NAME: Roll number:

Indian Institute of Science Education and Research Pune
Mid-semester Exam, Jan-Apr semester (2026).

Course name: Quantum Mechanics II — **Course code:** PH-3214
Date: 26.11.2024 (10:00 PM to 12:00 PM) — *Duration:* 2 hours
Instructor : M. S. Santhanam — Maximum marks: 60

- Answer all the questions. For 1-4, write the answers directly in the given box only.
- If you draw sketches as an answer, label the axes. No marks without axes labels.
- SHOW ALL THE STEPS CLEARLY in your calculations while arriving at an answer.
- Unless specified otherwise, all the symbols have their usual meanings.
- Use the same symbols and notation given in the question. Do not use your own.

1. If $\widehat{T}$ is the translation operator that takes $x \to x+\epsilon$, then $\widehat{T}^\dagger\,\widehat{x}\,\widehat{T} = \widehat{x}+\epsilon$. Using explicit form of $\widehat{T}$ leads us to which commutation relation? (4)

$[\hat{x}, \hat{p}] = i\hbar$

2. If $H(r,\theta,\phi)$ represents a Hamiltonian in three dimensions with the potential function $V(r,\theta,\phi) = r^2$, answer (do not derive) the following ; (2+2)

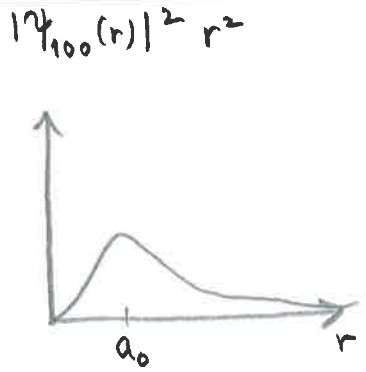
(i) What are the conserved quantities and the corresponding commutator relations.

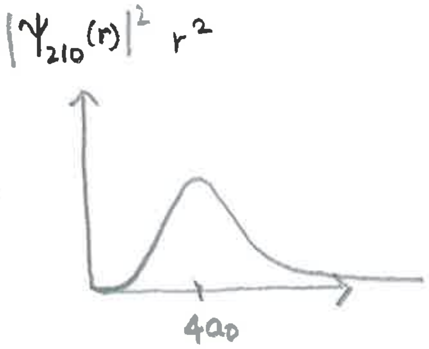
Energy, L^2, L_z. $[H, L^2] = 0$, $[H, L_z] = 0$, $[H, H] = 0$

(ii) In Hydrogen atom, what is the degree of degeneracy of a state with principal quantum number n.

n^2

3. Sketch the probability densities $|\psi_{100}(r)|^2\, r^2$ and $|\psi_{210}(r)|^2\, r^2$ of hydrogen atom ; (2+2)





4. If $\mathbb{P}$ is the parity operator, what are its possible eigenvalues ?
Consider 2-dimensional harmonic oscillator state with quantum numbers n_1 and n_2. What is the parity of its eigenstate $\Phi_{1,0}(x, y)$. (2+2)

Eigenvalues ± 1.
Parity of $\Phi_{1,0}(x,y)$ is -1, or odd parity.

5. Show that the anti-commutator $[L_x, L_y]_+$ can be reduced to $-\frac{i}{2}\left(L_+^2 - L_-^2\right)$. (6)

6. Evaluate the matrix element $\langle 0|p^2|0\rangle$, where $|0\rangle$ is the ground state of oscillator. The mass of oscillator is m and frequency is ω. (6)

7. For the hydrogen atom problem, with $l = n - 1$, the radial wavefunction has the form

$$\psi_{n,n-1,m} = A_n \, r^{n-1} \, e^{-r/na_0} \, Y_{n-1}^m(\theta, \phi).$$

In this, A_n is the normalisation constant (need not be calculated). What is the value of r at which the probability to find the electron is the maximum ? (6)

8. Assume that a particle has an orbital angular momentum with z component being $m\hbar$, and square of total angular momentum being $\hbar^2 l(l+1)$. Given that $\langle L_x\rangle = 0$, calculate $\langle L_x^2\rangle$. Use these answers to show that the uncertainty in L_x is (8)

$$\sigma_{L_x} = \frac{\hbar}{\sqrt{2}} \sqrt{l(l+1) - m^2}.$$

9. How does $Y_l^m(\theta, \phi)$ transform under parity operation? Let $\mathbb{P}\phi(x, y, z) = \phi(-x, -y, -z)$, where $\mathbb{P}$ is the parity operator. (a) Find how θ and ϕ transform under the parity operation defined here ? (b) Use this result to determine $\mathbb{P}\, Y_l^m(\theta, \phi)$ (c) Determine the parity of $\psi_{2,0,0}$ state of Hydrogen atom. (4+3+1)

Some useful information. You can use this information in your calculation (if neeeded):

Hydrogen atom wavefunction : $\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)\, Y_l^m(\theta, \phi)$.

Spherical harmonics :

$$Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)!}{4\pi}} \sqrt{\frac{(l+m)!}{(2l)!(l-m)!}} e^{im\phi} (\sin\theta)^{-m} \times \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l}$$

hydrogen atom eigenstates : $\psi_{100} = \sqrt{\frac{1}{\pi a_0^3}} e^{-r/a_0}$, and $\psi_{200} = \frac{2}{(2a_0)^{3/2}}\,(1 - r/2a_0)\, e^{-r/2a_0} Y_0^0(\theta, \phi)$

Ladder operators acting on $|lm\rangle$: $\quad L_\pm |lm\rangle = \hbar\left[(l \mp m)(l \pm m + 1)\right]^{1/2} |l, m \pm 1\rangle$

Creation and annihilation operators : $\quad a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

5. $[L_x, L_y]_+ = -\frac{i}{2}(L_+^2 - L_-^2)$

$$L_\pm = L_x + iL_y \qquad L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$$

$\therefore$ $[L_x, L_y]_+ = L_xL_y + L_yL_x$

$$= \frac{1}{4i}\left((L_+^2 - L_-^2) + (L_+^2 - L_-^2)\right) = \frac{1}{2i}(L_+^2 - L_-^2)$$

$$[L_x, L_y]_+ = -\frac{i}{2}(L_+^2 - L_-^2)$$

6. $\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)$

$$\hat{p}^2 = -\frac{m\omega\hbar}{2}(a^{\dagger 2} - a^\dagger a - a a^\dagger - a^2)$$

$$\langle 0|\hat{p}^2|0\rangle = \frac{m\omega\hbar}{2}\langle 0|a^{\dagger 2} - a^\dagger a - a a^\dagger - a^2|0\rangle$$

Using $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\langle 0|a^2|0\rangle = 0$ $\qquad$ $\langle 0|a a^\dagger|0\rangle = \langle 0|a|1\rangle = 1$

$\langle 0|a^\dagger a|0\rangle = 0$ $\qquad$ $\langle 0|a^{\dagger 2}|0\rangle = \langle 0|a^\dagger|1\rangle = 0$

$$\therefore \quad \langle 0|p^2|0\rangle = \frac{m\omega\hbar}{2}$$

7. $\psi_{n,n-1,m} = A_n\, r^{n-1}\, e^{-r/na_0}\, Y_{n-1}^m(\theta,\varphi)$

Probability to find electron in r and $r+dr$

$$P(r) = A_n \, r^{2(n-1)} \, e^{-2r/na_0} \, r^2$$
$$= A_n \, r^{2n} \, e^{-2r/na_0}$$

(r^2 is multiplied to account for Jacobian).

To find maximum of $P(r)$: $\frac{d}{dr}P(r) = 0$

$$\Rightarrow A_n \left[2n \, r^{2n-1} \, e^{-2r/na_0} + r^{2n} \left(-\frac{2}{na_0}\right) e^{-2r/na_0} \right] = 0$$

$$\Rightarrow \quad \frac{2n}{r} - \frac{2}{na_0} = 0 \quad \Rightarrow \quad r = n^2 a_0$$

Prob. to find electron is maximum at $n^2 a_0$

8. $\langle L_x \rangle = \langle l \, m | L_x | l m \rangle = 0 \quad$ (given)

$$\langle L_x^2 \rangle = \left\langle \left(\frac{L_+ + L_-}{2} \right)^2 \right\rangle = \frac{1}{4} \langle l m | L_+^2 + L_+ L_- + L_- L_+ + L_-^2 | l m \rangle$$

- $\langle l m | L_\pm^2 | l m \rangle = 0$

- $\langle l m | L_+ L_- | l m \rangle = \hbar \sqrt{(l+m)(l-m+1)} \, \langle l m | L_+ | l, m-1 \rangle$
$$= \hbar \sqrt{(l+m)(l-m+1)} \, \hbar \sqrt{(l-m+1)(l+m)} \, \langle l m | l m \rangle$$

- $\langle L_+ L_- \rangle = \hbar^2 (l+m)(l-m+1)$

- $\langle \ell m | L_- L_+ | \ell m \rangle = \hbar \sqrt{(\ell-m)(\ell+m+1)} \; \langle \ell m | L_- | \ell, m+1 \rangle$

$$= \hbar \sqrt{(\ell-m)(\ell+m+1)} \; \hbar \sqrt{(\ell+m+1)(\ell-m)} \; \langle \ell m | \ell m \rangle$$

- $\langle L_- L_+ \rangle = \hbar^2 (\ell-m)(\ell+m+1)$

- $\langle L_x^2 \rangle = \frac{\hbar^2}{4} \left[(\ell+m)(\ell-m+1) - (\ell-m)(\ell+m+1) \right]$

$$= \frac{\hbar^2}{4} \, 2\left(\ell(\ell+1) - m^2 \right) = \frac{\hbar^2}{2} \left(\ell(\ell+1) - m^2 \right)$$

$\therefore$ - $\sigma_{L_x} = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \frac{\hbar}{\sqrt{2}} \sqrt{\ell(\ell+1) - m^2}$

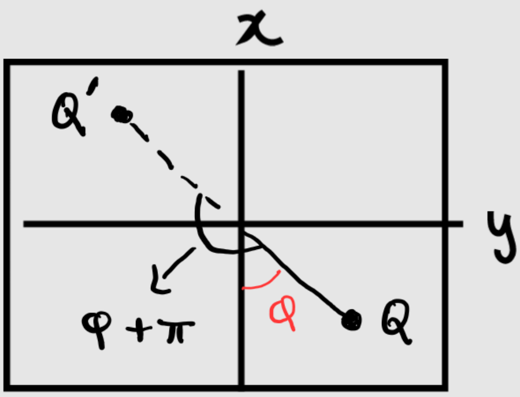
9. Parity operation $P\varphi(x,y,z) = \varphi(-x, -y, -z)$

Problem to be done in two parts.

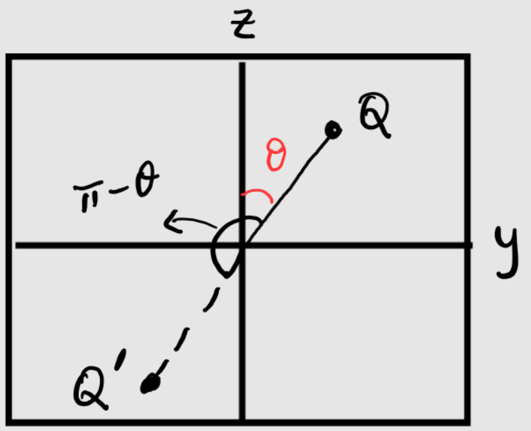
a) $(x,y,z) \rightarrow (-x, -y, -z)$

How does this reflection transform in (r, θ, φ) coordinates

Under parity, point Q goes to Q'



In x-y plane



In y-z plane

We need $x = r \sin\theta \cos\varphi$ $\quad \Rightarrow \quad$ $x' = -r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$ $\qquad\qquad$ $y' = -r \sin\theta \sin\varphi$

$z = r \cos\theta$ $\qquad\qquad$ $z' = -r \cos\theta$

That is,
$$x' = r\sin\theta(-\cos\varphi) = r\sin\theta\cos(\pi+\varphi)$$
$$y' = r\sin\theta(-\sin\varphi) = r\sin\theta\sin(\pi+\varphi)$$
$$z' = r(-\cos\theta) = r\cos(\pi-\theta)$$

(see figure).

The required transformation is $\theta \to \pi - \theta$ and $p \to \varphi + \pi$

b) To evaluate $\mathbb{P}\, Y_\ell^m(\theta,\varphi)$,

put $\theta \to \pi - \theta$ and $\varphi \to \varphi + \pi$ in the expression for $Y_\ell^m(\theta,\varphi)$

$$\mathbb{P}\, Y_\ell^m = Y_l^m(\pi-\theta,\pi+\varphi) = \frac{(-1)^\ell}{2^\ell\,\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\sqrt{\frac{(\ell+m)!}{(2\ell)!\,(\ell-m)!}}$$
$$e^{im(\varphi+\pi)}\,(\sin(\pi-\theta))^{-m}\frac{d^{\ell-m}}{d(\cos(\pi-\theta))^{\ell-m}}(\sin(\pi-\theta))^{2\ell}$$

$$= W(\ell,m)\,(-1)^m\,e^{im\varphi}\,(\sin\theta)^{-m}\,\frac{1}{(-1)^{\ell-m}}\,\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}(\sin\theta)^{2\ell}$$

$$= (-1)^{2m+1}\,W(\ell,m)\,e^{im\varphi}\,(\sin\theta)^{-m}\,\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}(\sin\theta)^{2\ell}$$

$$Y_l^m(\pi-\theta,\pi+\varphi) = (-1)^l Y_l^m(\theta,\varphi) \qquad \text{since } (-1)^{2m} = 1$$

c) Ψ_{200} is even parity state, using these results.