PHY 4154

Nuclear and Particle Physics

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- Time $\rightleftarrows$ Energy, Rotation $\rightleftarrows$ Angular momentum

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- Some symmetries are perfect (conservation laws hold exactly), some are broken (laws hold approximately).
- We aim to study angular momentum, discrete symmetries and so on. Let us first consider some additive quantum numbers like charge, and see some formalism.

Transformation

- Consider system described by time-indep H, where ψ satisfies Schrodinger equation

$$i\hbar \frac{d\psi}{dt} = H\psi \tag{1}$$

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- If $\hat{F}$ is an operator, then the observable F in state $\psi(t)$ is given by $\langle \hat{F} \rangle$.
- $\langle \hat{F} \rangle$ is conserved if

$$[H, \hat{F}] = 0 \Rightarrow \frac{d}{dt}\langle \hat{F} \rangle = 0 \tag{2}$$

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- If ψ' satisfies same Schrodinger equation, U is a symmetry operator. ($i\hbar d\psi'/dt = H\psi'$)
- We have

$$i\hbar \frac{d}{dt}(U\psi) = H(U\psi) \Rightarrow i\hbar \frac{d\psi}{dt} = U^{-1}HU\psi$$
$$\therefore H = U^{-1}HU = U^\dagger HU \Rightarrow [H, U] = 0$$

- The symmetry operator commutes with the Hamiltonian.

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- Transformations can be continuous (rotation) or non-continuous (parity).
- For continuous transformations we can write

$$U = e^{i\epsilon F}$$

where F is the generator of the transformation, and ϵ is a real parameter.

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- Invariance under continuous transform $\rightleftarrows$ Additive conservation law.
  Invariance under noncontinuous transform $\rightleftarrows$ Multiplicative conservation law.

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Let $\psi'(x) = U(\Delta)\psi(x)$ (Δ displacement along x). If system is invariant under translation, then ψ and ψ' both satisfy the Schrodinger equation, and $[H, U] = 0$.

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$$\hookrightarrow (1 - \Delta\frac{d}{dx})\psi = (1 - \Delta\frac{d}{dx})(1 + \Delta\frac{d}{dx})\psi'$$

$$(1 - \Delta\frac{d}{dx})\psi = (1)\psi'$$

ignoring terms of $\mathcal{O}(\Delta^2)$ in the last step.

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- Thus $U(\Delta) \sim (1 - \Delta \frac{d}{dx})$.
- Generator proportional to linear momentum operator $i\frac{d}{dx}$.
- Invariance under translation $\rightarrow$ conservation of momentum.
- Similarly we can show invariance under a local gauge transformation $\rightarrow$ conservation of charge.
- A local gauge transformation is $\psi' = e^{i\epsilon(\vec{x},t)Q}\psi$, where ϵ is real and arbitrary, and Q is charge operator with $[H, Q] = 0$.

Charge conservation

- Absence of $e \to \nu\gamma$ implies electric charge is conserved.
- Total charge conserved in a reaction, so if
  $a + b \to c + d + e$, then $N_a + N_b = N_c + N_d + N_e$
  where charge of a particle $q = Ne$ (an integral multiple of e).
- This is an additive conservation law. What is the symmetry principle?

Charge conservation

- Say, ψ described a state with charge q, and satisfies

$$i\hbar\frac{d\psi}{dt} = H\psi$$

- If Q is some operator, then $\langle Q \rangle$ is conserved and $[Q, H] = 0$ and thus Q and H have simultaneous eigenfunctions

$$Q\psi = \alpha\psi$$

and the eigenvalue q is also conserved.

- Here a gauge transformation is the symmetry

$$\psi' = e^{i\epsilon Q}\psi$$

where ϵ is real and arbitrary.

Charge conservation

We'll now illustrate that α is electric charge q using local gauge invariance.

- Let q be electric charge, and say there is a static E-field with $\vec{E} = -\vec{\nabla} A_0$, where A_0 is the scalar potential ($\vec{A}$ is the vector potential).
- We have $H = H_0 + qA_0$, where H_0 is the Hamiltonian when E-field is absent.

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- For free particle, $H_0 = p^2/2m = -\hbar^2\nabla^2/2m$
- The E, B fields are unchanged by gauge transformations $A_0 \to A_0'$, and $\vec{A} \to \vec{A}'$.
- If $\Lambda(\vec{x}, t)$ is an arbitrary function of position and time,

$$A_0' = A_0 - \frac{1}{c}\frac{\partial}{\partial t}\Lambda(\vec{x}, t)$$
$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda(\vec{x}, t)$$

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- Let us take $\Lambda(t), \epsilon(t)$ to simplify algebra (no $\vec{x}$ dependence).
- Impose invariance under local gauge transformation

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Put in A_0' and ψ'

$$i\hbar\frac{\partial}{\partial t}(e^{i\epsilon(t)Q}\psi) = \left(\frac{-\hbar^2}{2m}\nabla^2 + qA_0 - \frac{q}{c}\frac{\partial\Lambda}{\partial t}\right)e^{i\epsilon(t)Q}\psi$$

$$e^{i\epsilon(t)Q}\left(i\hbar\frac{\partial\psi}{\partial t} - \hbar Q\psi\frac{\partial\epsilon}{\partial t}\right) = e^{i\epsilon(t)Q}\left(\frac{-\hbar^2}{2m}\nabla^2 + qA_0 - \frac{q}{c}\frac{\partial\Lambda}{\partial t}\right)\psi$$

This gives

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Since $\epsilon(t)$ and $\Lambda(t)$ are arbitrary functions of space/time, say

$$\Lambda(t) = \hbar c \epsilon(t)$$

Together this gives us $Q\psi = q\psi$. Thus Q was the charge operator, and the symmetry transformation was $\psi' = e^{i\epsilon(\vec{x},t)Q}\psi$ (this leads to charge conservation)

As phase of the wavefunction varies as $\epsilon(\vec{x}, t)$, the variation is counteracted by corresponding changes in EM potential given by $\Lambda(\vec{x}, t) = \hbar c \epsilon(\vec{x}, t)$.

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$$\psi^R(\vec{x}^R) = \psi^R(\vec{x}) + \frac{\partial \psi^R(\vec{x})}{\partial \phi}\delta\phi = \left(1 + \delta\phi \frac{\partial}{\partial \phi}\right)\psi^R(\vec{x})$$

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As earlier, we can show now that

$$\psi^R(\vec{x}) = \left(1 - \delta\phi \frac{\partial}{\partial \phi}\right)\psi^R(\vec{x}^R) = \left(1 - \delta\phi \frac{\partial}{\partial \phi}\right)\psi(\vec{x})$$

where we have
(a) neglected terms of $\mathcal{O}(\delta^2)$ and
(b) used rotational invariance (i.e. $\psi(\vec{x}) = \psi^R(\vec{x}^R)$)

Rotational Symmetry

Looking at

$$\psi^R(\vec{x}) = \left(1 - \delta\phi \frac{\partial}{\partial\phi}\right) \psi(\vec{x})$$

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we can say $\epsilon = \delta\phi$ and $\hat{F} = i\partial/\partial\phi$.
Of course, the eigenfunctions and eigenvalues of $\hat{F}$ are known...

$$\hat{F} = -\frac{\hat{L}_z}{\hbar}$$

Angular momentum review

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Typically we measure $L^2 = \vec{L} \cdot \vec{L}$, and the z-component L_z.

$$L^2: \quad \ell(\ell+1)\hbar^2 \;\; \text{where } \ell = 0, 1, 2, 3, ...$$

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Similarly, for spin angular momentum $\vec{S}$ we have

$$S^2: \quad s(s+1)\hbar^2 \ \text{ where } s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, ...$$
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Fundamental particles have fixed spin.
Composite particles in addition have L, and several L states are possible. Thus electrons, protons have $s = \frac{1}{2}$, pions or kaons have $s = 0$, and photons or gluons have $s = 1$.

Addition of Angular momentum

Let us denote angular momentum states as follows:
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The total angular momentum, in situations where say orbital and spin angular momenta get coupled is obtained by $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

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Suppose we have two states $|j_1, m_1\rangle$, and $|j_2, m_2\rangle$, the sum of them will be denoted as $|j, m\rangle$.
In a straightforward way, we will have $m = m_1 + m_2$

The j will take values as $j = (j_1 + j_2), ..., |j_1 - j_2|$ in integer steps.

Addition of Angular momentum

Thus if the states are $|1,0\rangle$ and $|\frac{1}{2},\frac{1}{2}\rangle$,
then $m =$½, and j can take two values, 1+½ and 1−½
$j = \frac{3}{2}, \frac{1}{2}$.

Thus we can get two states, $|\frac{3}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},\frac{1}{2}\rangle$ from the addition of $|1,0\rangle$ and $|\frac{1}{2},\frac{1}{2}\rangle$

We write this as

$$|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle = \alpha|\frac{3}{2},\frac{1}{2}\rangle + \beta|\frac{1}{2},\frac{1}{2}\rangle$$

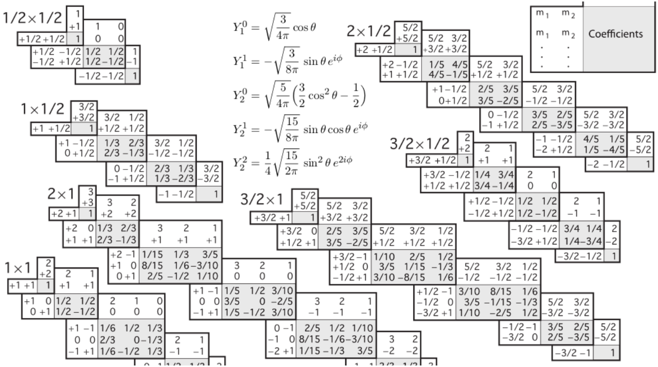
with the ability to calculate α and β.

Addition of Angular momentum

In general

$$|j_1, m_1\rangle|j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} C^{j\ j1\ \ j2}_{m\,m1\,m2}|j, m\rangle$$

Here the C's are Clebsch-Gordon coefficients, and we can look them up in a table.



Google clebsch site:pdg.lbl.gov/2024/, (pdf) is on our website.

Addition of Angular momentum

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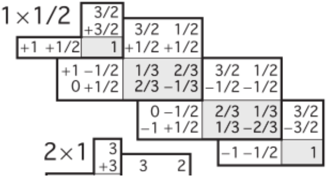

Here $m_1 = 0$ and $m_2 =$½, so look for the row corresponding to $0, 1/2$. This gives two numbers $2/3$ and $-1/3$

Remember to take the square root and we can write

$$|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle - \sqrt{\frac{1}{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Addition of Angular momentum: another example

$$|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle = \alpha|\frac{3}{2},\frac{1}{2}\rangle + \beta|\frac{1}{2},\frac{1}{2}\rangle$$

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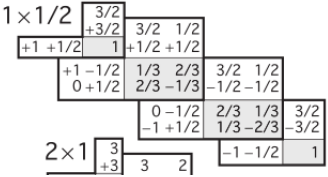

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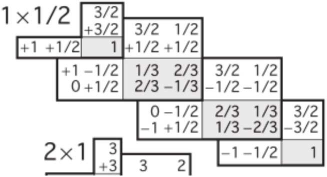


Here $m_1 = 1$ and $m_2 = -½$, so look for the row corresponding to $1, -1/2$.
This gives two numbers $1/3$ and $2/3$

Remember to take the square root and we can write

$$|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\frac{1}{3}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Addition of Angular momentum: and one more

$$|2,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle = \alpha|\frac{5}{2},-\frac{1}{2}\rangle + \beta|\frac{3}{2},-\frac{1}{2}\rangle$$

Here we are adding $j_1 = 2$ and $j_2 =$½, so look for the $2 \times 1/2$ table.

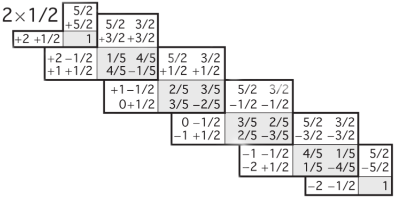
$2\times1/2$

m_1 m_2	J M		
	5/2 +5/2		
+2 +1/2	1		
	5/2 +3/2	3/2 +3/2	
+2 −1/2	1/5	4/5	
+1 +1/2	4/5	−1/5	
	5/2 +1/2	3/2 +1/2	
+1 −1/2	2/5	3/5	
0 +1/2	3/5	−2/5	
	5/2 −1/2	3/2 −1/2	
0 −1/2	3/5	2/5	
−1 +1/2	2/5	−3/5	
	5/2 −3/2	3/2 −3/2	
−1 −1/2	4/5	1/5	
−2 +1/2	1/5	−4/5	
	5/2 −5/2		
−2 −1/2	1		

Addition of Angular momentum: and one more

$$|2,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = \alpha|\tfrac{5}{2},-\tfrac{1}{2}\rangle + \beta|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

Here we are adding $j_1 = 2$ and $j_2 =$½, so look for the $2 \times 1/2$ table.



Here $m_1 = -1$ and $m_2 =$½, so look for the row corresponding to $-1, 1/2$.
This gives two numbers $2/5$ and $-3/5$

Remember to take the square root and we can write

$$|2,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\frac{2}{5}}\;|\tfrac{5}{2},-\tfrac{1}{2}\rangle - \sqrt{\frac{3}{5}}\;|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

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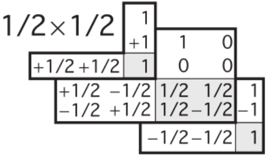

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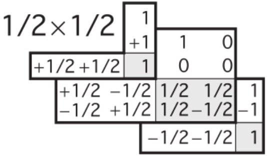

Inverting..

$$|1,1\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

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We observe $u\bar{u}$ bound states. (In practice it is a $\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$ bound state)
When $J = 0$ it is a π^0 meson ($m_{\pi^0} = 135$ MeV),
whereas in a $J = 1$ bound state, it is a ρ^0 meson ($m_{\rho^0} = 775$ MeV).

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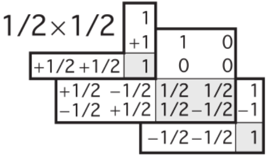

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(In practice this is more complicated).

Addition of Angular momentum

Consider the following two decays

$$\rho^0 \to \pi^0\pi^0$$
$$\rho^0 \to \pi^+\pi^-$$

Draw Feynman diagrams. Which one is more likely?

Addition of Angular momentum

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$$\rho^0 \to \pi^0 \pi^0$$
$$\rho^0 \to \pi^+ \pi^-$$

Only $\rho^0 \to \pi^+ \pi^-$ happens. Why?

Addition of Angular momentum

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The wavefunction of the $\pi^0\pi^0$ state would have to be symmetric (Bose statistics; two identical bosons). The ρ^0 has $J = 1$, while the right-hand $\pi^0\pi^0$, would need $L = 1$ .. but that is antisymmetric! Thus this decay is forbidden, and only the $\pi^+\pi^-$ decay happens.

Addition of Angular momentum

Now say we have three quarks bound together, with say $L = 0$, to form a baryon.

Thus $j = \frac{3}{2}, \frac{1}{2}$.
The baryon can have spin of $\frac{3}{2}$, or it can have spin of ½ in two ways
(i)first two add to 1 and we subtract ½ or
(ii)first two add to 0 and we add ½.

The baryon decuplet has $\frac{3}{2}$, while the baryon octet corresponds to one $\frac{1}{2}$.

Addition of Angular momentum

A quick point to note.

Its important to note the number of states before and after, and see that they match up.

If we add two spin ½ particles, the number of states before and after are four.

- $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$
- $\left|\frac{1}{2}, \frac{1}{2}\right\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$
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- $\left|\frac{1}{2}, \frac{1}{2}\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle$

- $|1, 1\rangle$
- $|1, 0\rangle$
- $|1, -1\rangle$
- $|0, 0\rangle$

One should check this when adding angular momenta.

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So we have on the left $|j = \frac{1}{2}, m_1\rangle_{\mathrm{n}}$ for the neutron.
On the right we have $|j_1 = \frac{1}{2}, m_2\rangle_{\mathrm{p}} |j_2 = \frac{1}{2}, m_3\rangle_{\mathrm{e}} |S, m_4\rangle_{\nu}$

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We need $j_1 =$½, $j_2 =$½, and S to "add" to give us $j =$½. (i.e. $j = j_1 + j_2 + S$)

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$j' = j_1 + j_2$ can take values 1, or 0.

This means that S has to take value of either $\frac{1}{2}$ or $\frac{3}{2}$, such that when we do $j = j' + S$, we will be able to get $j =$½.

Spin Angular momentum and Spinors

Spin and the corresponding formalism is fairly important and useful.
We can write a spin-up ($\uparrow$) state as $|½,½\rangle$ and a spin-down ($\downarrow$) as $|½,\text{-}½\rangle$

We can instead also define the two states ($\uparrow$ and $\downarrow$), by using two-component column vectors.

$$|½,½\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |½,\text{-}½\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are called as spinors.

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These are called as spinors.

Typically the most general state of a spin-$\frac{1}{2}$ particle is

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$ being the normalization condition.
Obviously, these form a basis in this spin-space.

Spin Angular momentum and Spinors

We have $\hat{S}_z|½,\pm½\rangle = \dfrac{\hbar}{2}|½,\pm½\rangle$

For the spinor notation, we define

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

This naturally gives us

$$\hat{S}_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$$
$$\hat{S}_z\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

Spin Angular momentum

We define the Pauli spin matrices as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define the components of spin as $\hat{\mathbf{S}} = \left(\frac{\hbar}{2}\right)\sigma$.

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Thus $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$.

The eigenvalues of $\hat{S}_x$ are $\pm\frac{\hbar}{2}$ and the eigenvectors are

$$\chi_+ = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \chi_- = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

These also form a basis.

Spin Angular momentum

Suppose an electron is in the state $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

If we measured $\hat{S}_x$, $\hat{S}_y$, or $\hat{S}_z$ what values will we get, with what probabilities?

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If we measured $\hat{S}_x$, $\hat{S}_y$, or $\hat{S}_z$ what values will we get, with what probabilities?

The eigenvectors of $\hat{S}_z$ are $\binom{1}{0}$ and $\binom{0}{1}$, and thus

$$\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus if we measure $\hat{S}_z$, we will get $\hbar/2$ with probability $1/5$ and $-\hbar/2$ with probability $4/5$.

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Thus if we measure $\hat{S}_x$, we will get $\hbar/2$ with probability $9/10$ and $-\hbar/2$ with probability $1/10$.

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Thus if we measure $\hat{S}_x$, we will get $\hbar/2$ with probability $9/10$ and $-\hbar/2$ with probability $1/10$.

HW: find the eigenvectors of $\hat{S}_y$, and find the corresponding probabilities.

Spin Angular momentum

Some useful properties of the Pauli spin matrices are

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

where δ_{ij} is the Kronecker delta, and ϵ_{ijk} is the Levi–Civita symbol.

The commutation relations are

$$\begin{aligned} [\sigma_i, \sigma_j] &= 2i\epsilon_{ijk}\sigma_k \\ \{\sigma_i, \sigma_j\} &= 2\delta_{ij} \end{aligned}$$

Also, for any two vectors a and b,

$$(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i\sigma \cdot (a \times b)$$

Rotating spinors

A spinor transforms as follows when we rotate the coordinate axes

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U(\theta) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{where } U(\theta) = e^{-i(\theta \cdot \sigma)/2}$$

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The vector θ points along the axis of rotation, and its magnitude is the angle of rotation about that axis.
So for a rotation around Z axis by angle δ, the operator would be

$$U(\delta) = exp(\frac{-i\delta\sigma_z}{2})$$

And for a general wave function, the operator would be

$$U(\theta) = e^{-i(\theta\cdot J)/\hbar}$$

Angular momentum

Let us rewrite the operator as

$$U(\theta) = exp(\frac{-i\theta \cdot J}{\hbar}) = exp(\frac{-i\delta\hat{n} \cdot J}{\hbar}) = U_n(\delta)$$

where δ is the magnitude of the rotation, and $\hat{n}$ is a unit vector along the axis of rotation.

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Now we note one thing: If a Hamiltonian is $H = H_0 + H_{mag}$.
If H_0 is isotropic, then we will have $[H_0, J] = 0$, but $[H, J] = [H_0 + H_{mag}, J] \neq 0$. The symmetry is broken.

Of course, the component of J along the magnetic field will still be conserved.