WEIL RESTRICTION

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These are notes of my talk given to Ph.D students at T.I.F.R. on 03-04-2012. Plan of the talk:

(1) Motivation
(2) Definition
(3) Examples
(4) Galois extensions
(5) Applications

1. Restriction of scalars functor or Weil Restriction.

Let L/K be a finite field extension. Suppose X is a L–variety then there is a canonical way of getting a K-variety from X say $R(X)$ called as Weil restriction of X. It was first introduced by A. Weil. If X is a variety over a number field F then by Weil restriction, one can work over a field of rationals rather than F. In the process the original variety now becomes complicated in the sense that dimension of the restricted variety may increase. In the case of the extension $\mathbb{C}/\mathbb{R}$, it can be interpreted as looking at a complex n -dimensional variety as $2n$-dimensional real variety.

The Weil restriction of abelian varieties were studied to solve mainly the problems in arithmetic algebraic geometry, prominant example is:
1. Milne's proof that the conjectures of Birch-Swinnerton Dyer for abelian varieties over the field of rationals implies the same conjectures over any number field.
2. Honda's theorem about classifications of isogeny classes of abelian varieties over finite fields.

We will restrict in this talk to an elementary setting of Weil restriction, the more general settings can be found in the book Néron Models.

For the rest of the talk fix the notation Var/K (resp. Var/L) for the category of varieties over K (resp. over L) and $Sets$ for the category of sets.

Definition 1.1. [Weil restriction] Let L/K be a finite extension of fields. For $X \in Var/L$, consider a functor

$$R_X : Var/K \to Sets$$

$$Z \mapsto \mathrm{Hom}_L(Z_L, X) \quad \text{where } Z_L := Z \times_K L$$

Since $\mathrm{Hom}(-, X)$ is a contravariant functor, R_X is a contravariant functor. i.e. Given a map $\phi : Z \mapsto Z' \in Var/K$, we get $R_X(\phi) : \mathrm{Hom}(Z'_L, X) \mapsto \mathrm{Hom}(Z_L, X)$.

Theorem 1.2. *R_X is a representable functor i.e. there exists a unique variety $R(X) \in Var/K$ such that*

$$R_X \cong Hom_K(-, R(X))$$

$R(X)$ is the object representing the functor R_X, and is called as Weil restriction of X.

Note that isomorphism of functors will always mean that they are naturally equivalent, i.e. to say given $Z \mapsto Z' \in Var/K$ the following square commutes,

$$\begin{array}{ccc} R_X(Z'_L) & \longrightarrow & R_X(Z_L) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_K(Z', R(X)) & \longrightarrow & \mathrm{Hom}_K(Z, R(X)) \end{array}$$

Also, by **Yoneda lemma**, giving a morphism between two representable functors is same as giving the morphism between their representing objects. So the uniqueness of $R(X)$ follows.

We can also consider the functor $F : Var/L \to Var/K$ given by $F(X) = R(X)$. Given a morphism $\phi : X \to Y \in Var/L$ we get a morphism between two representable functors $R_X \to R_Y$. Here we also observe that $X \to R_X$ is a covariant functor. By Yoneda lemma we get a morphism $R(X) \to R(Y)$. Thus the functor F is covariant.

Definition 1.3. [Weil restriction] The Weil restriction of $X \in Var/L$ can be thought of as a K-variety $R(X)$ such that for any $Z \in Var/K$, giving a K-morphism $Z \to R(X)$ is same as giving a L-morphism from $Z_L \to X$.

Examples.

(1) Let $X = SpecL$. Let $Z \in Var/K$ then $\mathrm{Hom}_L(Z_L, Spec(L))$ is a singleton set as $\mathrm{Spec}(L)$ is the terminal object in Var/L. By 1.3, $\mathrm{Hom}_K(Z, R(X))$ has to be a singelton set. By Yoneda lemma $R(X)$ should be $Spec(K)$ which is the terminal obejct in Var/K.

(2) Let $X = \mathbb{A}^n_L$. Claim: $R(X) = \mathbb{A}^{nd}_K$ where $[L : K] = d$.
By using the property $R(X \times_L Y) = R(X) \times_K R(Y)$ which is proved later in Property(1) it is enough to prove the claim when $n = 1$ So we are in the case $X = \mathbb{A}^1_L = Spec(L[X])$. Choose a basis $e_1, \cdots, e_d$ of L/K.

The variety $R(X)$ will be theone for which giving a map form $Z \to R(X)$ should be the same as giving a map from $Z_L \to X$ which is same as giving a map from $L(X) \to K(Z \times_K L)$. Write $x = \Sigma y_i e_i$ where y_i are variables. Consider $R(X) = Spec(K[y_1, \cdots, y_d])$ which is isomorphic to $\mathbb{A}^d_K$. Thus for $R(\mathbb{A}^n_L) = \mathbb{A}^{nd}_K$. This proves the claim. This example also shows that $\dim_K(R(X)) > \dim_L(X)$. when X is affine.

(3) Let X be a closed subvariety of $\mathbb{A}^n_L$ i.e. $X = Spec(L[x_1, \cdots, x_n]/J)$ where $J =< f_1, \cdots, f_r >$. Choose a basis $e_1, \cdots, e_d$ of L/K and write $x_i = \Sigma y_{ij} e_j$ where $y'_{ij}s$ are variables. Substitute in $f'_i s$ and solve. We get a system of equations in variables y_{ij} with coefficients in K. Then $R(X) = Spec(K[y_{ij}]/I)$ where I is generated by the equations we have obtianed. Thus $R(X)$ is a closed subariety of $\mathbb{A}^{nd}$.

(4) Let $L/K = \mathbb{C}/\mathbb{R}$ and $X = \mathbb{G}_m/\mathbb{C} = Spec(\mathbb{C}[x_1, x_2]/(x_1x_2 - 1))$. So X is a 1-dimensional complex variety. Choose a basis $1, i$ of $\mathbb{C}/\mathbb{R}$ and write $x_j = 1.y_{j1} + i.y_{j2}$ for $j = 1, 2$. Substitute in $x_1x_2 - 1$ and get

$$y_{11}y_{21} - y_{12}y_{22} = 1$$
$$y_{11}y_{22} + y_{21}y_{12} = 0$$

Let $R(X) = \mathbb{R}[X, Y, Z, W]/ < (XZ - YW - 1), (XW - YZ) >$ which is a two dimensional $\mathbb{R}$-variety. Later by Galois descent we will see that this is infact a torus isomporphic to $\mathbb{C}^*$.

Properties of Weil restriction.

(1) $R(X \times_L Y) = R(X) \times_K R(Y)$. Indeed,

$$\begin{aligned} R_{X\times Y} &= Hom_L(-, X \times_L Y) \\ &\cong Hom_L(-, X) \times Hom_L(-, Y) \\ &\cong Hom_K(-, R(X)) \times_K Hom_K(-, R(Y)) \\ &\cong Hom_K(-, R(X) \times_K R(Y)) \text{....by property of fiber product} \end{aligned}$$

Also, $R_{X\times Y} \cong Hom_K(-, R(X \times_L Y))$. Thus the claim follows by Yoneda lemma.

(2) $\dim_K(R(X)) = [L : K] \cdot \dim_L(X)$.

(3) For $Z = Spec(K)$ we get, $Hom_K(Z, R(X)) = Hom_L(Spec(L), X)$ which means that $R(X)(K) = X(L)$.

(4) If $K \subset L \subset M$ and $X \in Var/M$ then $R^L_K(R^M_L(X)) = R^M_K(X)$.

(5) Suppose $X \in Var/L$. Then how $R(X) \times_K L$ is related to X ?
Put $Z = R(X)$, then $Hom_K(Z, R(X)) \to Hom_L(Z_L, X)$ is an isomorphism and id $\mapsto$ projection onto$X$ as $\dim_K(Z) = \dim_L(Z_L) > \dim_L(X)$

(6) Suppose $Y \in Var/K$, then how $R(Y_L)$ is related to Y?
Put $Z = Y$ then $Hom_L(Z_L, Y_L) \to Hom_K(Z, R(Y_L))$ is an isomorphism where id $\mapsto$ closed embedding as $\dim_K(Z = Y) < \dim_K(R(Y_L))$.

(7) If X is affine(or smooth) then so is $R(X)$
(8) If G is a algebraic group over L then $R(G)$ is an algebraic group over K. This is because if F is a functor from Var/L to Var/K which preserves products and which takes $Spec(L)$ to $Spec(K)$ should take a group variety toa group variety. These properties are satisfied by the functor R_X (See Property 1. and e.g. 1)
(9) If G is an abelian variety over L then $R(G)$ is an abelian variety over K.
(10) If G is semisimple(unipotent) then so is $R(G)$.

The last proeprty can be explained by understanding Weil restriction in the case of Galois extensions.

2. Galois Case

For this section suppose that L/K is a Galois extension with the Galois group G. Most of the things here can be generalised to separable extensions but we will stick to Galois case for simplicity.
Advantage of considering a Galois extension is that one can explicitly know what the Weil restriction of a variety X over L. Let $X \in Var/L$. For an element $g \in G$, we define the conjugate variety X^g to be the fiber product,

$$\begin{array}{ccc} X^g & \longrightarrow & X \\ \downarrow & & \downarrow \\ Spec(L) & \xrightarrow{g} & Spec(L) \end{array}$$

In the affine case another way to see X^g is $\mathrm{Spec}(\frac{L[x_1,\cdots,x_n]}{<gf_1,\cdots,gf_r>})$

Theorem 2.1 (A. Weil)**.** *$X \in Var/L$ and L/K is a finite Galois extenison then $R(X) \times_K L \cong \prod_{g\in G} X^g$.*

Note that for $g \neq 1$, X^g is not isomorphic to X as L-variety. There is a semilinear Galois action on $R(X) \times_K L$. The corresponding semilinear action on $\prod_{g\in G} X^g$ is such that, for $h \in G$, $\phi_h : X^g \mapsto X^{gh}$ permutes the components of the product. By using Galois descent one can always recover $R(X)$ given this semilinear action on $\prod_{g\in G} X^g$. Following examples will show how Galois extensions are important in Weil restriction.

(1) Let $\mathbb{G}_m \in Var/L$, then $R(\mathbb{G}_m) \times_K L = \prod_{g\in G} \mathbb{G}_m$, so by definition of torus $R(\mathbb{G}_m)$ is a torus over K.
(2) Let $X \in Var/L$ be a semisimple algebraic group. Then $R(X) \times_K L = \prod_{g\in G} X$. Since the product of semisimple groups is again semisimple, $R(X)$ is semisimple over K.

3. Applications

(1) An important question is can we classify complex abelian varieties having $\mathbb{R}$-isomorphic Weil restrictions? Huismann proved the following:
Let E be an elliptic curve over $\mathbb{C}$ with complex multiplication (i.e. $End(E)$ is larger than $\mathbb{Z}$). Let $\rho(E) = \sharp\{\mathrm{E}'/\mathbb{C} \mid \mathrm{E}'\text{is nonisomorphic to E and } R(E) \cong R(E')\ \}$.
Then $R(E) \cong R(E')$ if and only if $End(E) \cong End(E')$, moreover $\rho(E) =$ class number of $End(E)$.

(2) Weil restriction is useful in constructing arithmetic algebraic groups such as Hilbert modular modular groups, Bianchi groups etc. Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field, hence Galois. Let SL_2 defined over F. Let $\mathbb{O}_F$ be the maximal order in F. F has two real embeddings and no complex embedding. $R(SL_{2,F})(\mathbb{R}) = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ which contains $SL_2(\mathbb{O}_F)$ as a discrete subgroup . It is called as Hilbert modular group from which $SL_2(\mathbb{O}_F)\backslash\mathbb{H}\times\mathbb{H}$ called as Hilbert modular surfaces are constructed where $\mathbb{H}$ is a upper half space.