PHY 4154

Nuclear and Particle Physics

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Klein Gordon equation

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We have $p_\mu = i\hbar\partial_\mu$, where $\partial_\mu = \frac{\partial}{\partial x_\mu}$. Explicitly, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x}$, and so on.

Klein Gordon equation

Take $p^\mu p_\mu - m^2c^2 = 0$ and put $p \to i\hbar\partial_\mu$

$$-\hbar^2\partial^\mu\partial_\mu - m^2c^2 = 0$$

$$-\partial^\mu\partial_\mu\psi = \left(\frac{mc}{\hbar^2}\right)\psi$$

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We can define the D'Alembertian operator $\partial^\mu\partial_\mu = \Box$, and along with $\hbar = c = 1$, we get a neat equation

$$(\Box + m^2)\psi = 0$$

$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$.

Problems with KG equation

- Second-order time derivative gives rise to negative energy plane wave solutions, and negative probabilities.
- Dirac tried to fix this by looking for equation that has first order derivatives in time.
- KG equation works for spin 0 particles, Dirac equation for spin-$\frac{1}{2}$ particles.

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$$\begin{aligned} p^\mu p_\mu - m^2c^2 &= (\beta^k p_k + mc)(\gamma^\lambda p_\lambda - mc) \\ &= \beta^k \gamma^\lambda p_k p_\lambda - mc(\beta^k - \gamma^k)p_k - m^2c^2 \end{aligned}$$

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$$\begin{aligned} (p^0)^2 - (p^1)^2 - \cdots = &(\gamma^0)^2(p^0)^2 - (\gamma^1)^2(p^1)^2 - (\gamma^2)^2(p^2)^2 - (\gamma^3)^2(p^3)^2 \\ &+ \gamma^0\gamma^1 p_0 p_1 + \gamma^0\gamma^2 p_0 p_2 + \gamma^0\gamma^3 p_0 p_3 \\ &+ \gamma^1\gamma^0 p_0 p_1 + \cdots \end{aligned}$$

We want to get rid of the cross terms, i.e. $(\gamma^0\gamma^1 + \gamma^1\gamma^0)p_0 p_1$.

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$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

This is the Clifford algebra and the γ's are matrices. The smallest set are 4×4 matrices.

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$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

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That is $\gamma^0\gamma^1 + \gamma^1\gamma^0 = 0$, $\gamma^0\gamma^0 = 1$, $\gamma^1\gamma^1 = \gamma^2\gamma^2 = \gamma^3\gamma^3 = -1$.

Define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, Note that $\{\gamma^\mu, \gamma^5\} = 0$.

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Conventionally we take $\gamma^\mu p_\mu - mc = 0$. Put in $p_\mu \to -i\hbar\partial_\mu$, and we get the Dirac equation

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Now this ψ is a four-element column vector, called Dirac spinor or bispinor.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

We write this as

$$\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

ψ_A and ψ_B are two component spinors representing electrons, positrons.

$$\psi = \begin{pmatrix} e^-\uparrow \\ e^-\downarrow \\ e^+\downarrow \\ e^+\uparrow \end{pmatrix}$$

Plane-wave solutions

Write plane-wave solutions to Dirac equation ($(i\gamma^\mu\partial_\mu - m)\,\psi = 0$) as

$$\psi(x) = ae^{-ip\cdot x}u(p)$$

Here x, p are 4-vectors and $u(p)$ is a bispinor, such that ψ satisfies Dirac equation.

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Putting it into Dirac equation gives

$$(\gamma^\mu p_\mu - m)u = 0$$

$$\gamma^\mu p_\mu = \gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} = \begin{pmatrix} p^0 & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -p^0 \end{pmatrix}$$

and

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

Plane-wave solutions

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

$$u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \qquad v^{(2)} = -N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

$\psi = ae^{-ip\cdot x}u$ (particles)
$$(\gamma^\mu p_\mu - m)u = 0$$

$\psi = ae^{ip\cdot x}v$ (antiparticles)
$$(\gamma^\mu p_\mu + m)v = 0$$

$$N \equiv \sqrt{E+m}$$

Adjoint spinor

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The quantity $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ is Lorentz invariant (i.e. a scalar).
(Note: $\bar{\psi}\psi = |\psi_1|^2+|\psi_2|^2-|\psi_3|^2-|\psi_4|^2$).

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We can thus write

$\bar{\psi}\psi$	scalar
$\bar{\psi}\gamma^5\psi$	pseudoscalar
$\bar{\psi}\gamma^\mu\psi$	vector (4-components)
$\bar{\psi}\gamma^\mu\gamma^5\psi$	pseudovector (4-components)

ψ transform

If ψ is a Dirac spinor, then if you go from a stationary frame to frame moving with speed v in x-direction

$$\psi \to \psi' = S\psi$$

where S is a 4×4 matrix

$$S = a_+ + a_- \gamma^0 \gamma^1$$

with

$$a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$$

Note that this last $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Lagrangian density

- In CM, we have $L(q_i, \dot{q}_i)$, and Euler Lagrange equations are

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  to get the EOM.
- Thus Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

  gives the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

- Notice the mass term.

Maxwell equations

- The 4-potential is $A^\mu = (\phi/c, \vec{A})$
- $F^{\mu\nu}$ is the field strength tensor.. explicitly

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_Z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

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- If we start with $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$, we get $\partial_\mu F^{\mu\nu} = j^\nu$.
- If we add to $\mathcal{L}$ a term like $\frac{1}{2}m^2 A_\mu A^\mu$, we get the KG equation $(\Box^2 + m^2)A^\mu = j^\mu$.
- Feynman rules follow from terms in the Lagrangian (quadratic in fields gives propagators, vertices from interactions).

Gauge invariance

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- Thus if $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$, this isn't invariant under local gauge transformation.

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$$\partial_\mu\psi = e^{i\theta(x)}\partial_\mu\psi + ie^{i\theta(x)}\psi\partial_\mu\theta$$
- To maintain invariance of the Lagrangian, what we need is some new covariant derivative such that $D_\mu\psi \to e^{i\theta(x)}D_\mu\psi$.

Gauge field

- Such a covariant derivative for QED is

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- The Lagrangian is now
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- Local phase invariance forced us to add vector gauge field coupling to particles.
- If A_μ is wavefunction of photon, need a term for its kinetic energy.
- Only way to add it is in terms of $F_{\mu\nu}$.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu \psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Gauge theories

Thus applying local phase invariance to Dirac Lagrangian generates all of ED.

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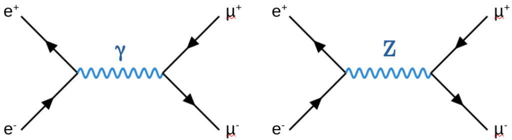
Thus applying local phase invariance to Dirac Lagrangian generates all of ED. The Feynman rules for QED look like this

- Notation (p's and q's)
- External lines: electrons get u (incoming) or $\bar{u}$ outgoing, positrons get $\bar{v}$ (incoming) or v (outgoing), and photons get a ϵ_μ (incoming), ϵ_μ^* (outgoing) ($A_\mu(x) = ae^{-(i/\hbar)p\cdot x}\epsilon_\mu^{(s)}$ is photon wave function)
- Each vertex gets $ig_e\gamma^\mu$
- Proagators are: electrons/positrons $\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$, and for photons $\frac{-ig_{\mu\nu}}{q^2}$
- Conservation of energy/momentum at each vertex
- Integrate over internal momenta
- Cancel the final δ function.

Gauge theories



$$\mathcal{M}_\gamma = -\frac{e^2}{k^2}(\bar{\mu}\ \gamma^\nu\ \mu)(\bar{e}\ \gamma_\nu\ e)$$

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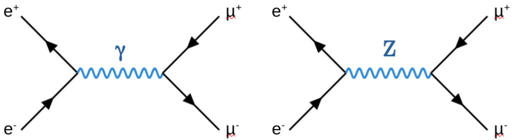

$$\mathcal{M}_\gamma = -\frac{e^2}{k^2}(\bar{\mu}\ \gamma^\nu\ \mu)(\bar{e}\ \gamma_\nu\ e)$$

$$\mathcal{M}_Z = -\frac{g^2}{4\cos^2\theta_W}\left[\bar{\mu}\ \gamma^\nu(c_V - c_A\ \gamma^5)\ \mu\right]\left(\frac{g_{\nu\sigma} - k_\nu k_\sigma/M_Z^2}{k^2 - M_Z^2}\right)\left[\bar{e}\ \gamma^\sigma(c_V - c_A\ \gamma^5)\ e\right]$$

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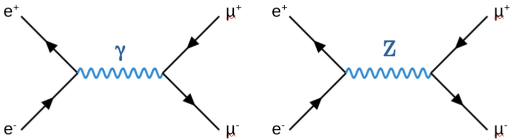

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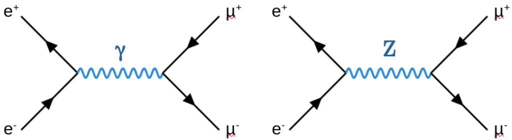
Here k is the four momentum of the virtual γ or Z $(s = k^2)$. The weak vertex factor is $-i\frac{g}{\cos\theta_W}\gamma^\mu\frac{1}{2}\left(c_V^f - c_A^f\gamma^5\right)$,

Gauge theories



$$\mathcal{M}_\gamma = -\frac{e^2}{k^2}(\bar{\mu}\ \gamma^\nu\ \mu)(\bar{e}\ \gamma_\nu\ e)$$

$$\mathcal{M}_Z = -\frac{g^2}{4\cos^2\theta_W}\left[\bar{\mu}\ \gamma^\nu(c_V - c_A\ \gamma^5)\ \mu\right]\left(\frac{g_{\nu\sigma} - k_\nu k_\sigma/M_Z^2}{k^2 - M_Z^2}\right)\left[\bar{e}\ \gamma^\sigma(c_V - c_A\ \gamma^5)\ e\right]$$

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$$A_\mu = B_\mu\cos\theta_W + W_\mu^3\sin\theta_W \qquad \text{(massless)}$$
$$Z_\mu = -B_\mu\sin\theta_W + W_\mu^3\cos\theta_W \qquad \text{(massive)}$$

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We can't add mass terms to Lagrangian like $M^2 W_\mu W^\mu$ because it makes the theory non-renormalizable. Thus while QCD is okay, for the weak interaction, to get massive bosons, we have to do something else.

Generating mass

- Consider $\mathcal{L} \equiv T - V = \frac{1}{2}(\partial_\mu \phi)^2 - (\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4)$

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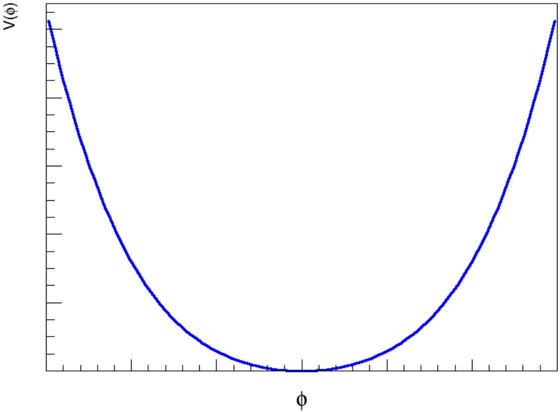

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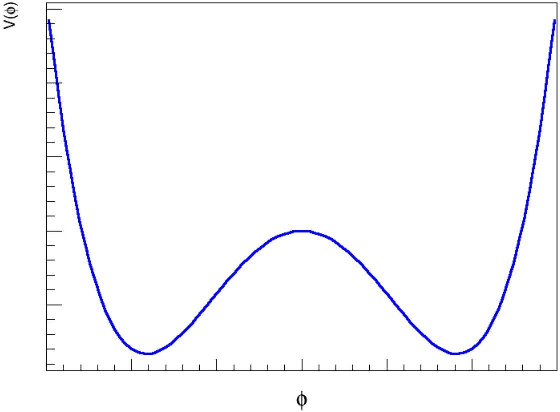

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- This mass was generated (or revealed) by spontaneous symmetry breaking.

SSB of Global symmetry

- Repeat for a complex scalar field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ with

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- SSB of global continous symmetry gives rise to one or more massless scalar spin-0 bosons called as Goldstone bosons.

Higgs mechanism

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- Unwanted massless Goldstone boson turned into longitudinal polarization of the massive gauge particles.
- Very approachable discussion in Halzen/Martin Chapter 13/14.