Name: **Roll Number:**

IISER Pune; PH-3214; Test : 2

6.4.2026 *Time: 45 minutes.* *Maximum Marks : 20.*

NOTE 1: Answer all the 7 questions. Use the same symbols/notation given in the questions. Choose only one correct answer for questions 1 to 3.
NOTE 2: If you are sketching a graph, label the axes. No marks if axes are not labelled. No partial marks for questions 1 to 3. Write clearly and legibly.

1. A system consists of 3 electrons; one each in $1s, 1p$ and $1f$ states. What is the Hilbert space dimension $\mathcal{N}$ and maximum angular momentum L_{max}? (2)

   (a) $\mathcal{N} = 9$ and $L_{\text{max}} = \sqrt{3}\hbar$.
   (b) $\mathcal{N} = 36$ and $L_{\text{max}} = 2\sqrt{6}\hbar$.
   (c) $\mathcal{N} = 4$ and $L_{\text{max}} = 2\sqrt{2}\hbar$.
   (d) $\mathcal{N} = 24$ and $L_{\text{max}} = 2\sqrt{5}\hbar$.
   (e) $\mathcal{N} = 18$ and $L_{\text{max}} = 3\sqrt{2}\hbar$.

2. Variational technique employs a trial state $|\phi_t(\beta)\rangle$ and a functional $\langle\phi_t(\beta)|H(\alpha)|\phi_t(\beta)\rangle$. In this, α and β are parameters. This method can be used to obtain? (2)

   (a) Upper bound to the ground state energy by parameters α.
   (b) Lower bound to the ground state energy by parameters α.
   (c) Exact ground state energy by varying parameter β.
   (d) Approximate ground state energy by varying parameters α and β.
   (e) Upper bound to the ground state energy by parameters α and β.

3. There are two particles in a harmonic oscillator potential. Consider *all* the states with energies $E \leq 5\hbar\omega$. Let M_d be the total number of states when particles are distinguishable, M_b be the number of states if the particles are bosons, and M_f be the number of states if the particles are fermions. Then, correct values of (M_d, M_b, M_f) is ? (2)

   (a) $(15, 8, 5)$.
   (b) $(15, 7, 6)$.
   (c) $(15, 9, 5)$.
   (d) $(15, 9, 6)$.
   (e) $(15, 8, 6)$.

4. Which of the following is the correct consequence of the Pauli Exclusion Principle? (2)

   (a) Two fermions cannot be in the same state even if they are at different positions.
   (b) Fermions need not have definite symmetry since their states may be different.
   (c) Two fermions can be in the same state provided they are at different positions.
   (d) Two fermions cannot have the same total energy.
   (e) Two bosons cannot be in the same position at the same time.

For the MCQs 1-4 given above, write your answer only inside the boxes here:

1 $\boxtimes$ 2 $\boxtimes$ 3 d 4 a

5. Consider three fermions in states $\psi_{n_1}(x)$, $\psi_{n_2}(x)$ and $\psi_{n_3}(x)$. What is the normalised state of this system. If $n_1 = n_2 = n$, then what is the corresponding state. (3)

$$\langle xyz|\varphi\rangle = \frac{1}{\sqrt{3!}} \begin{vmatrix} \varphi_{n_1}(x) & \varphi_{n_2}(x) & \varphi_{n_3}(x) \\ \varphi_{n_1}(y) & \varphi_{n_2}(y) & \varphi_{n_3}(y) \\ \varphi_{n_1}(z) & \varphi_{n_2}(z) & \varphi_{n_3}(z) \end{vmatrix}$$

If $n_1 = n_2 = n$, then $|\varphi\rangle = 0$.

6. For an electron in the spin state

$$\psi = A \begin{bmatrix} 3i \\ 4 \end{bmatrix}.$$

Find the expectation value S_x. (3)

7. For two spin system with $s_1 = \frac{1}{2}$, and $s_2 = \frac{1}{2}$. (a) Write down all the product states. (b) Write down all the coupled states in the notation $|s,\ m\rangle$. (c) Derive an expression for $|1, 0\rangle$ in terms of product states. (1+2+3)

Start writing your answers for questions 6 and 7 below

6) $\psi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$ Normalisation $\psi^*\psi = A^2(-3i\ \ 4)\begin{pmatrix} 3i \\ 4 \end{pmatrix} = 1$

$\therefore A = 1/5$

$$\langle S_x\rangle = \frac{\hbar}{2}\langle\sigma_x\rangle$$

$$= A^2(-3i\ \ 4)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} A = A^2(12i - 12i) = 0$$

$\therefore \langle S_x\rangle = 0$

7) (a) Product states

$|\tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_2$

$|\tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ -\tfrac{1}{2}\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_2$

$|\tfrac{1}{2}\ -\tfrac{1}{2}, \tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_2$

$|\tfrac{1}{2}\ -\tfrac{1}{2}, \tfrac{1}{2}\ -\tfrac{1}{2}\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_2$

(b) Coupled states:

$$s = 0, \quad 1$$
$$\downarrow \qquad \downarrow$$
$$m = 0 \quad -1, 0, 1$$

$\therefore\ |s, m\rangle$:

$|1, -1\rangle$

$|1, 0\rangle \qquad |0\ 0\rangle$

$|1, 1\rangle$

(c) $|1, 0\rangle = \alpha\, |\tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ -\tfrac{1}{2}\rangle + \beta\, |\tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$|1, 1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}; \tfrac{1}{2}\ \tfrac{1}{2}\rangle$ (top most state)

Use $S_\pm |s\ m\rangle = \hbar\sqrt{(s\mp m)(s\pm m+1)}\ |s, m\pm 1\rangle$

To get $|1, 0\rangle$: $\quad S_-|1, 1\rangle = \hbar\sqrt{2}\ |1, 0\rangle$

$$|1, 0\rangle = \frac{1}{\sqrt{2}\hbar} S_-|1, 1\rangle$$

$$= \frac{1}{\sqrt{2}\hbar} S_-|\tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$= \frac{1}{\sqrt{2}\hbar}(S_{1-} + S_{2-})\,|\tfrac{1}{2}\ \tfrac{1}{2}, \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

This gives

$$|1,0\rangle = \frac{1}{\sqrt{2}\hbar}\left[\left(S_{1-}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_1\right)\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_2 + \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_1\left(S_{2-}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_2\right)\right]$$

$$= \frac{1}{\sqrt{2}\hbar}\left[\hbar\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_1\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_2 + \hbar\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_1\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_2\right]$$

$$= \frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\,-\tfrac{1}{2},\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\,\tfrac{1}{2},\,\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right]$$

$$\boxed{|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|-+\rangle + |+-\rangle\right)}$$