Energy and Momentum

Chapter 3 : Relativistic Kinematics
– David J Griffiths

Proper time (τ) : Time measured by an object in it's own rest-frame. This becomes releavant when you try to measure momentum of an object.

The Lorentz scale factor : $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

- always greater than 1.
- approaches ∞ as $v \to c$

Now, proper time of an object :

$$\underbrace{d\tau}_{\text{actual half life of } \mu} = \frac{\overbrace{dt}^{\text{what we measure in lab.}}}{\gamma}$$

- $d\tau$: time inteval measured by a object which is moving from our perspective
- dt : what we measure in lab.
- $d\tau < dt$: that interval of time appears longer to us. Time dilation!

Proper time is a Lorentz invariant quantity.

Velocity and Proper velocity : Suppose an object $(\vec{x}, t)$ is moving.

3D

Velocity, $\vec{v} = \dfrac{d\vec{x}}{dt}$ → both $\vec{x}$ and t changes under Lorentz transformation.

Proper velocity, $\boxed{\vec{\eta} = \dfrac{d\vec{x}}{d\tau}}$ → only $\vec{x}$ changes under Lorentz transformation

$\Rightarrow \boxed{\vec{\eta} = \gamma \vec{v}}$

- $\vec{\eta}$ is a scaled version of $\vec{v}$
- It changes like $\vec{x}$ under Lorentz transformation.

Now, let's talk in terms of 4-vectors.
This will have some interesting consequences.

4D

Let's define : $\boxed{\eta^\mu = \dfrac{dx^\mu}{d\tau}}$ →

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{(1/\gamma)\,dt} = \gamma c$$

$$\eta^1 = \frac{dx^1}{d\tau} = \gamma v_x$$

... and so on.

$$\therefore \boxed{\eta^\mu = \gamma(c,\ v_x,\ v_y,\ v_z)}$$

Then, $\eta_\mu = \gamma(c, -v_x, -v_y, -v_z)$

$$\therefore \eta_\mu \eta^\mu = \gamma^2\{c^2 - (v_x^2 + v_y^2 + v_z^2)\} = \gamma^2 c^2 \{1 - v^2/c^2\} = c^2 \to \text{invariant!}$$

Why should I care about this new 'proper' velocity in 4D ?

Classically, momentum = mass x velocity.
In relativity, if we use $p = mv$, then the law of conservation of momentum breaks down (because v is changing).

You should read the following carefully, including the footnote.

Classically, momentum is mass times velocity. We would like to carry this over in relativity, but the question arises: Which velocity should we use—ordinary velocity or proper velocity? Classical considerations offer no clue, for the two are equal in the nonrelativistic limit. In a sense, it's just a matter of definition, but there is a subtle and compelling reason why ordinary velocity would be a bad choice, whereas proper velocity is a good choice. The point is this: If we defined momentum as mv, then the law of conservation of momentum would be inconsistent with the principle of relativity (if it held in one inertial system, it would not hold in other inertial systems). But if we define momentum as mq, then conservation of momentum is consistent with the principle of relativity (if it holds in one inertial system, it automatically holds in all inertial systems). I'll let you prove this for yourself in Problem 3.10. Mind you, this doesn't guarantee

..... and it goes on.

* Proper velocity is a hybrid quantity, in the sense that distance is measured in the lab frame, whereas time is measured in the particle frame. Some people object to the adjective "proper" in this context, holding that this should be reserved for quantities measured entirely in the particle frame. Of course, in its ***own*** frame the particle never moves at all—its velocity **is** zero. If my terminology disturbs you, call η the "four-velocity." I should add that although proper velocity is the more convenient quantity to calculate with, ordinary velocity is still the more natural quantity from the point of view of an observer watching a particle fly past.

(Griffith's book)

In relativity, momentum is defined in terms of proper velocity as follows.

$$\vec{p} = m\vec{\eta}$$

$$\boxed{p^{\mu} = m\eta^{\mu}}$$

* Remember :

$$p^{\mu} = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

What happens when we redefine momentum like this ?

$p^\mu = m\eta^\mu$

Space part	Time part
$\vec{p} = m\vec{\eta} = \gamma m\vec{u}$	$p^0 = m\eta^0 = \gamma mc$
$\Rightarrow \vec{p} = \dfrac{m\vec{u}}{\sqrt{1 - u^2/c^2}}$ — (A)	$\Rightarrow \dfrac{E}{c} = \gamma mc$
	$\Rightarrow \boxed{E = \gamma mc^2}$ — (B)

Now, the 4-momentum vector is defined as $p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$

$$\therefore \quad p_\mu p^\mu = \frac{E^2}{c^2} - (p_x^2 + p_y^2 + p_z^2)$$

$$= \frac{E^2}{c^2} - \vec{p}^2 \quad \{\text{putting values from (A) and (B)}\}$$

$$= m^2c^2$$

This result is often written as,

or,

$$\boxed{\begin{aligned} p_\mu p^\mu &= m^2c^2 \\ E^2 &= \vec{p}^2c^2 + m^2c^4 \end{aligned}}$$

We can expand E as follows :

$$E = \left\{p^2c^2 + m^2c^4\right\}^{1/2}$$

$$= mc^2\left\{1 + \frac{\gamma^2v^2}{c^2}\right\}^{1/2} \quad \{p = \gamma mv\}$$

$$\simeq mc^2\left\{1 + v^2/c^2\right\}^{1/2} \quad \left\{\begin{matrix}\text{for } v \ll c, \\ \gamma \simeq 1\end{matrix}\right\}$$

$$\simeq mc^2\left\{1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots\right\}$$

For $v \ll c$ (non relativistic case)

We only keep the first two terms.

$$\therefore \quad E \simeq mc^2 + \frac{1}{2}mv^2$$

→ Total E = mc^2 + classical KE !

This is where $E = mc^2$ came from ! This is widely misinterpeted in media.

The complete expression is

$$E^2 = p^2c^2 + m^2c^4.$$

$$(\text{or. } E = \gamma mc^2)$$

- For a massive object with negligibe momentum, $E \simeq mc^2$
- For photons, $E = pc$

Collisions

Energy-momentum relations:

Important relations!

$$E = \gamma mc^2, \quad T = E - mc^2 = (\gamma - 1)mc^2$$
$$p = \gamma mv, \quad E^2 = p^2c^2 + m^2c^4$$

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

$$p_\mu p^\mu = \frac{E^2}{c^2} - p_x \cdot p_x - p_y \cdot p_y - p_z \cdot p_z$$
$$= E^2/c^2 - \vec{p} \cdot \vec{p} = m^2c^2$$

$$\boxed{p_\mu p^\mu = m^2c^2} = \text{Lorentz invariant (Scalar)}$$

Under non-relativistic ($v \ll c$) limit, this reduces to the classical expression of Energy.

$$E \simeq mc^2 + \frac{1}{2}mv^2$$

Lorentz invariant quantities like $p_\mu p^\mu$ transforms like this ⇒

$$p'_\mu p'^\mu = \{\Lambda_\mu^{\ \nu} p_\nu\}\{\Lambda^\mu_{\ \alpha} p^\alpha\}$$
$$= \boxed{\Lambda_\mu^{\ \nu} \Lambda^\mu_{\ \alpha}} \, p_\nu p^\alpha$$
$$= \delta^\nu_\alpha \, p_\nu p^\alpha$$
$$= p_\nu p^\nu$$
$$= p_\mu p^\mu$$

→ both p'_μ and p'^μ transform under Lorentz transformation in "opposite" kind of way.

μ is just a dummy index.

$p_\mu p^\mu$ is a "scalar" quantity.

Types of Collisions:

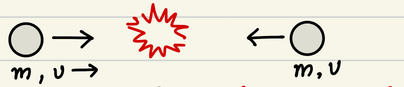
Classical Mechanics	Relativistic Mechanics.
• Mass is conserved.	• Each component of p^μ is conserved. (E and $\vec{p}$)
• Each component of momentum is conserved	* mass is not conserved in general.
• Subcases:	• Subcases:
(i) KE is conserved → Elastic collision	(i) KE is conserved → Elastic collision
(ii) KE is reduced → Sticky collision	(ii) KE is reduced → Sticky collision
(iii) KE is increased → Explosive collision	(iii) KE is increased → Explosive collision

Examples :

Sticky collision : Two lumps of clay, each of mass m, collide head on at $3/5\,c$, a form a new lump. What is the mass of the combined lump?

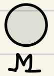
Sol^n The relative velocity between the two lumps are given $(3/5 c)$. Let's look at this in the COM frame. The final velocity of the combined lump = 0 in this frame. Now, we can consider two components of p^μ :

initial :



$p^\mu : \left(\frac{E_1}{c}, P_1, 0, 0\right)$ $\quad v = 3/5\,c \quad$ $\left(\frac{E_2}{c}, P_2, 0, 0\right)$

final :



$p^\mu :$ $\left(\frac{E_M}{c}, 0, 0, 0\right)$

Figure : Center of mass frame

$$p^1 : \quad P_1 + P_2 = 0 \quad \Rightarrow \quad \boxed{P_1 = -P_2}$$

$$p^0 : \quad E_1 + E_2 = E_M$$

Conservation of Energy : initial rest-mass energy = final rest-mass energy

$$\Rightarrow \frac{mc^2}{\sqrt{1-(3/5)^2}} + \frac{mc^2}{\sqrt{1-(3/5)^2}} = Mc^2$$

Solving for M : $\boxed{M = \frac{5}{4}(2mc^2)}$

Explosive collision : A lump of mass M at rest, explodes into two equal pieces of mass m. What is the speed of each mass m ?

Sol^n

From conservation of momentum (p^1) in the x direction, you can see that the two masses have equal and opposite momentum. Applying conservation of energy :

initial :

$p^\mu :$ $\left(\frac{E}{c}, 0, 0, 0\right)$

final :

$p^\mu :$ $\left(\frac{E_1}{c}, -P, 0, 0\right)$ $\quad$ $\left(\frac{E_2}{c}, P, 0, 0\right)$

$$E_1 + E_2 = Mc^2$$

$$\Rightarrow \frac{mc^2}{\sqrt{1-v^2/c^2}} + \frac{mc^2}{\sqrt{1-v^2/c^2}} = Mc^2 \quad \Rightarrow \quad \boxed{v = c\sqrt{1-(2m/M)^2}}$$

Pion decay : A π^- at rest decays to a μ^- and a $\bar{\nu}_\mu$. What is the speed of μ?

Sol^n

(i) Conservation of energy : $E_\pi = E_\mu + E_\nu$

(ii) Conservation of momentum : $P_\mu + P_\nu = 0$

In order to calculate velocity, we need to find P_μ first.

initial : π^-

$\left(\frac{E_\pi}{c}, 0, 0, 0\right)$

For the first equation, we can find out the energies of the individual particles.

(we assume that neutrinos are massless)

final : $\mu^- \leftarrow$ ✳ $\rightarrow \bar{\nu}_\mu$

$\left(\frac{E_\mu}{c}, P_\mu, 0, 0\right)$ $\left(\frac{E_\nu}{c}, P_\nu, 0, 0\right)$

Using $E = \sqrt{p^2c^2 + m^2c^4}$,

$$E_\pi = \sqrt{0^2c^2 + m_\pi^2 c^4} = m_\pi c^4 \qquad \text{—— (a)}$$

$$E_\mu = \sqrt{\vec{P}_\mu^2 c^2 + m_\mu^2 c^4} = c\sqrt{\vec{P}_\mu^2 + m_\mu^2 c^2} \qquad \text{—— (b)}$$

$$E_\nu = \sqrt{\vec{P}_\nu^2 c^2 + 0^2 c^4} = |\vec{P}_\nu| c = |P_\mu| c \qquad \text{—— (c)} \quad \{\text{from (ii)}\}$$

Putting this in (i),

$$c\sqrt{\vec{P}_\mu^2 + m_\mu^2 c^2} + |\vec{P}_\mu| c = m_\pi c^2$$

$$\Rightarrow \vec{P}_\mu^2 + m_\mu^2 c^2 = \left\{ m_\pi c - |\vec{P}_\mu| \right\}^2$$

$$\Rightarrow \cancel{\vec{P}_\mu^2} + m_\mu^2 c^2 = m_\pi^2 c^2 + \cancel{\vec{P}_\mu^2} - 2|\vec{P}_\mu| m_\pi c$$

$$\Rightarrow |\vec{P}_\mu| = \left\{ \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \right\} c$$

Also, $$E_\mu = \left\{ \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \right\} c^2 \qquad \left\{ \text{using } E = \sqrt{p^2c^2 + m^2c^4} \right\}$$

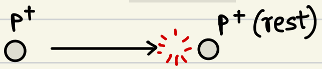
We know that $E = \gamma mc^2$ and $p = \gamma mv$. Therefore, $v = \frac{E}{pc^2}$.

In this case,

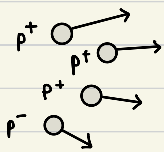
$$\boxed{v_\mu = \left\{ \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \right\} c \simeq 0.271c}$$

P-P collision: A target proton is at rest. Another high energy proton strikes it. What is the threshold energy for the following interaction?

initial



final



Sol^n Let us consider the four vectors:

initial : $\left(\frac{E}{c}, p, 0, 0\right)$ and $\left(\frac{mc^2}{c}, 0, 0, 0\right)$

final : Energy = at least $4mc^2$ → This is the minimum energy of the final state. Since we are calculating threshold, we need an expression for E, when the four ps in the final state has minimum total energy.

$\therefore$ Total p^μ(initial) $= \left(\frac{E}{c} + mc, p, 0, 0\right)$

Total p^μ (final) $= \left(4mc, 0, 0, 0\right)$

Since $p_\mu p^\mu$ is conserved,

$$\left\{\frac{E}{c} + mc\right\}^2 - p^2 = \{4mc\}^2$$

→ Since we want E, we eleminate p^2 by using $E^2 = p^2c^2 + m^2c^4$

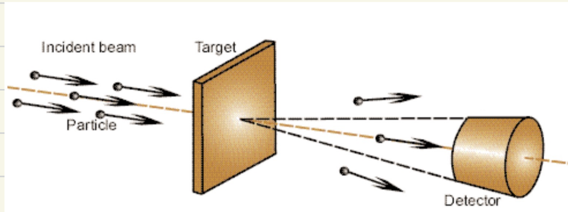
$$\Rightarrow \cancel{\frac{E^2}{c^2}} + m^2c^2 + 2Em - \left\{\cancel{\frac{E^2}{c^2}} - m^2c^2\right\} = 16\, m^2c^2$$

$$\Rightarrow 2Em + 2m^2c^2 = 16\, m^2c^2$$

$$\Rightarrow E + mc^2 = 8mc^2$$

$$\Rightarrow \boxed{E = 7mc^2}$$

$$\simeq 6.57 \text{ GeV}$$

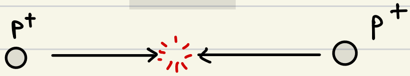


Collider v.s. Fixed target experiment: Which one is better?

Consider the previous example, but both protons are approaching each other at equal and opposite momentum.

initial

final



$\left(\frac{E}{c}, p, 0, 0\right)$ $\left(\frac{E}{c}, -p, 0, 0\right)$ $\left(\frac{4mc^2}{c}, 0, 0, 0\right)$

In this case,

$$p^{\mu}_{total}(\text{initial}) = \left(\frac{2E}{c}, 0, 0, 0\right)$$

$$p^{\mu}_{total}(\text{final}) = \left(\frac{4mc^2}{c}, 0, 0, 0\right)$$

Since $p_\mu p^\mu$ is conserved,

$$\left(\frac{2E}{c}\right)^2 = (4mc)^2$$

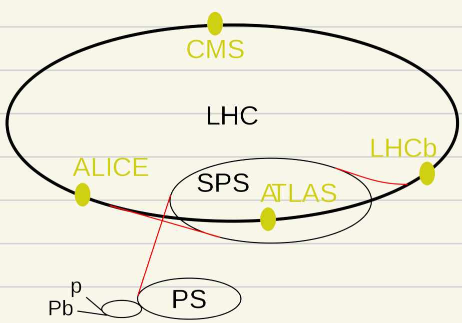
$$\Rightarrow \frac{2E}{c} = 4mc$$

$$\Rightarrow \boxed{E = 2mc^2} \simeq 1.88 \text{ GeV}$$

You may have expected this result to be half of the previous result, but this is even less!

The threshold energy of the individual protons in the collider experiment is 1.88 GeV, while that of the proton beam in the fixed target experiment is 6.57 GeV.

Instead of having 1 proton beam hitting a fixed target, having 2 proton beams hitting each other is much more efficient.



The LHC is a collider experiment.