IISER Pune; PH-4273/6423; Nonlinear Dynamics; Test: 1

8.9.2026 Time: 45 minutes Maximum Marks: 30

Name: ______________________________ **Roll Number:** ______________________________

NOTE 1: Answer all the questions. Use the same symbols/notation given in the questions. For questions 1 to 5, more than one option can be correct. No partial marks for questions 1 to 5, and full marks only if all correct options are chosen.
NOTE 2: If you are sketching a graph, label the axes. No marks if axes are not labelled. Write clearly and legibly.
NOTE 3: For questions 5 to 7, answers must be written within the given boxes. Anything outside will not be considered.

1. Let $\dot{x} = f(x)$ be the system of interest. What is the characteristic timescale in the neighbourhood of a fixed point x^*? (3)

   (a) $\tau = \sqrt{\dfrac{1}{|f'(x^*)|}}$

   (b) $\tau = |f'(x^*)|$

   (c) $\tau = \dfrac{1}{f'(x^*)}$

   (d) $\tau = \dfrac{1}{|f'(x^*)|}$

   (e) $\tau = \dfrac{1}{|f''(x^*)|}$

   Characteristic timescale determines the time required for $x(t)$ to vary significantly in the neighbourhood of x^*. $\delta x(t) = \delta x(0)e^{f'(x^*)t}$

2. Which of the following statements is correct about the time period T of the nonlinear pendulum of length L whose trajectory is on the separatrix? (3)

   (a) T depends on initial condition.

   (b) $T = 2\pi\sqrt{\dfrac{L}{g}}$

   (c) $T = 0$

   (d) $T = \infty$

   (e) T can become large if initial condition is a multiple of $\pi/4$.

   For the nonlinear pendulum, the separatrix approaches the unstable equilibrium asymptotically. Therefore the time required to complete a periodic orbit on the separatrix diverges.

3. For the system (with two parameters α and β) given by

$$\dot{x} = 2x + \frac{\alpha x}{\beta(1 + x^2)}.$$

   What is the type of bifurcation, and what is the critical value α_c at which bifurcation takes place? (3)

   (a) Subcritical pitchfork bifurcation, $\alpha_c = 2/\beta$

   (b) Transcritical bifurcation, $\alpha_c = 2\beta$

(c) Supercritical pitchfork bifurcation, $\alpha_c = -2\beta$

(d) Supercritical pitchfork bifurcation, $\alpha_c = -2/\beta$

(e) Subcritical pitchfork bifurcation, $\alpha_c = -2\beta$

Equation is invariant under $x \to -x$, hence it is pitchfork bifurcation. To determine the critical value, we first find the fixed points of the system.
$\dot{x} = 0 \implies x^* = 0$ or $x^* = \sqrt{-(1 + \frac{\alpha}{2\beta})}$
Second fixed point is valid only when $(1+\frac{\alpha}{2\beta})$ is negative i.e when $\alpha < -2\beta$. This implies that the critical value of alpha is $\alpha_c = -2\beta$. To determine whether the bifurcation is subcritical or supercritical check the stability of the fixed point $x^* = 0$ with respect to α.

$$f'(x) = 2 + \frac{\alpha}{\beta}\frac{(1-x^2)}{(1+x^2)^2}$$

$$f'(0) = 2 + \frac{\alpha}{\beta} \begin{cases} < 0 \text{ for } \alpha < -2\beta \implies \text{stable} \\ > 0 \text{ for } \alpha > -2\beta \implies \text{unstable} \end{cases}$$

The two arms corresponding to the other fixed point exist only for $\alpha < -2\beta$ and in this region $x^* = 0$ is stable, thus the two arms will be unstable. Thus the bifurcation is subcritical pitchfork.

4. For the 1D flow given by

$$\frac{dx}{dt} = x \ln x - be^b, \qquad b \in \mathbb{R}.$$

Which statement correctly describes the behavior of linearised perturbation in the neighbourhoods of all possible fixed points? (3)

(a) Linear stability is insufficient to determine stability in this case.

(b) Irrespective of whether perturbation grows or decays, the behaviour is independent of b.

(c) Perturbations can only exponentially decay for $b > 1$.

(d) Perturbations can exponentially grow for $b < -1$.

(e) Perturbations can only oscillate if $|b| < 1$.

A fixed point for this system is given by $x^* = e^b$. Therefore, $f^{'}(x^*) = 1 + b$.
For $b > -1, f^{'}(x^*) > 0$ and perturbations grow exponentially around this fixed point.
For $b < -1, f^{'}(x^*) < 0$ and perturbations decay exponentially around this fixed point.
However, for $b = -1$, $f^{'}(x^*) = 0$ and hence there is at least one value of b for which linearisation is insufficient to determine stability.

The other fixed point can be obtained by equating $x \ln x = b\ e^b$ for certain b values. Now, $y = x \ln x$ has a minima at $x = \frac{1}{e}$ and and intersects the x-axis as $x = 0, 1$. Hence, for $b < -1$, $y = b\ e^b$ intersects $y = x \ln x$ twice giving two fixed points. Let us call these fixed points p_1 and p_2 such that $0 < p_1 < \frac{1}{e}$ and $\frac{1}{e} < p_2 < 1$. We can see that $f^{'}(p_1) < 0$ and $f^{'}(p_2) > 0$. Thus it is possible that, for some fixed points, perturbations can grow for $b < -1$.

For this question, full marks will be given if at least one among (a) or (d) is chosen as the correct option.

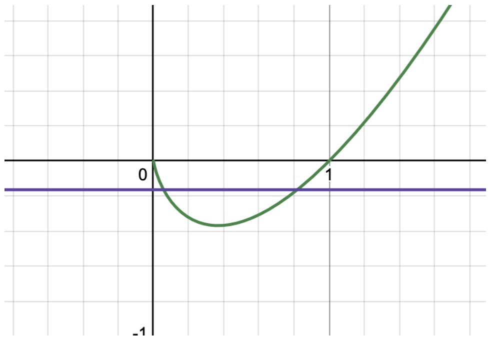


Figure 1: For $b = -0.2$. Green curve corresponds to $y = x \ln x$ and purple curve corresponds to $y = be^b$. The intersections of these two curves correspond to two fixed points $x = p_1, p_2$.

5. Which among the following is/are non-linear equations? (3)

   (a) $\dot{x} = \dfrac{1}{a+x}$
   (b) $\dot{x} = -\omega^2 x + \cos(\omega t)$
   (c) $\dot{x} = -x + x^3$
   (d) $\dot{x} = -\sin(\omega t)$
   (e) $\dot{x} = x - \cos(x)$

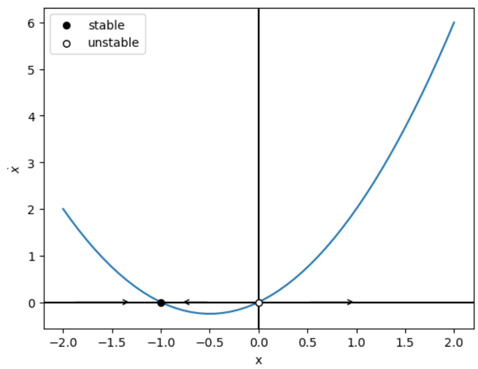
   In options (a), (c) and (e) the dependent variable x is in nonlinear form, hence these equations are nonlinear.

6. Consider the potential
$$V(x) = -\frac{x^2}{2} - \frac{x^3}{3}.$$

   (a) Write an expression for its vector field.
   (b) Sketch state space graphically.
   (c) Mark the fixed points on the state space diagram and indicate the stability.

(2+1+2)

   (a) The potential of a system $\dot{x} = f(x)$ is defined as $-\frac{dV}{dx} = f(x)$.
   $f(x) = -\frac{d}{dx}(-\frac{x^2}{2} - \frac{x^3}{3}) = x + x^2$
   The vector field is $\dot{x} = x + x^2$
   (b) State space diagram



(c) The fixed points of the system can be found by solving $x^* + (x^*)^2 = 0$. From this we will get $x^* = 0, -1$
$x^* = 0$ unstable fixed point
$x^* = -1$ stable fixed point

7. Consider the flow on the circle for the system given by

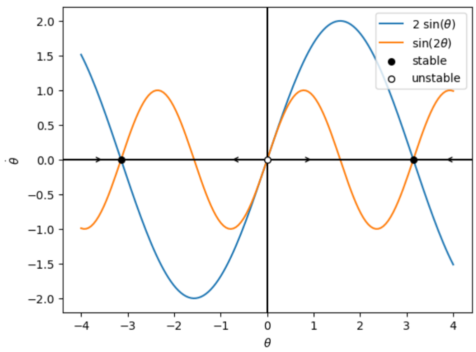
$$\dot{\theta} = \mu \sin\theta - \sin 2\theta.$$

(a) Sketch the state space for this system for $\mu = 2$. On the same figure, indicate directions of flow, and the fixed points.

(b) What are the critical values of μ at which bifurcation takes place.

(3+2)

(a) The fixed points for the system when $\mu = 2$ are the below:
$2\sin(\theta^*) - \sin(2\theta^*) = 0 \implies \theta^* = 0, \pi$



(b)

$$\begin{aligned} \mu\sin(\theta) - \sin(2\theta) &= 0 \\ \mu &= 2\cos(\theta) \\ \cos(\theta) &= \frac{\mu}{2} \end{aligned}$$

This implies $|\mu| \leq 2$ so the bifurcation takes place at $\mu = 2$ or -2.

8. The system

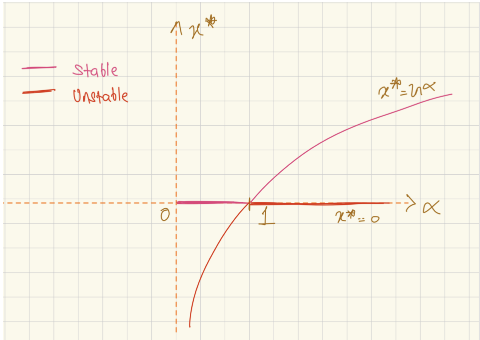
$$\dot{x} = x(\alpha - e^x)$$

displays a transcritical bifurcation. Sketch the bifurcation diagram. Show some relevant numbers on the bifurcation diagram. (5)

The fixed points for the above equations are $x^* = 0, \ln\alpha$. We can verify that given system shows transcritical bifurcation by expanding $f(x)$ near $x^* = 0$.

$\therefore \dot{x} = x(\alpha - (1 + x + \frac{x^2}{2} + ...) \approx x(\alpha - 1) - x^2 \implies \alpha_c = 1.$

$f^{'}(0) = \alpha - 1 \implies$ stable fixed point for $\alpha < 1$ and unstable for $\alpha > 1$.
$f^{'}(\ln\alpha) = -\alpha\ln\alpha \implies$ stable fixed point for $\alpha > 1$ and unstable for $0 < \alpha < 1$.


Stable
Unstable
x^*
$x^* = \ln\alpha$
0
1
$x^* = 0$
α