Mid-term : IDC 206 Mathematical Methods

Maximum Marks: 50 Time: 11 AM — 12:30 PM

Please note: You will be provided with answer booklets. Books, papers and communication devices such as cellphones are **not permitted.** *While answering questions, you need to show all the steps in your computation.*

Notation used: $y'(x) = dy/dx, y''(x) = d^2y/dx^2$.

1. Find a second-order differential equation for $y(x)$ that has the solution $y(x) = e^{-3x} + 2xe^{-3x}$. What are the initial conditions $y(0)$ and $y'(0)$? (*10 marks*)

2. Solve the Initial value problem:

$$y''(x) - 3y'(x) + 2y(x) = 14 \sin 2x,$$
$$y(0) = 1, y'(0) = 0.$$

(*10 marks*)

3. Consider the differential equation

$$(1 - x^2)y''(x) - xy'(x) + p^2y(x) = 0,$$

where p is a constant. (i) Explain why you can find a power series solution to this equation around $x = 0$. (*5 marks*)
(ii) Find the first three non-zero terms in the general power series solution around $x = 0$.(*10 marks*)
(iii) Find the minimum radius of convergence of this series solution. (*5 marks*)

4. Consider the differential equation

$$x^4y''(x) + (2x^3 - 1)y'(x) + \frac{y}{x} = 0.$$

What is the behaviour of the most general solution to this equation as $x \to \infty$? (*10 marks*)

IDC 206 MIDTERM FORMULA SHEET:-

Separable Equations	$\dfrac{dy}{dx} = g(x)f(y)$
Step 1: Separate x's and y's on different sides.	$\dfrac{1}{f(y)}dy = g(x)dx$
Step 2: Integrate both sides.	$\displaystyle\int \frac{1}{f(y)}dy = \int g(x)dx + C$
Step 3: Express y in terms of x where possible.	If $\lvert y\rvert = h(x)$, then $y = \pm h(x)$. If $y = \pm e^C\, h(x)$, then $y = A\, h(x)$ where A is *any* real number (including zero).
Step 4: Check that constant solutions $y = C$ where $f(C) = 0$ are not missed.	

First Order Linear Equations	$y' + P(x)y = Q(x)$
Step 1: Find integrating factor.	$I(x) = e^{\int P(x)dx}$
Step 2: Write differential equation as	$\big(I(x)y\big)' = I(x)Q(x)$
Step 3: Integrate both sides.	$\displaystyle I(x)y = \int I(x)Q(x)dx + C$
Step 4: Divide both sides by $I(x)$	$\displaystyle y = \frac{1}{I(x)}\left(\int I(x)Q(x)dx + C\right)$

Remark: Be careful with the sign of $P(x)$. For instance,
If $y' + \frac{1}{x}y = 1$, the integrating factor is $I(x) = e^{\int 1/x\, dx} = x$.
If $y' - \frac{1}{x}y = 1$, the integrating factor is $I(x) = e^{\int -1/x\, dx} = \frac{1}{x}$.

Second Order Linear Homogeneous Equations	$ay'' + by' + cy = 0$
Step 1: Write down the auxiliary equation.	$ar^2 + br + c = 0$ (CHARACTERISTIC EQN.)
Step 2: Solve the auxiliary equation.	$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Step 3: Depending on the roots:	GENERAL SOLUTION:
(i) $b^2 - 4ac > 0$. Two real roots r_1, r_2.	$y = c_1\, e^{r_1 x} + c_2\, e^{r_2 x}$
(ii) $b^2 - 4ac = 0$. One real root $r = r_1 = r_2$.	$y = c_1\, e^{rx} + c_2\, x\, e^{rx}$
(iii) $b^2 - 4ac < 0$. Two complex roots $\alpha \pm i\beta$.	$y = c_1\, e^{\alpha x} \cos \beta x + c_2\, e^{\alpha x} \sin \beta x$

Second Order Linear Non-homogeneous Equations	$ay'' + by' + cy = G(x)$

Method of Undetermined Coefficients

Step 1: Solve the homogeneous equation. $ay_h'' + by_h' + cy_h = 0$

$y_h = c_1\, y_1(x) + c_2\, y_2(x)$

Step 2: Write down a trial solution:

IF	GUESS
(i) $G(x) = P(x)$	$y_p = Q(x)$
(ii) $G(x) = P(x)e^{kx}$	$y_p = Q(x)\, e^{kx}$
(iii) $G(x) = P(x)e^{kx}\cos mx$ or $P(x)e^{kx}\sin mx$	$y_p = Q(x)\, e^{kx}\cos mx + R(x)\, e^{kx}\sin mx$

[If your guess in (i) or (ii) solves the homogeneous eqn. Multiply guess by x. If this does not work, by x^2]

Here, $P(x), Q(x)$ and $R(x)$ are polynomials of the same degree.

In (iii), Multiply y_p by x if $k+im$ is a root of $ar^2+br+c=0$.

Step 3: Substitute y_p into the differential equation, group terms of the same form together, e.g. $x^n e^{kx}\cos mx$, $x^n e^{kx}\sin mx$ and solve for the unknown coefficients. $ay_p'' + by_p' + cy_p = G(x)$

Step 4: Write down the general solution. $y(x) = y_h(x) + y_p(x)$

Second Order Linear Non-homogeneous Equations	$ay'' + by' + cy = G(x)$

Method of Variation of Parameters

Step 1: Solve the homogeneous equation. $ay_h'' + by_h' + cy_h = 0$

$y_h = c_1\, y_1 + c_2\, y_2$

Step 2: The particular solution has the form: $y_p = u_1\, y_1 + u_2\, y_2$

Write down the two conditions:

$$u_1'\, y_1 + u_2'\, y_2 = 0$$
$$u_1'\, y_1' + u_2'\, y_2' = G(x)/a$$

Solve the conditions for u_1' and u_2'.

$$u_1' = \frac{G(x)y_2}{a(y_1'y_2 - y_2'y_1)} \qquad u_2' = \frac{G(x)y_1}{a(y_2'y_1 - y_1'y_2)}$$

Step 3: Integrate u_1', u_2' to get u_1, u_2.

$$u_1 = \int u_1'dx + c_1 \qquad u_2 = \int u_2'dx + c_2$$

Step 4: Write down the general solution.

$$y = (\textstyle\int u_1'dx + c_1)y_1 + (\int u_2'dx + c_2)y_2$$

SOLUTIONS: MID-TERM:

1. $y(x) = e^{-3x} + 2xe^{-3x}$

$= c_1 e^{-3x} + c_2 x e^{-3x}. = c_1 e^{rx} + c_2 x e^{rx}$ $(r = -3)$.

This is the solution to the DE

$$ay'' + by' + cy = 0$$

where $ar^2 + br + c = 0$ has the double root $r = -3$.

Clearly, an easy example is

$$ar^2 + br + c = (r+3)^2$$

$$= r^2 + 6r + 9$$

$$\Rightarrow a = 1, \quad b = 6, \quad c = 9$$

A DE for which the given $y(x)$ is the solution is

$$y'' + 6y' + 9y = 0$$

Clearly $c_1 = 1$, $c_2 = 2$, & $y(0) = c_1 = 1$.

$$y'(x) = -3e^{-3x} + 2e^{-3x} - 6xe^{-3x}$$

$$y'(0) = -3 + 2 = -1$$

So initial conditions are $y(0) = 1$
$y'(0) = -1$.

2. The given IVP is

$$y'' - 3y' + 2y = 14 \sin 2x$$

$$y(0) = 1$$

$$y'(0) = 0$$

First, guess a form for the particular solution: 2

Let $y_p(x) = A\sin 2x + B\cos 2x$.

$$y_p' = 2A\cos 2x - 2B\sin 2x$$

$$y_p'' = -4A\sin 2x - 4B\cos 2x$$

Substitute back in the DE.

$$-4(A\sin 2x + B\cos 2x)$$
$$-6(A\cos 2x - B\sin 2x)$$
$$+2(A\sin 2x + B\cos 2x) = 14\sin 2x$$

$$-2A\sin 2x - 2B\cos 2x + 6B\sin 2x$$
$$-6A\cos 2x = 14\sin 2x$$

$$(6B - 2A)\sin 2x - (6A + 2B)\cos 2x = 14\sin 2x.$$

$$6A + 2B = 0 \Rightarrow B = -3A$$

$$6B - 2A = 14 \Rightarrow A = -7/10$$

$$B = 21/10.$$

$$y_p = -7/10\sin 2x + \frac{21}{10}\cos 2x.$$

Now,

the associated homogeneous eqn. is

$$y_h'' - 3y_h' + 2y_h = 0$$

Characteristic eqn: $r^2 - 3r + 2 = 0$

Roots : $r_1 = 2$, $r_2 = 1$

General soln.. $y_h = c_1 e^{2x} + c_2 e^{+x}$

General solution to nonhomogeneous eqn.

$$y'' - 3y' + 2y = 14 \sin 2x$$

is

$$y = y_p + y_h$$

$$y(x) = -\frac{7}{10} \sin 2x + \frac{21}{10} \cos 2x + c_1 e^{2x} + c_2 e^{x}.$$

$$y'(x) = -\frac{14}{10} \cdot \cos 2x - \frac{42}{10} \sin 2x + 2c_1 e^{2x} + c_2 e^{x}.$$

$$y(0) = \frac{21}{10} + c_1 + c_2 = 1$$

$$\Rightarrow \quad c_1 + c_2 = -11/10$$

$$y'(0) = 0 \Rightarrow \quad -\frac{14}{10} + 2c_1 + c_2 = 0$$

$$\Rightarrow \quad 2c_1 + c_2 = 14/10$$

$$\Rightarrow \quad c_1 = 5/2 \quad , \quad c_2 = -18/5$$

$$y(x) = \frac{7}{10} - \frac{7}{10}\sin 2x + \frac{21}{10}\cos 2x + \frac{5}{2}e^{2x} - \frac{18}{5}e^{x}$$

is the solution to the IVP.

3. $(1-x^2)\,y''(x) - x\,y'(x) + p^2 y(x) = 0.$

In standard form

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

$$y'' - \frac{x}{(1-x^2)}\,y' + \frac{p^2}{(1-x^2)}\,y = 0$$

(a) $P(x)$ & $Q(x)$ are analytic at $x=0$. (Rational functions are analytic except when denominator $=0$).

Therefore, we can find a power series solution around $x=0$.

(b) Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots$

Substitute in the DE

$(1-x^2)\left[2a_2 + 6a_3 x + 12a_4 x^2 + \cdots\right]$

$- x\left[a_1 + 2a_2 x + 3a_3 x^2 + \cdots\right]$

$+ p^2\left[a_0 + a_1 x + a_2 x^2 + \cdots\right] = 0$

$$2a_2 \quad - 2a_2 x^2 + 6a_3 x + 12a_4 x^2 + \cdots$$

$$- \quad a_1 x \quad - 2a_2 x^2 \quad + \cdots\cdots$$

$$+ p^2 a_0 + p^2 a_1 x + p^2 a_2 x^2 + \cdots = 0$$

(explicitly writing only terms up to x^2)

$$\Rightarrow (2a_2 + p^2 a_0) + (6a_3 - a_1 + p^2 a_1) x$$

$$+ (-2a_2 + 12a_4 - 2a_2 + p^2 a_2) x^2 + \cdots = 0$$

$$\Rightarrow p^2 a_0 = -2a_2 \Rightarrow a_2 = -\frac{p^2 a_0}{2}$$

$$6a_3 - a_1 + p^2 a_1 = 0$$

$$6a_3 = (1-p^2) a_1$$

$$a_3 = \frac{(1-p^2) a_1}{6}$$

$$12a_4 + (p^2 - 4) a_2 = 0$$

$$\Rightarrow \quad 12a_4 = -(p^2-4) a_2$$

$$a_4 = -\frac{(p^2-4) a_2}{12} = \frac{(p^2-4) p^2 a_0}{24}$$

$$y(x) = a_0 + a_1 x - \frac{p^2 a_0}{2} x^2 + \frac{(1-p^2) a_1}{6} x^3$$

$$+ \frac{(p^2-4) p^2 a_0}{24} x^4 + \cdots$$

(a) becomes

$$\sum_{i=0}^{\infty} (i+2)(i+1)\, a_{i+2}\, x^i - \sum_{i=2}^{\infty} i(i-1)\, a_i\, x^i - \sum_{i=1}^{\infty} i a_i x^i + p^2 \sum_{i=0}^{\infty} a_i x^i = 0$$

$$2a_2 + 6a_3 x + \sum_{i=2}^{\infty} (i+2)(i+1) a_{i+2} x^i - \sum_{i=2}^{\infty} i(i-1) a_i x^i - a_1 x - \sum_{i=2}^{\infty} i a_i x^i + p^2 a_0 + p^2 a_1 x + p^2 \sum_{i=2}^{\infty} a_i x^i = 0$$

$$\left(2a_2 + 6a_3 x - a_1 x + p^2 a_0 + p^2 a_1 x\right) + \sum_{i=2}^{\infty} \left[(i+2)(i+1) a_{i+2} - i(i-1) a_i - i a_i + p^2 a_i\right] x^i = 0$$

$$\Rightarrow \quad 2a_2 + p^2 a_0 = 0 \quad \Rightarrow a_2 = -p^2 a_0 / 2$$

$$\Rightarrow \quad 6a_3 - a_1 + p^2 a_1 = 0 \quad \Rightarrow a_3 = \frac{(1-p^2) a_1}{6}$$

$$\Rightarrow (i+2)(i+1)\, a_{i+2} + [p^2 - i^2]\, a_i = 0 \quad \boxed{i \geq 2}$$

$$a_{i+2} = \frac{(i^2 - p^2)}{(i+2)(i+1)}\, a_i \quad \boxed{i \geq 2}$$

let $i = 2$

$$a_4 = \frac{(4-p^2)}{4 \cdot 3} a_2$$

$$= \frac{(4-p^2)(-p^2)\, a_0}{24}$$

etc.

$$y(x) = a_0 \left[1 - \frac{p^2}{2} x^2 + \frac{p^2(p^2-4)}{24} x^4 + \cdots \right]$$
$$+ a_1 \left[x + \frac{(1-p^2)}{6} x^3 + \cdots \right]$$

(c) Writing the DE in standard form

$$y'' \underbrace{- \frac{x}{1-x^2}}_{P(x)} y' + \underbrace{\frac{p^2}{1-x^2}}_{Q(x)} y = 0$$

$$P(x) = \frac{-x}{1-x^2} = -x \left[1 + x^2 + x^4 + \cdots \right]$$

for $x < 1$.

$P(x)$ & $Q(x)$ ~~cont~~ are analytic for $x < 1$. Therefore the minimum radius of convergence of the series expansion for $y(x)$ is $R = 1$.

4. ~~(a)~~

$$x^4 y'' + (2x^3 - 1) y'(x) + \frac{y}{x} = 0 \quad \rightarrow (b)$$

$x > 0$.

Now to find the behaviour of solns. to this DE as $x \to \infty$, we have to rewrite the DE as an equn. for a new variable $z = 1/x$.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz}\left(-\frac{1}{x^2}\right).$$

$$y'(x) = -z^2 \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-z^2 \frac{dy}{dz}\right) = -z^2 \frac{d}{dz}\left(-z^2 \frac{dy}{dz}\right).$$

$$y''(x) = z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz}$$

Substituting back in equn. (b),

$$\frac{1}{z^4}\left[z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz}\right] + \left(\frac{2}{z^3} - 1\right)\left(-z^2 \frac{dy}{dz}\right)$$

$$+ yz = 0$$

$$\frac{d^2y}{dz^2} + \frac{2}{z}\frac{dy}{dz} - \frac{2}{z}\frac{dy}{dz} + z^2 \frac{dy}{dz} + yz = 0$$

$$\frac{d^2y}{dz^2} + z^2\frac{dy}{dz} + zy = 0$$

in the new variable z, thus DE is of the form

$$\frac{d^2y}{dz^2} + P(z)\frac{dy}{dz} + Q(z)y = 0$$

where $P(z) = z^2$

& $Q(z) = z$

are analytic everywhere.

So we can look for a solution

$$y(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{around } z=0$$

$$y(x) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

$$\frac{dy}{dz} = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots$$

$$\frac{d^2y}{dz^2} = 2a_2 + 6a_3 z + 12a_4 z^2 + \cdots$$

Substituting back in DE,

$$(2a_2 + 6a_3 z + 12a_4 z^2 + \cdots) + z^2(a_1 + 2a_2 z + 3a_3 z^2 + \cdots)$$

$$+ z(a_0 + a_1 z + a_2 z^2 + \cdots) = 0$$

$$\Rightarrow 2a_2 + 6a_3 z + 12 a_4 z^2 + \cdots$$

$$+ z^2 a_1 + \cdots$$

$$+ a_0 z + a_1 z^2 + \cdots = 0$$

(explicitly writing only terms up to z^2)

$$\Rightarrow 2a_2 = 0 \quad \Rightarrow a_2 = 0$$

$$\Rightarrow a_0 + 6a_3 = 0 \quad \Rightarrow a_3 = -\frac{a_0}{6}$$

$$2a_1 + 12 a_4 = 0 \quad \Rightarrow a_4 = -\frac{a_1}{6} \text{ etc.}$$

$$y(z) = a_0 + a_1 z - \frac{a_0}{6} z^3 - \frac{a_1}{6} z^4 + \cdots$$

As $x \to \infty$, $z \to 0$

$y(x) \longrightarrow a_0$ (some finite constant).

Another solution to Q.1.

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$$y(x) = e^{-3x} + 2x e^{-3x}$$

Since the question only asks you to find a second order DE with this solution:

$$\text{So } y'(x) = -3e^{-3x} + 2e^{-3x} - 6x e^{-3x}$$

$$y''(x) = 9e^{-3x} - 6e^{-3x} - 6e^{-3x} + 18x e^{-3x}$$

$$= -3e^{-3x} + 18x e^{-3x}$$