previous example, with the vertices being $\pi, -\pi, \infty$.

Definition 4.23. An *edge pairing* is a partition of the set of edges in pairs $\{e, e'\}$ such that:

- e and e' have the same length.
- There exists an isometry g which takes e to e'.

Points w and w' on e and e' respectively are said to be identified if $g(w) = w'$.

Definition 4.24. An *identification space* (denoted as S_Π) is defined as a hyperbolic polygon with each edge pairing identified with each other.

The points of the identification space are:

- The interior points of Π
- Pair's of w, w' on the interior points of Π identified together.
- Cycles of proper vertices identified together $\{v_1, v_2, ..., v_k\}$

A nice way to think about identification spaces via polygons is via the symbol of a polygon.

Definition 4.25. The symbol of a polygon is obtained by listing sides of the polygon in clockwise fashion, with labelling sides identified together with the same alphabet. If the direction associated with the side is counterclockwise, it is labelled with the inverse of the alphabet. Otherwise, it is labelled with just the alphabet.

The symbol can also be thought of as unique upto cyclic permutation as, depending on where you start, your string gets shifted appropriately.

Example 4.26. The symbol of the polygon below is $abd^{-1}dba^{-1}$.

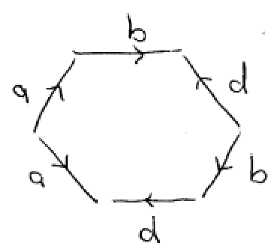


Figure 6: Symbol of a Polygon

Theorem 4.27. *The identification space S_Π has a distance function making it a hyperbolic surface, when the angles of each vertex cycle add up to 2π.*

Proof. We define the distance function between P and Q on S_Π as the infimum distance of all polygonal paths from P to Q. This is done as follows:

Simply decompose p_{PQ} into paths $p_{w_1,w_1'}p_{w_2,w_2'}...p_{w_k,w_k'}$ where w_1 denotes the first point where p_{PQ} cuts an edge of Π and w_1' is where it reenters Π ($w_1 = P$ and $w_k' = Q$). Define $len(p_{PQ})$ as $\sum_k H^2length(p_{w_i.w_i'})$. Thus we define the distance function as follows:

$$d_s(P, Q) = inf\{len(p_{PQ}) \mid p_{PQ} \text{ is a polygonal path from P to Q}\}$$

.

- **Case 1**: If A is an interior point of S_Π, an ϵ ball around A where $\epsilon < \frac{1}{2}$ (min length of A to an edge of A). Since the shortest path between any two points in this ball is line segment between them, which lies in the ball, this means that it is isometric to $D_\epsilon(u)$ where u is the center of the ball.
- **Case 2**: If $A = \{w, w'\}$ is a pair of identified points on an edge of S_Π via the side pairing isometry g, choose ϵ ¡ $\frac{1}{4}$(min distance or w or w' to another edge.) Clearly, $D_\epsilon(A)$ is the union of half disks of radius ϵ around w and w'. Therefore, using this it can be seen that for $B, C \in D_\epsilon(A)$,

$$d_{S_\Pi}(B, C) = \begin{cases} d_{S_\Pi}(B, C) & B, C \in samehalfdisc \\ d_{S_\Pi}(B', C) & B \in D_\epsilon(w) and C \in D_\epsilon(w') \\ d_{S_\Pi}(B, C') & B \in D_\epsilon(w') and C \in D_\epsilon(w) \end{cases}$$

  Hence, with this information we can conclude that $D_\epsilon(A)$ is isometric to $D_\epsilon(w')$ as the identity on their common half and g on the other half.

- **Case 3**: If $\{v_1, v_2, ..., v_k\}$ is a vertex cycle, consider ϵ ¡ quarter of any edge in Π. Therefore, we have:

$$D_\epsilon(\{v_1, v_2, ..., v_k\}) = (D_\epsilon(v_1) \cap \Pi) \cup (D_\epsilon(v_2) \cap \Pi)...$$

  When the angle sum is 2π, this can be thought of as decomposing $D_\epsilon(A)$ into sectors for each v_i, each in same order as that of $\{v_1, v_2, ..., v_k\}$. Therefore for any points $B, C \in D_\epsilon(A)$, we have that $d_{S\Pi}(B, C) = d_{H^2}(B^{(i)}, C^{(i)})$ where $B^{(i)}$ and $C^{(i)}$ are the images of B and C in the corresponding sectors. Thus since the union of all sectors can be mapped isometrically to a disc, $D\epsilon(A)$ is isometric to $D_\epsilon(v_j)$ for any j.

□

With the above proposition we have mapped all identification spaces to a hyperbolic space. With that we come to the following proposition:

Theorem 4.28. *If the polygon Π is compact, then S_Π is complete.*

Proof. We know that each line segment in S_Π is an image of a sequence of line segments on Π, via the map identifying edge pairings $\Pi \to S_\Pi$. Therefore, if we continue a line segment on S_Π it corresponds to extending a line segment in Π until it hits an edge and then resumes at the identified point. Therefore, if we look at the preimage of a line L, it is an infinite sequence of line segments $L_1, L_2, ...$

If the total length of these segments is ∞, then the line is complete. Let us assume the total length is finite. This implies that as $n \to \infty$, then length of L_n goes to 0. This means that $\ni N$ such that for all $n > N$, L_n will lie in an ϵ ball around a vertex of S_Π, $\{v_1, v_2, ..., v_k.\}$ (if it is outside, there exists a positive lower bound on the length of the segment). But taking ϵ small enough, we can make sure that there is only finitely many such possible line segments. Hence, it induces a contradiction. Therefore length of L is ∞. Hence S_Π is complete. □

Corollary 4.29. *Given an identification space S_Π, it is can be written in the form $\frac{H^2}{\Gamma}$*

Therefore, with this proposition, we have that S_Π is a hyperbolic surface, which is compact and connected and complete. Hence via the Killing-Hopf theorem, we see that there exists an isometry subgroup Γ of H^2 such that:

$$S_\Pi = \frac{H^2}{\Gamma}$$

We now check whether the converse of the theorem holds. It first starts with the below proposition:

Theorem 4.30 (Rado)**.** *Any compact surface, is homeomorphic to the identification space of a polygon S_Π.*

We will not prove the above theorem, but we shall use it to prove the following proposition:

Theorem 4.31. *All compact surfaces can be realised as a Euclidean, Spherical or Hyperbolic surface.*

Proof.

Lemma 4.32. *A polygon Π can always be replaced by another polygon Π' with $S_\Pi = S'_\Pi$, where all vertices glue to the same point.*

Proof. Suppose all vertices glue to two vertices, either P or Q. Consider the following picture:

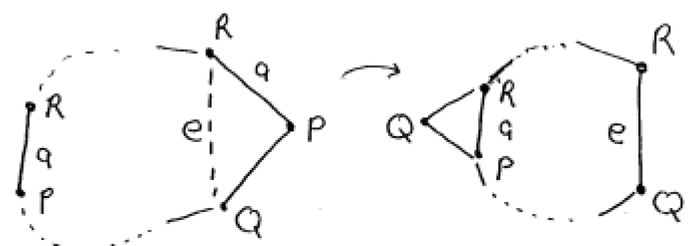


Figure 7: Cut and Paste

Clearly from the figure, Q is a Q vertex and P is a P vertex. Let R be the previous vertex to P as in the figure. It can be a P or Q vertex. It is evident that by cutting along the line e between R and Q and pasting along a, we obtain a new polygon Π' with 1 fewer P vertex. Therefore, we can do this recursively to obtain a polygon with only 1 P vertex.

In this case, via the construction above, two edges connected to P have to have the sub-symbol aa^{-1}. Hence by joining them in the interior of the polygon, we get a polygon with the same S_Π such that all vertices map to the same point. Hence proved.

□

From the previous theorem, it suffices to check the theorem for S_Π for any Π. Since the edges are identified in pairs in Π, the number of edges is even. We can also assume from the lemma that all vertices map to the same point. Therefore, lets look at the trial cases for $n = 2$. The corresponding polygons are as follows:

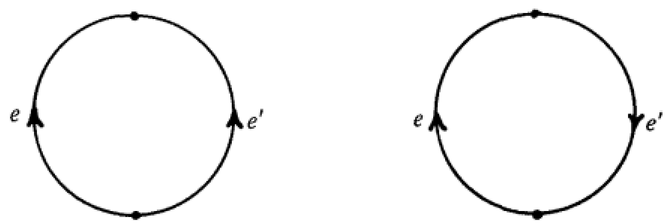


Figure 8: 2-gon's

Clearly, from the previous section, we know that these are S^2 and RP^2 respectively. Hence they form spherical surfaces. Similarly for a 4-gon, it can be shown that it can be identified as a Sphere, Projective Plane, Torus or Klein bottle, all of which are Spherical and Euclidean surfaces respectively.

For any general 2n-gon, we know from the previous theorem that it can be given the structure of a Hyperbolic surface if it has an angle sum of 2π. Since we are only homeomorphically identifying a compact surface with Π via Rado's theorem, we can limit ourselves to regular polygons. It is easy to show that for a regular hyperbolic polygon, the following is true:

$$(2n-2)\pi - area(\Pi) = anglesum.$$

Since we can vary the diameter of the polygon continuously, the area also will vary continuously. Hence via intermediate value theorem, there will exist a diameter such that $(2n-2)\pi - area(\Pi) = 2\pi$. Hence, it will form a hyperbolic surface. □

We shall now state the final proposition of the converse without proof. It is given as follows:

Theorem 4.33. *For any compact hyperbolic surface* $\frac{H^2}{\Gamma}$*, there exists a polygonal fundamental domain.*

With the above proposition, we have completely classified compact complete and connected hyperbolic surfaces, via identification spaces of compact polygons. It must be mentioned that there do exist other hyperbolic surfaces which aren't compact which cannot be classified via this method. In fact hyperbolic surfaces can also arise via identification spaces of polygons which are not compact. These correspond to Γ which are finitely generated but not finite. Hence, there are further classifications to all the possible surfaces obtained via the Killing Hopf theorem.

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