PHY 4154

Nuclear and Particle Physics

Sourabh Dube

Aug 2025

Nuclear Physics

- Nuclei arise when protons and neutrons bind to each other due to strong force.
- A nuclear species is characterized by total positive charge and the total number of mass units. The nuclear charge $= +Ze$, where Z is the atomic number. That is, the nucleus has Z protons.
- The mass number A of a nuclear species is the total number of nucleons. $A = Z$ protons $+(A - Z)$ neutrons. Sometimes N used for number of neutrons.

A specific nuclear species, or nuclide, is written as ${}^{A}_{Z}X_{N}$.
For example, ${}^{238}_{92}\mathrm{U}_{146}$ or ${}^{1}_{1}\mathrm{H}_{0}$, or ${}^{56}_{26}\mathrm{Fe}_{30}$.

But sometimes just ${}^{A}\mathrm{X}$, so ${}^{238}\mathrm{U}$, where U tells us that $Z = 92$, and thus $A - Z = N = 146$.

Terms

- Isotopes: Nuclides with same proton number, but different neutron numbers.
  For example, $^{35}\mathrm{Cl}$ and $^{37}\mathrm{Cl}$
- Isotones: Nuclides with same neutron number, but different proton numbers.
  For example, $^{2}\mathrm{H}$ and $^{3}\mathrm{He}$
- Isobars: Nuclides with same mass number A
  For example, $^{3}\mathrm{He}$ and $^{3}\mathrm{H}$

More than 5000 nuclides have been characterized, with about 251 stable nuclides.
Heaviest stable nucleus is $^{210}_{82}\mathrm{Pb}$.

Properties

The properties of nuclides that we would be interested in would be

Mass, Radius, Relative abundance (for stable nuclides), Decay modes and half life (for unstable nuclides), Reaction modes and cross sections, spin, magnetic moment, etc.
We are also interested in what makes a particular nuclide stable, and what are the patterns of stable nuclei.

Nuclear masses are measured in terms of unified atomic mass unit u. It is defined such that the mass of an atom of $^{12}\mathrm{C} = 12u$.

The nucleons are thus typically of mass $1u = 931.502 \ \ \mathrm{MeV}$.

Mass of proton $m_p = 938.27 \ \ \mathrm{MeV} = 1.00727 \ u$

Size

We need an operational definition of size. We can characterize nuclear shape with two parameters.

- mean radius: where density is half of central value
- skin thickness: density drops from near max to its min.

These depend on the experiment.

So we shall consider two things:
distribution of nuclear charge (which is probed using EM interaction), and
distribution of nuclear matter (probed using strong interaction)

Nuclear charge distribution

- We can scatter electrons elastically from a nuclear target - this acts like diffraction from a circular disk of diameter D, with first minimum at $\theta = \sin^{-1}(1.22\lambda/D)$. This roughly gives $^{16}\mathrm{O}$ as 2.6 fm, and $^{12}\mathrm{C}$ as 2.3 fm.
- We can do something qualitative: let the initial electron wavefn be $e^{ik_i r}$, and final electron be $e^{ik_f r}$. where $p = hk$.
- The interaction potential $V(r)$ says that probability of transition from initial i to final f is proportional to

$$F(k_i, k_f) = \int \psi_f^* V(r) \psi_i dV$$

- Rewrite with $q = k_f - k_i$, and using $V(r)$ as EM interaction (nuclear charge density $Ze\rho_e(r')$) gives us

$$F(q) = \int e^{iqr'} \rho_e(r') dv'$$

Nuclear charge distribution

Here $F(q)$ is known as the form factor for the nucleus.

For radial $\rho_e(r')$, we get

$$F(q) = \frac{4\pi}{q} \int \sin qr' \rho_e(r') r' dr'$$

We measure scattering prob. between p_i and p_f as function of scattering angle, and for various nuclei.

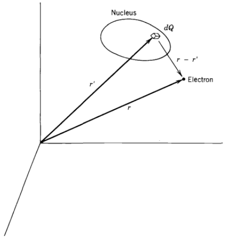


Figure 3.3 The geometry of scattering experiments. The origin of coordinates is located arbitrarily. The vector ***r***' locates an element of charge *dQ* within the nucleus, and the vector ***r*** defines the position of the electron.

We see that number of nucleons per unit volume is roughly constant. This gives us a relatively simple empirical formula for size

$$R(A) \sim 1.2 A^{\frac{1}{3}} \text{ fermi}$$

The skin thickness is typically 2.3 fm.

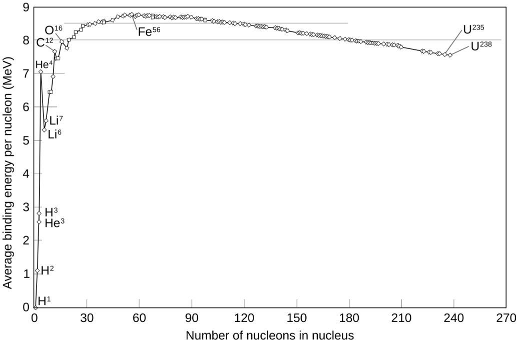
Nuclear matter distribution

- Scatter an α-particle ($^4\mathrm{He}$) from say $^{197}\mathrm{Au}$. Low energies $\rightarrow$ Coulomb force, (Rutherford scattering). At higher energies, Coulomb repulsion is overcome and nuclear forces act.
- Radioactive decay (α-particle emission). The α must escape nuclear potential and penetrate a Coulomb barrier. α-decay probabilities calculated from standard barrier-penetration approach depend on nuclear matter radius R.
- π-mesic X rays: Fire pions at the nucleus. The π meson wavefn overlaps with nucleus, and shifts energy levels from values calculated using Coulomb interaction. Pions can also be absorbed into the nucleus, and one can study "disappearance rate"

Turns out that charge and matter radii are nearly equal to within 0.1 fm. Inside the nucleus, neutrons and protons are mixed well.

Binding energy

- Given A and Z, can we derive the mass of a nuclide?
- Experimentally, we find mass of nucleus $<$ sum of its constituents. Thus B.E is negative.
- $B(A, Z) = M(A, Z) - Zm_p - (A - Z)m_n$
- Then average binding energy per nucleon is $|B|/A$



Some observations

- $|B|/A$ is about 1 $\mathrm{MeV}$for ${}^2_1\mathrm{H}$, but rises and is later constant around 8 MeV
- Peaks at about 9 $\mathrm{MeV}$for ${}^{56}_{26}\mathrm{Fe}$, ${}^{62}\mathrm{Ni}$ (very abundant in planetary cores)
- $|B|/A$ curve drops at small A: for small nuclei, greater number of nucleons at surface, and are not surrounded by other nucleons.
- $|B|/A$ drops at large A: Coulomb effect depens on no. of proton pairs (Z^2 increases faster than A and Coulomb repulsion makes these nuclei less tightly bound.)

Isotope vs Half-life

$$N(t) = N_0 e^{-t/\tau}$$

where N_0 is the number of initial objects, and $N(t)$ is the number surviving after time t, and τ is the mean lifetime.

The half-life is the time $t_{1/2}$ such that $N(t_{1/2}) = \frac{1}{2}N_0$.

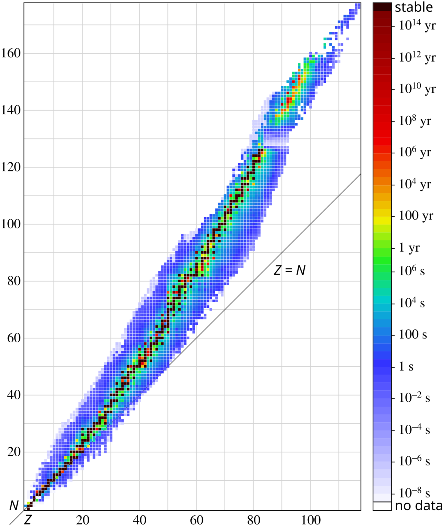
This gives us $t_{1/2} = \tau \ln 2$.

Deuteron ${}^2_1\mathrm{H}$ is stable, but Tritium nucleus ${}^3_1\mathrm{H}$ has a half-life of 12.32 years with the decay mode being

$$ {}^3_1\mathrm{H} \to {}^3_2\mathrm{He} + e^- + \bar{\nu}_e \tag{1}$$

Why 12.32 years? Neutron lifetime is minutes...
Here $\Delta M = 18.6\ \ \mathrm{keV}$ (but $m_n - m_p = 1.29\ \ \mathrm{MeV}$)

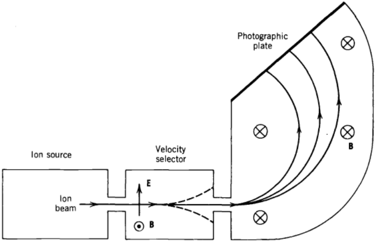
Isotope vs Half-life



Measuring mass

Measuring mass of a nuclide is tied into measuring its binding energy. A Mass spectrometer is an important technique.

Ion source: produce ionized atoms/molecules.
Velocity selector: crossed $\vec{E}$ and $\vec{B}$ fields.
Momentum selector: just a $\vec{B}$ field.



$\vec{E}$ has force upward, and $\vec{B}$ has force downward. Setting the two fields such that the forces are balanced gives $qE = qvB$, and thus ions of velocity $v = E/B$ pass undeflected.
In the momentum selector, $mv = qBr$. This gives

$$m = \frac{qrB^2}{E}$$

Mass doublet method

Measuring all terms in $m = qrB^2/E$ to high precision is difficult for different ions. Suppose we want to measure $^{1}\mathrm{H}$ in terms of u, and then measure $^{14}\mathrm{N}$. Resetting apparatus can be hard. Or finding the right ions.

- Fix apparatus for mass 128, and measure difference between C_9H_{20} (nonane) and $C_{10}H_8$ (napthalene). We get $\Delta = 0.09390032 \pm 0.00000012$ u
- Ignoring molecular binding energy ($\sim 10^{-9}$u), we can write

$$\Delta = m(C_9H_{20}) - m(C_{10}H_8) = 12m(^{1}\mathrm{H}) - m(^{12}\mathrm{C})$$

Thus

$$m(^{1}\mathrm{H}) = \frac{1}{12}\left[m(^{12}\mathrm{C}) + \Delta\right] = 1.000000 + \frac{1}{12}\Delta$$
$$= 1.00782503 \pm 0.00000001u$$

Mass doublet method

- Now set apparatus for mass 28, and measure difference between C_2H_4 and N_2. Obtain an expression for $m(N_2)$ in terms of known masses ($^1\mathrm{H}$ etc.).
  ($\Delta = m(C_2H_4) - m(N_2)$)

$$\begin{aligned}\Delta &= m(C_2H_4) - m(N_2) \\ &= 2m(^{12}\mathrm{C}) + 4m(^{1}\mathrm{H}) - 2m(^{14}\mathrm{N}) \\ &= 0.025152196 \pm 0.000000030u\end{aligned}$$

$$\begin{aligned}m(^{14}\mathrm{N}) &= m(^{12}\mathrm{C}) + 2m(^{1}\mathrm{H}) - \frac{1}{2}\Delta \\ &= 14.00307396 \pm 0.00000002u\end{aligned}$$

Mass spectrometer also allows us to measure relative abundances by measuring the ion current for different masses.
One can also use it to collect a large quantity of a particular isotope by setting for a single mass.

Nuclear spins

- Nuclei are also classified according to whether the Z and N are even or odd.
- Nuclei are classified as even-even (both Z, N even) or odd-odd (both Z, N odd), or even-odd, odd-even.
- In even-even or odd-odd cases, A is even and thus spin of the nucleus is integer.
- In odd-even or even-odd, A is odd, and nucleus has half-integer spin.

For example, ${}^4_2\mathrm{He}$ is a boson, while ${}^3_2\mathrm{He}$ is a fermion.
${}^4_2\mathrm{He}$ exhibits Bose-Einstein condensation and thus superfluidity when cooled to low temperature ($\sim$ 2.2K). ${}^3_2\mathrm{He}$ has to be cooled further ($\sim$ 2.5mK), where first bosonic Cooper pairs are formed before condensing.

Nuclear spins

- In principle, a nucleus with 100 or 200 nucleons can have a spin as high as $100\hbar$.

  In fact, all even-even nuclei have spin 0. Thus half spins are paired with other half!

- The range of spins can be deduced by measuring magnetic dipole moments (μ).

- We have for a current loop due to a single particle of charge e and mass m with radius r, $\mu = \frac{e}{2m}L$. QM says $\mu = \frac{e\hbar}{2m}\ell$.

- For fundamental particles, in analogy we write $\mu = g\frac{e\hbar}{2m}s = \frac{g}{2}\mu_B$. Here g is a dimensionless number called g-factor, and μ_B is the Bohr magneton.

- The classical Dirac equation allows us to predict $g = 2$ which agrees very well with experimental value. Additional quantum corrections to the Dirac equation using QED allow us to reproduce the experimentally measured value $g = 2.002319$.

Nuclear spins

- For nuclei we define the nuclear magneton $\mu_N = \frac{e_p \hbar}{2m_p}$ and we get $\mu_p = 2.7928456\mu_N$ and $\mu_n = -1.9130419\mu_N$ for the proton and neutron.
- The g-factors for them are $g_p = 5.5856912 \pm 0.0000022$ and $g_N = -3.8260837 \pm 0.0000018$. These are considerably different from 2! This can't be just quantum corrections.. in some sense this is evidence that the proton/neutron are composite particles.
- Nuclear magnetic moments are much smaller than electron.. thus at atomic physics scale one is dominated by electron. Only completely ionized atoms allow us to see the nuclear magnetic moment.
- Experimentally, nuclear magnetic moments range from $-3\mu_N$ to $+10\mu_N$. This is very different from naively expecting $200\mu_N$, and thus large cancellations must be taking place.
- Note that all even-even nuclei are spinless and thus have no magnetic moment.

Nuclear stability

- For $Z < 40$, we see that $N \approx Z$. For $^{56}_{26}\mathrm{Fe}$, $N = 1.15Z$. For $Z > 40$, the number of neutrons rises. For $^{222}_{86}\mathrm{Rn}$, $N = 1.58Z$.
- As number of protons increases, protons repel each other.. so we need a proportionately larger number of neutrons to "dilute" the protons.
- There are 156 stable even-even nuclei, only 5 stable odd-odd, and about 50 each of odd-even, and even-odd. This suggests there is some pairing mechanism at work.

Recall that we saw strong force is same whether its *pp* or *pn* or *nn*. We also saw there are no *nn* bound states (motivated this using isospin). We saw that only singlet state existed ($|0, 0\rangle = \left(\frac{1}{\sqrt{2}}\right)(pn - np)$)

Nuclear Stability

- Consider deuteron (pn). Its spin S can be ? $1, 0$.
- We observe is that deuteron has $s = 1$ and $\ell = 0$ (or 2). There is no spinless deuteron.
- Binding tighter for aligned spins ($\uparrow\uparrow, \downarrow\downarrow$) rather than anti-aligned ($\uparrow\downarrow$).
- Thus pair of neutrons must also need aligned spins to make bound state.
- This violates Pauli's exclusion principle (wave-function antisymmetric under exchange for two-fermion system).
- Similarly no pp nuclei.
- Even-even are remarkably stable, consider ${}^4_2\mathrm{He}$. Here the two neutrons are $\uparrow\downarrow$, two protons are $\uparrow\downarrow$ and these combine. 'Pairing' accounts for spinless nature of all even-even nuclei.

Nuclear Instability

- Unstable nuclei emit particles and aim to get stable: Radioactivity
- Typical well known types of radiation: α, β, γ.
- α-decay is a form of nuclear fission

$$
\begin{gathered}
{}^{A}_{Z}\mathrm{X} \rightarrow {}^{A-4}_{Z-2}\mathrm{X}' + {}^{4}_{2}\mathrm{He} \\
{}^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}
\end{gathered}
$$

- Typical KE of α particles is 5 MeV.
- α-decay is a tunnelling of a part of the nucleus out of the potential barrier of the rest of the nucleus. It is a strong interaction.

Nuclear Instability

- Already seen β decay ($n \to p + e^- + \bar{\nu}_e$).
- Nuclear process is

$$ {}^{A}_{Z}\mathrm{X} \to {}^{A}_{Z+1}\mathrm{X}' + e^- + \bar{\nu}_e \ ; \quad {}^{14}_{6}\mathrm{C} \to {}^{14}_{7}\mathrm{N} $$

- Positron β decay

$$ {}^{A}_{Z}\mathrm{X} \to {}^{A}_{Z-1}\mathrm{X}' + e^+ + \nu_e \ ; \quad {}^{23}_{12}\mathrm{Mg} \to {}^{23}_{11}\mathrm{Na} $$

- "Electron-capture" (a nuclear reaction appearing as decay)

$$ {}^{A}_{Z}\mathrm{X} + e^- \to {}^{A}_{Z-1}\mathrm{X}' + \nu_e $$
$$ {}^{81}_{36}\mathrm{Kr} + e^- \to {}^{81}_{35}\mathrm{Br} + \nu_e $$

- Typical energy release is of order 1 MeV.

All possibilities depend on the relative B.E. : need at least 0.511 $\mathrm{MeV} + m_\nu$. Often β-decay will follow from daughter of α-decay.

Nuclear Instability

- γ-radiation is emission of photon from excited nucleus.
- EM process, energy range from 100 keV to 10 MeV.
- can happen soon after an α- or β-decay.
- We also have induced nuclear fission (α is spontaneous)

$$^{235}_{92}\mathrm{U} + n \to {}^{89}_{36}\mathrm{Kr} + {}^{144}_{56}\mathrm{Ba} + 3n$$

- This releases ~ 177 MeV (large energy), and can trigger a chain reaction. (The $3n$ carry kinetic energy and further induce fission in a sufficiently dense sample.

Mass models

- Nuclear mass models are typically empirical or semi-emipirical.
- We shall examine three of them: Liquid drop model, Fermi gas model, and Shell model
- Note that mass defect

$$\Delta M(A, Z) = M(A, Z) - Zm_p - (A - Z)m_n$$

  Binding energy per nucleon

$$-\frac{B}{A} = -\frac{\Delta M(A, Z)\ c^2}{A}$$

Liquid drop model

Observations:

- We see $|B|/A$ is nearly constant in range $12 < A < 240$ and that size $R \sim A^{\frac{1}{3}}$
- The constancy of $|B|/A$ is a hint: if nuclear force due to A particles interacting pairwise, then total energy of collection of A objects would scale as $\sim A^2$ at large A.
  Thus $|B|/A$ would grow linearly with A (energy would scale as A^2).
- But nuclear force is short-range: objects interact with only some nearest neighbors - the force is "saturated"
- Let us treat the nucleus as incompressible liquid droplet of $R \sim A^{\frac{1}{3}}$.
- $|B|/A$ peaks around 60. For $A < 60$, its favorable to assemble two nuclei into larger one (fusion). For $A > 60$, its favorable for nucleus to split (fission).

Liquid drop model

Broadly we have

- More number of nucleons, more binding energy.
- More surface, less binding energy.
  (less nearest neighbours)
- Higher Z, less binding energy.
  (repulsion between protons)
- If $N \neq Z$, less binding energy.
- Then we have even-even, odd-odd, and others.

Liquid drop model

Broadly we have

- More number of nucleons, more binding energy. Volume term.
- More surface, less binding energy. Surface term.
  (less nearest neighbours)
- Higher Z, less binding energy. Coulomb term.
  (repulsion between protons)
- If $N \neq Z$, less binding energy. Symmetry term.
- Then we have even-even, odd-odd, and others.

Liquid drop model

- First term is just one for the saturated force. $B = -a_v A$
- Now we account for decrease of $|B|/A$ for small A. Let us say that there is an outer layer which is less tightly bound - this term will be important at small A since surface to volume ratio will be large. This term scales as R^2, thus $A^{\frac{2}{3}}$

  $$B = -a_v A + a_s A^{\frac{2}{3}}$$

  This term reduces the B.
- Now the slow decrease of $|B|/A$ at large A due to Coulomb repulsion. This scales as $Z(Z-1)/R$, so

  $$B = -a_v A + a_s A^{\frac{2}{3}} + a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}}$$

This is the form of the energy for a liquid drop. To go further we need to incorporate other effects.

Liquid drop model

- We know that for small A we have $A = 2Z$, and at large A, we have $A \sim 2.5Z$. We want to put in a term that reduces B when $A \neq 2Z$.
- So we have symmetry term

$$B = -a_v A + a_s A^{\frac{2}{3}} + a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} + a_{sym} \frac{(A-2Z)^2}{A}$$

Now we need to add one last term to prefer even-even (more stable) and penalize odd-odd (less stable)

Liquid drop model

Bethe-von Weizsäcker semi-empirical binding energy formula

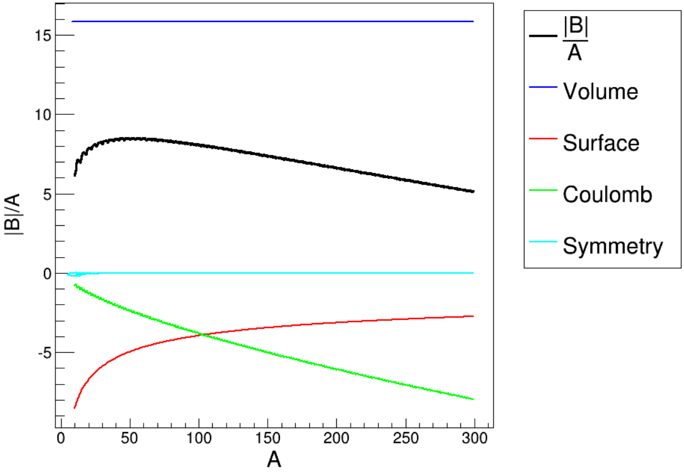
$$B = -a_v A + a_s A^{\frac{2}{3}} + a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} + a_{sym} \frac{(A-2Z)^2}{A} + \frac{a_p}{A^{\frac{1}{2}}}$$

where experimentally we have

$$\begin{aligned} a_v &= 15.8 \ \ \mathrm{MeV}, \qquad a_s = 18.3 \ \ \mathrm{MeV}, \qquad a_c = 0.714 \ \ \mathrm{MeV} \\ a_{sym} &= 23.2 \ \ \mathrm{MeV}, \qquad a_p = \pm 12 \text{ or } 0 \ \ \mathrm{MeV} \end{aligned}$$

where for a_p, the $+$ is for odd-odd (less stable) and $-$ for even-even (less stable) and 0 for others.

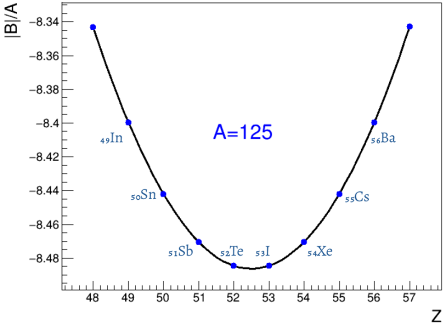
Liquid-drop model



Liquid-drop model

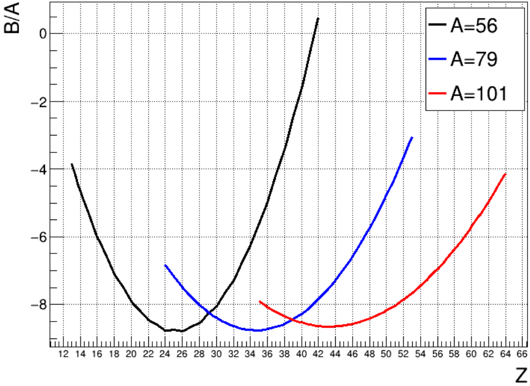
β-stability valley: Given A, how does B vary as function of Z?

${}_{49}\mathrm{In}$: β^--decay to ${}_{50}\mathrm{Sn}$
${}_{56}\mathrm{Ba}$: β^+-decay to ${}_{55}\mathrm{Cs}$
${}_{52}\mathrm{Te}$: stable.
${}_{53}\mathrm{I}$: Electron capture to excited ${}_{52}\mathrm{Te}$.



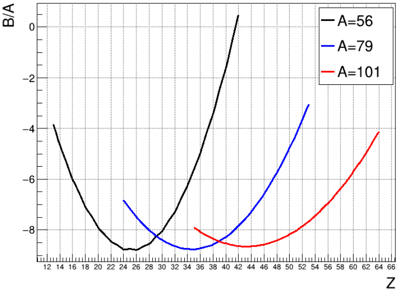
Liquid-drop model

β-stability valley: Given A, how does B vary as function of Z?



Liquid-drop model

β-stability valley: Given A, how does B vary as function of Z?



$^{56}_{26}\mathrm{Fe}$: very stable!
$^{79}_{35}\mathrm{Br}$ stable. $^{79}_{34}\mathrm{Se} \to {}^{79}_{35}\mathrm{Br} + e^- + \bar{\nu}_e$
$^{101}_{44}\mathrm{Ru}$ stable.
$^{101}_{43}\mathrm{Tc}$ has β^- decay to $^{101}\mathrm{Ru}$.
$^{101}_{42}\mathrm{Mo} \to {}^{101}_{43}\mathrm{Tc} \to {}^{101}_{44}\mathrm{Ru}$
$^{101}_{45}\mathrm{Rh}$ undergoes EC to $^{101}\mathrm{Ru}$.
$^{101}_{46}\mathrm{Pd}$ has β^+ decay to $^{101}\mathrm{Rh}$

Liquid-drop model

- β-stability valley: Given A, how does B vary as function of Z?
- Add mass of constitutents $M_0 = (A - Z)m_n + Z(m_p + m_e)$.

Liquid-drop model

- β-stability valley: Given A, how does B vary as function of Z?
- Add mass of constitutents $M_0 = (A - Z)m_n + Z(m_p + m_e)$.
- Then mass of bound state (for given A) is
  $M = M_{constituents} - B$

Liquid-drop model

- β-stability valley: Given A, how does B vary as function of Z?
- Add mass of constitutents $M_0 = (A - Z)m_n + Z(m_p + m_e)$.

- Then mass of bound state (for given A) is

$$
\begin{aligned}
M &= [Am_n - Z(m_n - m_p - m_e)] \\
&- a_v A + a_s A^{\frac{2}{3}} + a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} + a_{sym} \frac{A^2 - 4AZ + 4Z^2}{A} + \frac{a_p}{A^{\frac{1}{2}}} \\
&= \alpha - \beta Z + \gamma Z^2
\end{aligned}
$$

where α, β, γ are coefficients.

Liquid-drop model

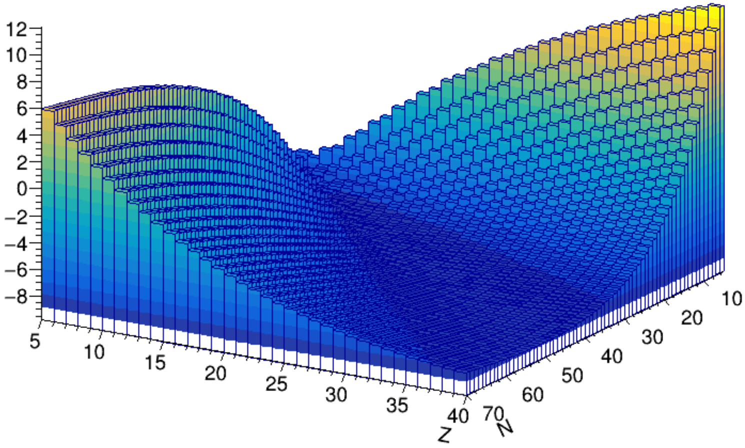
- Parabola with minimum at $\frac{\partial M}{\partial Z} = 0$, i.e. $Z_{min} = \beta/2\gamma$.
- Thus Z_{min} is integer closest to $\beta/2\gamma$.
- Say A is odd (odd-even/even-odd nuclei). β-decay takes one to the other. Here a_p=0 and

$$Z_{min} = \frac{(4a_{sym} + m_n - m_p - m_e)A}{2(4a_{sym} + a_c A^{\frac{2}{3}})}$$

- This gives $Z_{min} \leq \dfrac{A}{2}$, ie. $N \geq Z$
- Given Z nucleus, β-decay if $Z + 1$ is closer to Z_{min}.
- If $Z > Z_{min}$, then nucleus can undergo positron emission, or electron capture to go to $Z - 1$.
- Those nuclei whose Z (given A) are closest to Z_{min} will be stable to β-decay, forming what is called β-stability valley on plot of M vs Z.

Liquid-drop model

β-stability valley: $|B|/A$ as function of N and Z



Liquid-drop model

- Similarly determine stability under α-decay,

$$|B(A,Z)| < |B(A-4,Z-2)| + 28.3 \ \ \mathrm{MeV}$$

- Nuclei with $A > 165$ are unstable to α-decay, but half-lives are very long.
  Consider stable up to $^{209}_{83}\mathrm{Bi}$.
- At larger A, either α-decay or fission or both are energetically favourable. This is why at some point, the periodic table of elements comes to an end!

Fermi gas model

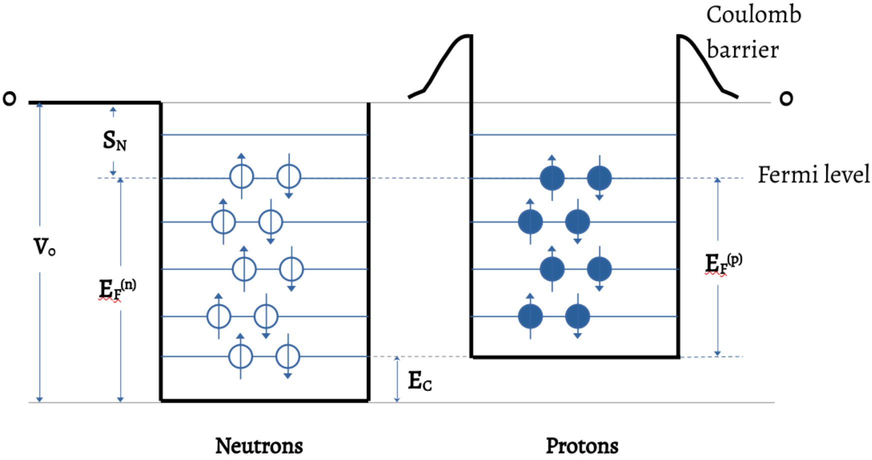
Liquid drop model allowed us to derive binding energies, but told us nothing about dynamics, spin-alignment etc. We expect this to be quantum mechanical in nature.

Let's now assume the opposite - let the nucleons be a QM system, where each nucleon moves independently in the average potential because of the other nucleons.

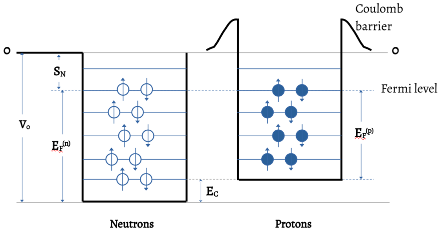
We can consider several potentials, let us take the square well type.

Fermi gas model

These wells are radial. Nucleus is like a box enclosed in a hard wall that requires finite energy to overcome.

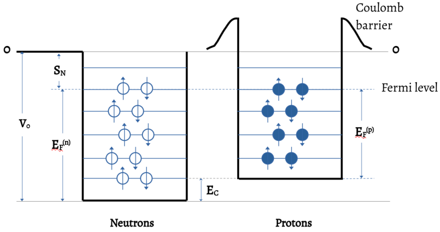


Fermi gas model



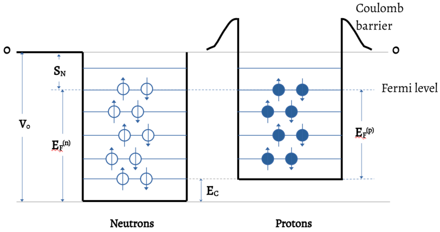
- Neutron well has depth V_0 (determined from experiment), and corresponding discrete energy levels.
- Each level can have two neutrons with anti-aligned spins.
- Level at the last filled level is called the "Fermi energy" $E_F^{(n)}$
- Once levels are filled upto $E_F^{(n)}$, we need additional S_n energy to liberate a neutron.
- Thus $E_F^{(n)} = V_0 - S_n$

Fermi gas model



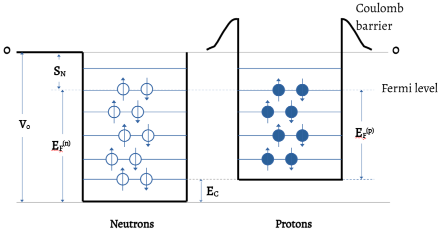
- S_n is called the neutron separation energy.
- $S_n(N, Z) = |B(N, Z) - |B(N-1), Z|$
- Typical value of S_n is about 8 MeV.

Fermi gas model



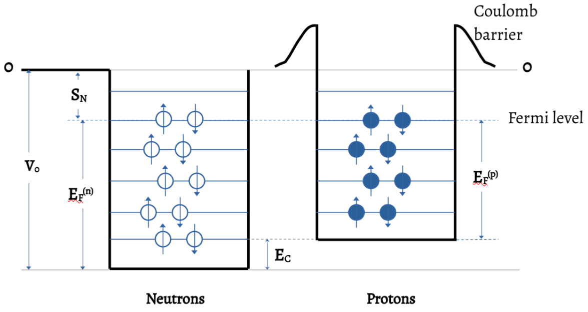
- Proton well is similar except for couple of differences
- It has a Coulomb barrier at top.
- Bottom of the proton well is at a higher level (a smaller number of bound states for protons is observed)
- Both wells are finite (finite number of bound states), and the proton well is higher by E_c

Fermi gas model



- Thus $E_F^{(p)} + E_C = V_0 - S_p$
- $S_p(N, Z) = |B(N, Z)| - |B(N, Z - 1)|$
- S_p is the proton separation energy. We expect S_p to be slightly smaller than S_n due to Coulomb repulsion
- The wells are filled up independently by nucleons (At most two for each level).
- For example for $A = 4$, we may have four neutrons (two levels filled) or two protons and two neutrons.

Fermi gas model



- Typically Fermi energies of the two wells are independent.
- But for β-stable nuclei, we want to keep them the same. (Why?)
- Otherwise one nucleon would decay to other (via β^--decay of β^+-decay)

Fermi gas model

To do some quantitative stuff, let us at the moment consider both wells to be identical. Consider the neutron well. Let p_{max} be the maximum momentum in the well.

$$\frac{p_{max}^2}{2m_n} = E_F^{(n)} \quad \mathrm{p_{max}} \text{ is called Fermi momentum}$$

The total number of particles is given by

$$n = \frac{1}{(2\pi\hbar)^3} \int d^3x \; d^3p$$

The $\int d^3x$ is the volume, and

$$\int_0^{p_{max}} d^3p = \int_0^{p_{max}} 4\pi |p|^2 d|p| = \frac{4}{3}\pi p_{max}^3$$

Thus

$$n = \frac{V p_{max}^3}{6\pi^2\hbar^3}$$

Fermi gas model

- The maximum momentum is given by $p^2/2m = E_F$
- The total number of nucleons is $2n$.
- Thus total number of neutrons N is given by

$$N = \frac{V p_N^3}{3\pi^2\hbar^3}$$

where V is the volume. $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A$, and we get

$$p_N = \frac{\hbar}{R_0}\left(\frac{9\pi N}{4A}\right)^{1/3}$$

Similarly

$$p_Z = \frac{\hbar}{R_0}\left(\frac{9\pi Z}{4A}\right)^{1/3}$$

Recall $R_0 \sim 1.2$ fm

Fermi gas model

- Consider nuclei with $N = Z = A/2$. Then

$$p_N = \frac{\hbar}{R_0}\left(\frac{9\pi}{8}\right)^{1/3}$$

giving $E_F \sim 40\ \ \mathrm{MeV}$.

- The average KE per nucleon is

$$\langle E\rangle = \frac{\int_0^{p_f} E\ d^3p}{\int_0^{p_f} d^3p} = \frac{3}{5}\frac{p_F^2}{2m} \sim 24\ \ \mathrm{MeV}$$

- Both are much less than $\Lambda_{QCD} = 220\ \ \mathrm{MeV}$.

Fermi gas model

- The average KE for a nucleus is

$$\langle E(N,Z)\rangle = N\langle E_N\rangle + Z\langle E_Z\rangle = \frac{3}{5}\left(N\frac{p_N^2}{2m} + Z\frac{p_Z^2}{2m}\right)$$

- Assuming equal masses for p and n and equal radii for both wells

$$\langle E(N,Z)\rangle = \frac{3}{10m}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}+Z^{5/3}}{A^{2/3}}$$

- For given A, this has minimum for $N = Z = A/2$.

Fermi gas model

- Let us study behavior around this minimum. Set $Z - N = \epsilon$, with $Z + N = A$. Thus

$$Z = \frac{1}{2}A(1 + \frac{\epsilon}{A}) \quad N = \frac{1}{2}A(1 - \frac{\epsilon}{A})$$

  with $\epsilon/A \ll 1$. Using

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ...$$

  we get

$$\langle E(Z, N) \rangle = \frac{3}{10m}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(A + \frac{5}{9}\frac{(Z-N)^2}{A} + ...\right)$$

- We get the volume and symmetry terms

Fermi gas model

- Consider the coefficient of the symmetry term

$$\frac{1}{6}\left(\frac{9\pi}{8}\right)^{2/3}\frac{\hbar^2}{mR_0^2}\frac{(Z-N)^2}{A} \approx 11 \ \ \mathrm{MeV} \times \frac{(Z-N)^2}{A}$$

- This is only about half as much as the coefficient we have seen earlier ($a_{sym} = 23.2 \ \ \mathrm{MeV}$)
- For the rest, assume that well depth V_0 has additional term $\propto (Z-N)/A$, with coefficient $\sim 30 \ \ \mathrm{MeV}$.
- One can also then include effect of Coulomb interactions (proton well is a bit shallower, changing $E_F^{(p)}$ and so on.
- In this model, the notion of excited nuclei also makes sense.

Shell model

It is observed that nuclei with N, or Z or both with

$$2, 8, 20, 28, 50, 82, 126$$

are especially stable.

- From considering the S_n or S_p - the separation energies (the gap between the Fermi energy and the continuum energy).

Shell model

For example consider S_n for isotopes of ${}_{58}\mathrm{Ce}$ (Cerium)

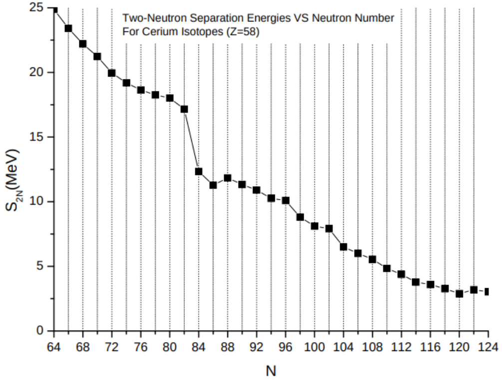
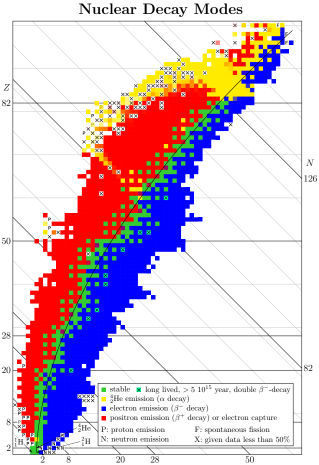
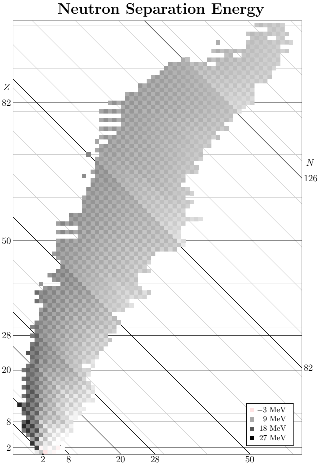


FIG. 6: Two-neutron separation energies (S_{2N}) along the cerium isotopic chain. This quantity is defined as $S_{2N}(A, Z, N) = Bind(A, Z, N) - Bind(A-2, Z, N-2)$ where the binding energy $Bind(A, Z, N)$ is given by Eq. (1). Note that in our approache the neutron drip line (where $S_{2N} \approx 0$) can be extrapolated around $N = 128$ for Cerium isotopes.

(Progress in Physics, vol. 11 (2015), issue 3 (July), arXiv:1504.07726 [nucl-th])

Shell model

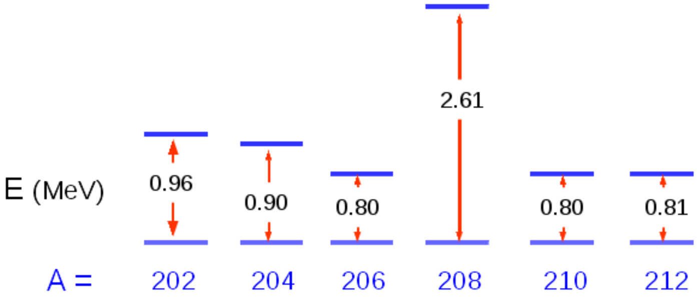




(QM for Engineers, Prof. Dommelen)

Shell model

Can also test this idea by considering excited states of nuclei, for example the doubly magic $^{208}_{82}\mathrm{Pb}$, with $Z = 82, N = 126$.



Excitation energies for different isotopes of lead with even-even nuclei.

Shell model

- In Fermi gas model, we didn't do details of the spectrum in the neutron/proton well.
- Consider Schrodinger equation for finite potential well in 3d.
- Assume nucleons orbit in some common potential ($H\psi = E\psi$). Assume a central potential.
- Non-relativistic Schrodinger equation

$$\left[\vec{\nabla}^2 + \frac{2m}{\hbar^2}\left(E - V(\vec{r})\right)\right]\psi(\vec{r}) = 0$$

- For central potential, $[H, J] = 0$.
- Energy eigenstates are angular momentum eigenstates,
- Energy states can be labelled with angular momentum quantum numbers.

Shell model

In hydrogen atom, $\vec{J} = \vec{L} + \vec{s}$, and we use the notation

$$n\ L_J^P$$

The spectroscopic notation is

l	0	1	2	3	4	5
	S	P	D	F	G	H

where degeneracies are given by $2(2l + 1)$.

Shell model

A finite potential well like Fermi gas model is hard. So let us consider an infinite well.

$$\psi(r,\theta,\phi) = u_{n,\ell}(r) Y_{\ell,m}(\theta,\phi)$$

Here the radial part solves the free Schrodinger equation. For neutrons

$$\frac{-\hbar^2}{2m_n}\frac{1}{r}\frac{d^2}{dr^2}(ru_{n,\ell}(r)) + \frac{\hbar}{2m_n}\frac{\ell(\ell+1)}{r^2}u_{n,\ell}(r) = Eu_{n,\ell}(r)$$

The boundary conditons are $u_{n,\ell}(r=0)=$finite, and $u_{n,\ell}(r=R)=0$ where R is boundary of nucleus.

Solutions for $\ell = 0$ are easiest. We find that $ru_{n,\ell=0}(r)$ is a linear combination of $\cos kr$ and $\sin kr$. Given the boundary conditions, only $\sin kr$ is allowed, and $k = n\pi/R$

$$\therefore u_{n,\ell=0}(r) = \frac{\sin kr}{kr} \quad \text{with} \quad E_{n,\ell=0} = \frac{\hbar^2}{2m_n}\left(\frac{n\pi}{R}\right)^2$$

Shell model

- Generally for $\ell \neq 0$, solutions are spherical Bessel fn $j_\ell(kr)$.
- Finite at origin, and vanish for special values of argument $x_{n,\ell}$.
- Allowed values of k are $k_{n,\ell} = x_{n,\ell}/R$. Energy levels are

$$E_{n,\ell} = \frac{\hbar^2}{2m_n}\left(\frac{x_{n,\ell}}{R}\right)^2$$

.

- Energy eigenvalue is n^{th} zero of ℓ^{th} Bessel function.

The first few ordered values of $x_{n,\ell}$ are

$$x_{1,0} < x_{1,1} < x_{1,2} < x_{2,0} < x_{1,3} < x_{2,1} < x_{1,4} < x_{2,2} < x_{1,5} < x_{3,0} < \ldots$$

Shell model

- For each n, ℓ combination, the quantum number m takes $2\ell + 1$ values.

  Each state is occupied by two particles with spin $\uparrow$ and spin $\downarrow$
- Thus the degeneracy of each state is $2(2\ell + 1)$.
- We can count also as total number of states lower than given energy...

	1S	1P	1D	2S	1F	2P	1G	2D	1H	3S
(n, ℓ)	$(1, 0)$	$(1, 1)$	$(1, 2)$	$(2, 0)$	$(1, 3)$	$(2, 1)$	$(1, 4)$	$(2, 2)$	$(1, 5)$	$(3, 0)$
degeneracy	2	6	10	2	14	6	18	10	22	2
total	2	8	18	20	34	40	58	68	90	92

Shell model

- Got magic numbers $2, 8, 20$, but did not get $28, 50, 82$.
- Also got other numbers, which are not magic $(18, 34, 40, \ldots)$
- We can take one more step (?)

The key point is to introduce spin-orbit coupling. This will split these levels, and perhaps reproduce the magic numbers.

Shell model

- Introduce to Hamiltonian $U_{s-o}(r)\vec{L}\cdot\vec{s}$.
- Single-particle Hamiltonian, r is distance of nucleon from center, $\vec{L}$ is orbital ang. mmtm about center and $\vec{s}$ is nucleon's own spin operator.
- States can no longer be labeled by m (eigenvalue of $\vec{L}_z$) or by s_z (eigenvalue of $\vec{s}_z$). $[\vec{L}\cdot\vec{s},\vec{L}_z] \neq 0 \neq [\vec{L}\cdot\vec{s},\vec{s}_z]$
- However, $\vec{L}^2$ and $\vec{s}^2$ do commute with $\vec{L}\cdot\vec{s}$, and thus we can still use ℓ and s.
- $\ell = 0, 1, 2, 3, ...$ and $s = \frac{1}{2}$
- Also $\vec{J} = \vec{L} + \vec{s}$ commutes, $[\vec{J}, \vec{L}\cdot\vec{s}] = 0$, so eigenvalues of $\vec{J}^2$ and $\vec{J}_z$, viz j and j_z are good quantum numbers.
- States labelled as $|\ell, s, j, j_z\rangle$ with $s = \frac{1}{2}$ always.

Shell model

- States are labelled as $|\ell, s, j, j_z\rangle$ where $s = \frac{1}{2}$.
- Moreover, $j = \ell \pm \frac{1}{2}$ (since $\vec{J} = \vec{L} + \vec{s}$).
- The degeneracy now comes from different values of j_z
- For $j = \ell + \frac{1}{2}$, it is $2(\ell + \frac{1}{2}) + 1 = 2\ell + 2$
- For $j = \ell - \frac{1}{2}$, it is $2(\ell - \frac{1}{2}) + 1 = 2\ell$
- In addition, radial part is labelled by n

States are labelled as

$$n, \ell_{j=\ell+\frac{1}{2}}, \qquad n, \ell_{j=\ell-\frac{1}{2}}$$

with degeneracy of $2\ell + 2$, 2ℓ.

Shell model

The degenerate states will split in energy
by expectation value of $\vec{L} \cdot \vec{s}$.
Using

$$\vec{L} \cdot \vec{s} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{s}^2)$$

we write

$$\begin{aligned}
\langle \ell, s, j, j_z | \vec{L} \cdot \vec{s} | \ell, s, j, j_z \rangle &= \frac{1}{2} \langle \ell, s, j, j_z | (\vec{J}^2 - \vec{L}^2 - \vec{s}^2) | \ell, s, j, j_z \rangle \\
&= \frac{1}{2} \left(j(j+1) - \ell(\ell+1) - s(s+1) \right) \\
&= \frac{1}{2} \left(\pm(\ell + \tfrac{1}{2}) - \tfrac{1}{2} \right) \text{ for } j = \ell \pm \frac{1}{2}
\end{aligned}$$

Shell model

- The amount of splitting of $n, \ell_{\ell+\frac{1}{2}}$ from $n, \ell_{\ell-\frac{1}{2}}$ depends on expectation of $U_{s-o}(r)$.
- U_{s-o} is chosen to fit experiment... and is found to be negative
- Thus $n, \ell_{\ell+\frac{1}{2}}$ has lower energy than $n\ell_{\ell-\frac{1}{2}}$
- Method:
  - Calculate ordering of unsplit levels based on zeros of spherical Bessel functions.
  - Then calculate magnitude/sign of the splitting due to spin-orbit coupling
  - Add this to unperturbed energy and now find reordered levels
- Thus for example, earlier $(1, 2)$ was lower than $(2, 0)$. But now we see $(1, 2)_{\frac{5}{2}}$ remains below $(2, 0)_{\frac{1}{2}}$, but $(1, 2)_{\frac{3}{2}}$ is above it.

Shell model

Degeneracies are $2\ell+2$ for $(n,\ell)_{j=\ell+\frac{1}{2}}$, and 2ℓ for $(n,\ell)_{j=\ell-\frac{1}{2}}$.

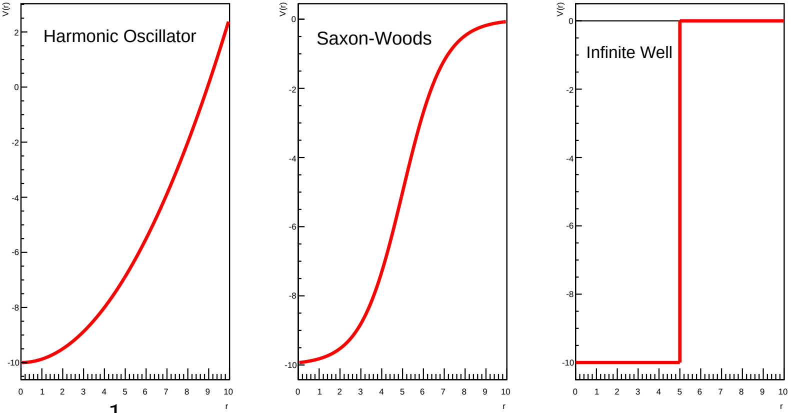
$(n,\ell)_j$	$(1,0)_{\frac{1}{2}}$	$(1,1)_{\frac{3}{2}}$	$(1,1)_{\frac{1}{2}}$	$(1,2)_{\frac{5}{2}}$	$(2,0)_{\frac{1}{2}}$	$(1,2)_{\frac{3}{2}}$	$(1,3)_{\frac{7}{2}}$	$(2,1)_{\frac{3}{2}}$
degeneracy	2	4	2	6	2	4	8	4
total	$\underline{2}$	6	$\underline{8}$	14	16	$\underline{20}$	$\underline{28}$	32

$(n,\ell)_j$	$(1,3)_{\frac{5}{2}}$	$(2,1)_{\frac{1}{2}}$	$(1,4)_{\frac{9}{2}}$	$(2,2)_{\frac{5}{2}}$	$(1,4)_{\frac{7}{2}}$	$(1,5)_{\frac{11}{2}}$	$(3,0)_{\frac{1}{2}}$	$(2,2)_{\frac{3}{2}}$
degeneracy	6	2	10	6	8	12	2	4
total	38	40	$\underline{50}$	56	64	76	78	$\underline{82}$

Now we have all magic numbers (we will get 126 too if we keep going). Other shells too, but at magic numbers, energy gap to next shell is large.

This is simply from infinite square well.. we can consider other potentials..

Shell model



$$V(r) = \frac{1}{2} m \omega^2 r^2, \qquad V(r) = \frac{-V_0}{1+\exp(\frac{r-R}{a})}$$

Nuclear physicists use the Saxon-Woods potential. Then the Schrodinger equation has to be solved numerically.