PHY4154 NUCLEAR AND PARTICLE PHYSICS

Assignment 1

(1) Find the speed at which a meter stick is moving if its length is observed to shrink to 0.5 m.

Solution: Let L be the length of the meter stick in the stationary observer's frame, and $L' = 1$ m be the length of the meter stick in its own frame of reference (thus L' is the proper length of the meter stick). We know that moving objects appear to be contracted by a factor of γ, i.e. $L = L'/\gamma$. Here, $L = 0.5$ m is given.

This gives us $\gamma = L'/L = 2$. Given that $\gamma = 1/\sqrt{1-\beta^2}$, we get $\beta = 0.866$, and thus $v = 0.866c$.

(2) Two bodies of mass m, each with speed $\frac{3}{5}c$, collide head on and stick together. What is the mass of the final clump?

Solution: Let $\vec{p}_1 = \vec{p}$ be the initial momentum of one body. Thus the other body has momentum $\vec{p}_2 = -\vec{p}$. Let $\vec{p}_M$ be the momentum of the final clump of mass M. Conservation of momentum gives us $\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_M$.

The initial energy of both clumps is $E_1 = E_2 = \gamma mc^2$. If $E_M = Mc^2$ is the energy of the final clump, then conservation of energy gives us $E_M = 2E_1$. Thus $Mc^2 = 2\gamma mc^2$, and $M = 2\gamma m$.

$\gamma = 1/\sqrt{1-\left(\frac{3}{5}\right)^2} = 5/4$, and thus $M = 5m/2$.

(3) A body of rest mass m moving at speed v collides with and sticks to an identical body at rest. What is the mass and momentum of the final clump?

Solution: The initial 4-momentum of the moving mass m is $p_1 = (\gamma mc, \gamma m\vec{v})$. The initial 4-momentum of the stationary mass is $p_2 = (E/c, 0)$. p_2 can be rewritten using $E = mc^2$ as $(mc, 0)$. Note that $p_1^2 = p_2^2 = m^2c^2$, given that they are energy-momentum 4-vectors. Thus the total initial 4-momentum is (adding components of vectors)
$p_i = p_1 + p_2 = ((\gamma+1)mc, 0)$.
The final 4-momentum of the clump of mass M is $p_f = (E_f/c, \vec{p})$.
We use conservation of energy-momentum before and after collision, and equate the invariants

before and after the collision to write $p_i^2 = p_f^2$.

$$\begin{aligned} p_f^2 = M^2c^2 &= p_i^2 \\ &= (p_1 + p_2)^2 \\ &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ &= 2m^2c^2 + 2\gamma m^2c^2 \\ &= 2m^2c^2(1+\gamma) \end{aligned}$$

This gives us $M = m\sqrt{2(1+\gamma)}$.

(4) A muon has a proper lifetime of $2.0\ \mu$s. It is created 100 km above the earth and moves towards earth at 2.97×10^8 m/s. At what altitude does the muon decay? According to the muon, how far did it travel in its life?

Solution: The proper lifetime of the muon (the average time for decay in its own frame) is $\tau_\mu = 2.0\mu$s. For a stationary observer on earth, the muon lives for a lifetime of $t = \gamma\tau_\mu$. In this time, the muon travels a distance of $d = vt = v\gamma\tau_\mu$. Thus $d = \frac{2.97\times10^8\times2\times10^{-6}}{\sqrt{1-0.99^2}} \simeq 4210$ m.

Thus the muon travels 4.2 km before decaying, and decays at an altitude of $100 - 4.2 = 95.8$ km. In its own frame, the muon travelled $d/\gamma = v\tau_\mu = 594$ m.

(5) The muon decays as $\mu \to e\bar{\nu}_e\nu_\mu$. If the number of muons at $t = 0$ is N_0, the number N at time t is $N = N_0e^{-t/\tau}$, where $\tau = 2.0\ \mu$s is the proper lifetime of the muon. Suppose the muons move at speed $0.95c$. What is the observed lifetime of the muons? How many muons remain after traveling a distance of 3.0 km.

Solution: The observed lifetime of the muons is $\tau = \gamma\tau_\mu$. Given that $\beta = 0.95c$, thus $\gamma = 3.202$, and $\tau = 3.202 \times 2 = 6.405\ \mu$s. This is the time constant to be used in the given decay equation for N.

The time taken to travel 3.0 km is $t = \frac{3\times10^3}{0.95\times3\times10^8} = 10.53\ \mu$s.

The number at time t is

$$N(t = 10.53) = N_0e^{-t/\tau} = N_0e^{-10.53/6.405} = 0.193N_0$$

(6) Show that the energy-momentum relationship $E^2 = p^2c^2 + m^2c^4$ follows from the relations $E = \gamma mc^2$, and $p = \gamma mv$.

Solution:

$$p = \gamma m v$$
$$p^2 = \frac{m^2 v^2}{1 - v^2/c^2}$$
$$\frac{p^2}{m^2c^2} = \frac{v^2/c^2}{1 - v^2/c^2}$$
$$1 + \frac{p^2}{m^2c^2} = 1 + \frac{v^2/c^2}{1 - v^2/c^2}$$
$$1 + \frac{p^2}{m^2c^2} = \gamma^2$$

$$E = \gamma m c^2$$
$$E^2 = \gamma^2 m^2 c^4$$
$$E^2 = \left(1 + \frac{p^2}{m^2c^2}\right) m^2 c^4$$
$$E^2 = m^2c^4 + p^2c^2$$

(7) A pion at rest decays to a muon and a neutrino. What is the speed of the muon? (You may answer in terms of m_π, m_μ etc.). On average how far will the muon travel (in vacuum) before disintegrating? (Use $m_\pi = 139.6\ \mathrm{MeV}/c^2$, $m_\mu = 105.7\ \mathrm{MeV}/c^2$ to give an answer in metres.)

Solution: We know that $p = \gamma m v$ and $E = \gamma m c^2$. Thus $p/E = v/c^2$ and $v = pc^2/E$. If we find the momentum and energy of the outgoing muon, we shall know its speed.

Let p_π, p_μ, and p_ν be the 4-momenta of the π, μ and ν respectively. Since the pion is at rest initially, we have $p_\pi = (E_\pi/c, 0)$. Let $p_\mu = (E_\mu/c, \vec{p}_\mu)$.

By conservation of momentum (initial=zero, so final=zero) the muon and neutrino emerge with opposite momentum equal in magntitude, i.e. $|\vec{p}_\mu| = |\vec{p}_\nu|$.

The neutrino is massless, thus its energy $E_\nu = |\vec{p}_\nu|c = |\vec{p}_\mu|c$

Conservation of 4-momentum gives us $p_\pi = p_\mu + p_\nu$.

$$p_\nu = p_\pi - p_\mu$$
$$p_\nu^2 = p_\pi^2 + p_\mu^2 - 2p_\pi p_\mu$$
$$0 = m_\pi^2c^2 + m_\mu^2c^2 - 2m_\pi E_\mu$$
$$E_\mu = \frac{\left(m_\pi^2 + m_\mu^2\right)c^2}{2m_\pi}$$

$$p_\mu = p_\pi - p_\nu$$
$$p_\mu^2 = p_\pi^2 + p_\nu^2 - 2p_\pi p_\nu$$
$$m_\mu^2c^2 = m_\pi^2c^2 + 0 - 2m_\pi E_\nu$$
$$m_\mu^2c^2 = m_\pi^2c^2 + 0 - 2m_\pi |\vec{p}_\mu| c$$
$$|\vec{p}_\mu| = \frac{\left(m_\pi^2 - m_\mu^2\right)c}{2m_\pi}$$

where I have used the following: $p_\pi p_\mu = \frac{E_\pi}{c}\frac{E_\mu}{c} - 0 = \frac{E_\pi}{c^2}E_\mu = m_\pi E_\mu$. (Note that the pion is at rest, so all three $\vec{p}_\pi$ components are zero.) Similarly $p_\pi p_\nu = m_\pi E_\nu$.

Now $v = pc^2/E$, thus

$$v = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}c$$

This gives γ as

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{m_\pi^2 + m_\mu^2}{2m_\pi m_\mu}$$

.

The lifetime in the lab frame is $\gamma\tau$ and the distance is $v\gamma\tau$. Putting in the numbers gives me a distance of 169 m.

(8) The Bevatron at Berkeley produced antiprotons by the reaction $p+p \to p+p+p+\bar{p}$, where a high energy proton strikes a proton at rest on the LHS. What is the minimum energy required for the striking proton?
Now assume that both the initial protons are moving (for a head-on collision). Now what is the minimum energy required by each initial proton for this reaction?

Solution: The minimum energy required by the initial proton will be that energy that barely produces the necessary particles, i.e. produces the necessary particles at rest, without 'wasting' any energy for the kinetic energy of the particles on the right side.

The total 4-momentum before the collision is obtained by adding the 4-momentum of the moving proton $(E/c, |p|, 0, 0)$ and the stationary target proton $(mc^2/c, 0, 0, 0)$. [Here I have chosen to align the momentum to x-axis.] The initial 4-momentum is thus $((E + mc^2)/c, |p|, 0, 0)$.

The final 4-momentum of the four particles (all at rest for threshold) is $(4mc^2/c, 0, 0, 0)$.

Equating the invariants gives

$$\begin{aligned} \left(\frac{E}{c} + mc\right)^2 - p^2 &= (4mc)^2 \\ \frac{E^2}{c^2} + m^2c^2 + 2Em - \left(\frac{E^2}{c^2} - m^2c^2\right) &= 16m^2c^2 \\ 2Em &= 14m^2c^2 \\ E &= 7mc^2 \end{aligned}$$

where I have used $E^2 - p^2c^2 = m^2c^4$ to give $p^2 = E^2/c^2 - m^2c^2$.

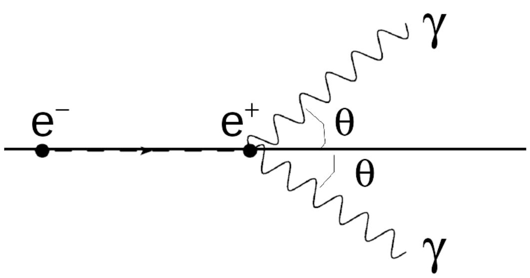
Thus the total energy needed by the proton is $7mc^2$, and thus $6mc^2$ of kinetic energy is needed to produce an additional proton-antiproton pair.

Now let us do the same exercise for when the initial protons have opposite and equal momenta and energies of E each. The initial 4-momentum is $p_i = (2E/c, 0, 0, 0)$, and to produce the final particles at rest the final 4-momentum is $p_f = (4mc, 0, 0, 0)$. Equating invariants $p_i^2 = p_f^2$ gives us $4E^2/c^2 = 16m^2c^2$, i.e. $E = 2mc^2$.

Note the total energy needed here is $4mc^2$. We see that we need to add only mc^2 worth of kinetic energy to each proton and we see that collisions between particles that are both moving are much more efficient than hitting a stationary target.

(9) An electron annhilates with a positron as follows: $e^- + e^+ \to \gamma + \gamma$. Let the positron be at rest initially, and the electron have kinetic energy of 1.0 MeV. The emitted photons travel at angle θ with the electron's direction of motion. Determine the energy E, momentum p and angle of emission θ of each photon. (Note: $m_e = 0.511$ MeV$/c^2$, $E_\gamma = pc$). What is the angle of emission θ if the electron has kinetic energy of 1.0 GeV?

Solution: The setup for the problem is shown in the figure.



An electron comes in with kinetic energy of K and annhilates with a positron at rest to produce two photons. The two photons each move with angle θ with the original electron direction (which we can take as x-axis).

We shall use a result from Problem 6, viz. $\gamma^2 = 1 + p^2/(m^2c^2)$. The kinetic energy can be written as total energy − rest energy. Thus $K = E - mc^2 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$. This gives $\gamma = 1 + K/mc^2$. Equating the two expressions for γ^2

$$\left(1 + \frac{K}{mc^2}\right)^2 = 1 + \left(\frac{p}{mc}\right)^2$$
$$1 + \frac{K^2}{m^2c^4} + \frac{2K}{mc^2} = 1 + \frac{p^2}{m^2c^2}$$
$$p^2 = \frac{K^2}{c^2} + 2Km$$
$$p = \frac{\sqrt{K(K + 2mc^2)}}{c}$$

This gives us an expression for the momemtum, once kinetic energy is known.

Now the total energy before is $E = K + 2mc^2$. By conservation of energy, and symmetry, the two photons must share the energy, and thus $E_\gamma = E/2$. And the momentum for the photons is $p_\gamma = E_\gamma/c = E/(2c)$.

The momentum for any photon makes an angle θ with the x-axis. Conservation of momentum along x-axis gives us $p_\gamma \cos\theta = p/2$ and thus $\cos\theta = p/(2p_\gamma)$.

Now let us put in some numbers.

Initial kinetic energy $K = 1.0$ MeV.
Thus initial momentum $p = 1.422$ MeV$/c$.
Now $E = K + 2mc^2 = 2.022$ MeV
and $p_\gamma = E_\gamma/c = 1.011$ MeV$/c$
Thus $\cos\theta = 1.422/2.022$
and $\theta = 45.3°$.

Initial kinetic energy $K = 1.0$ GeV.
Thus initial momentum $p \simeq 1.0$ GeV$/c$.
Now $E = K + 2mc^2 \simeq 1.0$ GeV
and $p_\gamma = E_\gamma/c = 0.5$ GeV$/c$
Thus $\cos\theta \simeq 1.00/1.00$
and $\theta \simeq 0°$.

What we can see is that as the initial kinetic energy increases, the opening angle between the produced photons becomes smaller and smaller.

(10) Compton scattering: A photon of wavelength λ collides elastically with a charged particle of mass m. If the photon scatters at angle θ, show that its outgoing wavelength λ' is $\lambda' = \lambda + (h/mc)(1 - \cos\theta)$.

Solution: Let the incoming photon come along x-axis, the charged particle is at rest. After scaterring, the photon scatters at an angle θ with incoming photon direction, let the charged particle (of mass m) scatter at angle ϕ with incoming photon direction. Let E and E' be the energies of the incoming and outgoing photon respectively.

Conservation of momentum tells us that for the outgoing particles, the components of momentum perpendicular to x-axis should cancel out.
Thus $p_m \sin\phi = p_\gamma \sin\theta \Rightarrow \sin\phi = (E'/cp_m)\sin\theta$.

Along the x-axis, conservation of momentum gives us

$$
\begin{aligned}
\frac{E}{c} &= p_\gamma \cos\theta + p_m \cos\phi \\
&= \frac{E'}{c}\cos\theta + p_m\sqrt{1 - \left(\frac{E'\sin\theta}{cp_m}\right)^2} \\
\therefore E &= E'\cos\theta + \sqrt{p_m^2 c^2 - (E'\sin\theta)^2} \\
(E - E'\cos\theta)^2 &= p_m^2 c^2 - (E'\sin\theta)^2 \\
\therefore p_m^2 c^2 &= E^2 + E'^2\cos^2\theta - 2EE'\cos\theta + E'^2\sin^2\theta \\
&= E^2 + E'^2 - 2EE'\cos\theta
\end{aligned}
$$

Applying conservation of energy gives us

$$
\begin{aligned}
E + mc^2 &= E' + \sqrt{m^2c^4 + p_m^2c^2} \\
&= E' + \sqrt{m^2c^4 + E^2 + E'^2 - 2EE'\cos\theta} \\
\therefore (E - E' + mc^2)^2 &= m^2c^4 + E^2 + E'^2 - 2EE'\cos\theta \\
E^2 + E'^2 + m^2c^4 - 2EE' + 2Emc^2 - 2E'mc^2 &= m^2c^4 + E^2 + E'^2 - 2EE'\cos\theta \\
Emc^2 - E'mc^2 &= EE' - EE'\cos\theta \\
(E - E')mc^2 &= EE'(1 - \cos\theta) \\
\therefore mc^2hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) &= \frac{h^2c^2}{\lambda\lambda'}(1 - \cos\theta) \\
mc(\lambda - \lambda') &= h(1 - \cos\theta) \\
\therefore \lambda' &= \frac{h}{mc}(1 - \cos\theta) + \lambda
\end{aligned}
$$

where we have used $E = hc/\lambda$ for the photon.