Quiz : IDC 206 Mathematical Methods

Maximum Marks: 50 Time: 10 AM — 11:30 AM

Please note: You will be provided with answer booklets. Books, papers, calculators and communication devices such as cellphones are **not permitted.** *While answering questions, you need to show all the steps in your computation.*

1. Using the appropriate method in each case, find the most general solution(s) or list all possible solutions *in explicit form* to each of the following differential equations :

(a) $dy/dx = (x+1)(y^2-1)$. (*7 marks*)
(b) $x(dy/dx) - 3y = x^4, x > 0$. (*7 marks*)

2. (a) Solve the following initial value problem (IVP): (*5 marks*)

$$\begin{aligned} \frac{dy}{dx} &= x^2 e^{-y}; \\ y(0) &= \ln 2. \end{aligned} \tag{1}$$

(b) Determine the interval of x for which the solution in part (a) exists. (*5 marks*)

3. Radioactive decay of a radioactive substance (such as Carbon-14) can be modelled (as discussed in class) by the differential equation

$$\frac{dx(t)}{dt} = -kx(t), \tag{2}$$

where $x(t)$ is the amount (in grams) of the substance at time t. $k > 0$ is a constant whose value depends on the particular radioactive substance studied.

(a) Find the general solution to this differential equation. (*4 marks*)

(b) The time taken for half of a sample of Carbon-14 to decay (its half-life) is 5730 years. From this information, determine k for this system. (*6 marks*)

(c) Suppose in an ancient biological sample, the percentage of original Carbon-14 remaining at the present day is 14 %. How old is the sample? (*6 marks*)

4. (a) Show that applying the *change of variable* $z(x) = [y(x)]^3$ to the differential equation

$$xy^2\frac{dy}{dx} + y^3 = x\cos x, \tag{3}$$

converts it into a *linear* differential equation for $z(x)$. (*5 marks*) 6

(b) Use this to find the most general solution to the differential equation given in part (a). (*5 marks*)

$\frac{-\operatorname{cosec}\theta\ \cancel{\cot\theta}\ d\theta}{\cot^2\theta}$

$-\frac{1}{\sin}\ \frac{\sin}{\cos}$

SOLUTIONS: QUIZ

1. (a) $$\frac{dy}{dx} = (x+1)(y^2-1)$$

Two obvious solutions: (i) $y = +1$,
(by inspection) (ii) $y = -1$

To get the other solutions, we use separation of variables.

$$\int \frac{dy}{y^2-1} = \int (x+1)\,dx$$

$$\frac{1}{2}\left[\int \frac{dy}{y-1} - \int \frac{dy}{y+1}\right] = \int (x+1)\,dx$$

$$\ln\left|\frac{y-1}{y+1}\right| = x^2+2x+C$$ ← integration constant.

$$\Rightarrow \left|\frac{y-1}{y+1}\right| = e^{x^2+2x+C}$$

We have two possibilities: Either

(iii) $$\frac{(y-1)}{(y+1)} = e^{x^2+2x+C}$$

$$\Rightarrow y(x) = \frac{1+e^{(x^2+2x+C)}}{1-e^{(x^2+2x+C)}}$$

(iv) or $$\frac{(y-1)}{(y+1)} = -e^{x^2+2x+C}$$

$$\Rightarrow y(x) = \frac{1-e^{(x^2+2x+C)}}{1+e^{(x^2+2x+C)}}$$

These are all the possible solutions.

(b) Write the DE in standard form:

$$\left(\frac{dy}{dx}\right) - \frac{3y}{x} = x^3$$

This is of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

$$P(x) = -3/x \quad , \quad Q(x) = x^3$$

As derived in class,

$$y(x) = e^{-\int P dx} \left[\int Q(x) e^{\int P dx'} dx + C \right]$$

$$\int P(x)\, dx = -3 \int \frac{dx}{x} = -3 \ln x = \ln x^{-3}$$

$$e^{\int P dx} = \frac{1}{x^3} ;$$

$$y(x) = x^3 \left[\int x^3 \cdot \frac{1}{x^3} dx + c \right]$$

$$= x^3 [x + c] = x^4 + Cx^3$$

2(a)

$$\frac{dy}{e^{-y}} = x^2 dx \qquad \text{(separation of variables)}$$

$$\int e^y\, dy = \int x^2\, dx$$

$$e^y = \frac{x^3}{3} + C$$

Taking logarithms on both sides,

$$y = \ln\left(\frac{x^3}{3} + c\right).$$

$$y(0) = \ln 2$$

$$\Rightarrow \ln c = \ln 2, \quad \Rightarrow c = 2.$$

The solution to the IVP is

$$y(x) = \ln\left(\frac{x^3}{3} + 2\right).$$

(b) Clearly this solution exists only when

$$\frac{x^3}{3} + 2 = e^y > 0$$

or argument of $\ln\left(\frac{x^3}{3} + 2\right)$ is positive.

$$\Rightarrow x^3 > -6$$

$$x > (-6)^{1/3}$$

Interval of x for which solution exists is $((-6)^{1/3}, \infty)$

3. $$\frac{dx}{dt} = -kx$$

(a) $$\int \frac{dx}{x} = -k \int dt$$

$$\cancel{x(t) =} \quad \ln|x| = -kt + C$$

Since $x(t) \geq 0$, we drop the modulus. ($x(t)$ is the amount in grams)

$$\Rightarrow \quad x(t) = e^{c} e^{-kt}$$

$$x(0) = x_0 = e^{c}$$

$$\Rightarrow \quad x(t) = x_0 e^{-kt}$$ is the general solution.

(b) After the half life T, $x(t) = x_0/2$.

$$\Rightarrow \quad \frac{x_0}{2} = x_0 e^{-kT}$$

$$\Rightarrow \quad e^{kT} = 2$$

$$kT = \ln 2$$

$$T = 5730 \text{ yrs.}$$

$$k = \frac{\ln 2}{5730} \text{ yr}^{-1}$$

(c) In the sample, % of original Carbon-14 remaining is 14%. ~~At~~ At present time t,

$$\Rightarrow \quad x(t) = \frac{14}{100} x_0 .$$

$$\frac{14}{100} x_0 = x_0 e^{-kt}$$

$$e^{kt} = \frac{100}{14}$$

$$kt = \ln\left(\frac{100}{14}\right)$$

$$t = \ln\left(\frac{100}{14}\right) \cdot \frac{1}{k}$$

$$= \ln\left(\frac{100}{14}\right) \cdot \left(\frac{5730}{\ln 2}\right) \text{ yrs.}$$

4(a)

$$xy^2 \frac{dy}{dx} + y^3 = x \cos x$$

Let $z = y^3$

$$dz = 3y^2\, dy \quad ; \quad \frac{dy}{dz} = \frac{1}{3y^2}$$

$$xy^2 \cdot \frac{1}{3y^2} \cdot \frac{dz}{dx} + y^3 = x \cos x$$

$$\frac{1}{3} x\, z'(x) + z(x) = x \cos x.$$

$$\Rightarrow \quad z'(x) + \frac{3z}{x} = 3 \cos x \quad \text{(Linear DE for } z\text{)}$$

(b) Let $3/x = P(x)$; $3 \cos x = Q(x)$.

Then, solution to the DE for $z(x)$ is

$$z(x) = e^{-\int P\,dx}\left[\int Q(x)\, e^{\int P\,dx'}\,dx + C\right]$$

$$\int P\,dx = 3\int \frac{dx}{x} = 3\ln x \quad \text{for } x > 0$$
$$= \ln x^3$$

$$e^{\int P\,dx} = x^3$$

$$z(x) = \frac{1}{x^3}\left[3\int (\cos x)\cdot x^3\,dx + C\right]$$

$$= \frac{1}{x^3}\left[3x^3 \sin x - 18x\sin x + 9x^2\cos x - 18\cos x + C\right]$$

$$z = \left[3\sin x - \frac{18}{x^2}\sin x + \frac{9}{x}\cos x - \frac{18}{x^3}\cos x + C/x^3\right]$$

$y(x) = [z(x)]^{1/3}$ solves the original DE.

We have a similar solution for $x < 0$.