PH-3214; Test : 1

IISER, Pune. (2.2.2026) *Time: 50 minutes. Maximum Marks : 20.*

NOTE 1: Answer all the 7 questions. Use the same symbols/notation given in the questions. Choose only one correct answer for questions 1 to 3. **NOTE 2:** If you are sketching a graph, label the axes. No marks if axes are not labelled. No partial marks for questions 1 to 3. Write clearly and legibly.

1. For a parity-symmetric Hamiltonian in one dimension, which of the following statements is correct about degeneracy of bound states? (2)

   (a) Bound states are non-degenerate only for even states, odd states can be degenerate.
   (b) Bound states are always at least doubly degenerate due to parity.
   (c) Bound states are non-degenerate; each energy eigenvalue corresponds to an even or odd state.
   (d) Each energy eigenvalue corresponds to both an even and an odd eigenstate with the same energy.
   (e) Parity symmetry guarantees infinite degeneracy for each energy level.

2. For the 1D harmonic oscillator, how does the Bohr correspondence principle manifest in the behavior of high-n energy eigenstates $\psi_n(x)$, (n is the quantum number) ? (2)

   (a) The probability density $|\psi_n(x)|^2$ approaches the classical probability distribution $F(x)$, concentrating in regions where the classical particle spends more time.
   (b) The high-n eigenstates become sharply concentrated near $x = 0$, behaving like point particles around $x = 0$.
   (c) The spacing between adjacent quantised energy levels $E_{n+1} - E_n$ increases without bound, matching the unbounded kinetic energy of a classical harmonic oscillator.
   (d) The probability density becomes completely uniform over all space, reflecting that the oscillator can be anywhere with equal likelihood.
   (e) Classical probability distribution $F(x)$ is not related to $|\psi_n(x)|^2$ since classical and quantum dynamics show very different features.

3. Which of the statements is correct about time-reversal and time translation invariance ? (2)

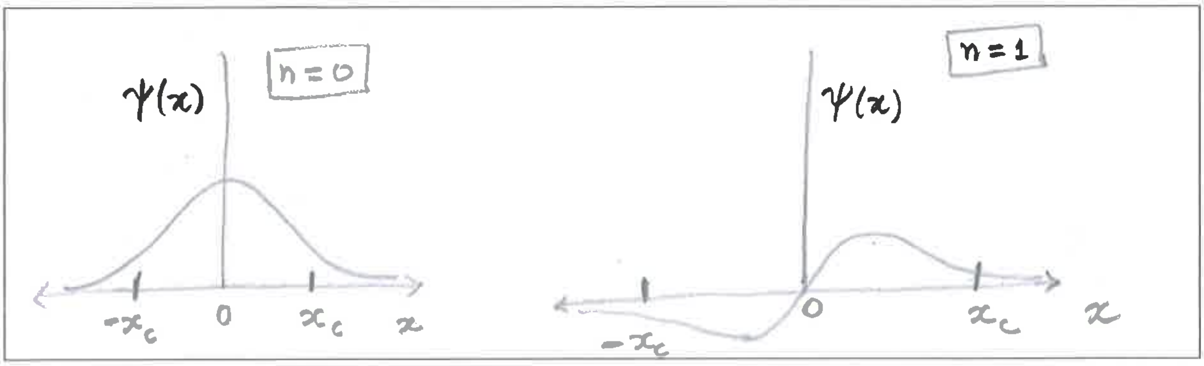
   (a) Time translation invariance of H requires that $[H, L_z] = 0$.
   (b) If $\psi(x,t)$ is a complex quantum state and it satisfies time-dependent Schrodinger equation, then $\psi(x,-t)$ will also be satisfied by the same Schrodinger equation and hence the system is time-reversal invariant.
   (c) Time-reversal invariance requires a Hamiltonian H such that $H^* = H$.
   (d) Time-reversal invariance is the property of eigenstates of the system.
   (e) Time translation invariance means that the initial states do not evolve when translated from t to $t + \delta t$.

For the MCQs 1-3 given above, write your answer only inside the three boxes here:

1 $\boxed{c}$ 2 $\boxed{a}$ 3 $\boxed{c}$

4. Consider the oscillator $H = p^2/2m + m\omega^2 x^2/2$.

(a) Sketch the $n = 0$ and $n = 1$ state of the harmonic oscillator (n is the quantum number). On the same graph, show the location of the classical turning points x_c for both the states. (3)



(b) Write (do not derive) an expression for the classical turning point x_c in terms of n and $\hbar$. (2)

$$x_c = \pm\sqrt{(2n+1)\frac{\hbar}{m\omega}}$$

5. Compute $\langle m|\widehat{X}|n\rangle$, where m and n are harmonic oscillator states. Take $\hbar/2m\omega = 1$. (3)

6. Let $H = p^2/2m + \beta$, where $\beta > 0$ is some constant, be a system that is invariant under translation in 1D. If $\widehat{U} = e^{i\hat{H}t/\hbar}$ is used to used to time evolve some state, then show that $\widehat{U}$ is also invariant under translation. (3)

7. Let $\widehat{U}$ be the operator for infinitesimal rotation by an angle ϵ about z-axis, and $\widehat{X}$ is the position operator. Then, starting from $\widehat{U}^\dagger\ \widehat{X}\ \widehat{U} = \widehat{X} - \epsilon\widehat{Y}$, determine the commutator $[\widehat{X}, \widehat{L}_z]$. (3)

Start writing your answers for questions 5 to 7 below

5. position operator : $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$.

$$\therefore \langle m|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle m|a|n\rangle + \langle m|a^+|n\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\langle m|n-1\rangle + \sqrt{n+1}\,\langle m|n+1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right)$$

$$= \left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right)$$ Since $\frac{\hbar}{2m\omega} = 1$.

6. $H = \frac{p^2}{2m} + \beta \qquad (\beta > 0)$.

T is the translation operator such that $\hat{T}|x\rangle = |x+\epsilon\rangle$

If H is invariant under spatial translation, then $[H, \hat{T}] = 0$

If $\hat{U} = e^{-iHt/\hbar}$, then we need to show that $[T, \hat{U}] = 0$

To show this, write $\hat{U} = \sum_{n=0}^{\infty} \frac{(-it/\hbar)^n}{n!} H^n$

We need to show that $[T, U] = 0$

Now, $[\hat{U}, \hat{T}] = \sum_{n=0}^{\infty} \frac{(-it/\hbar)^n}{n!} [H^n, T]$

Implies that we must show that $[H^n, T] = 0$ for all n,

given that it is true for $n = 1$.

By induction: Assume $[H^k, T] = 0$ for $k \leq n$.

Then, $[H^{n+1}, T] = H^n [H, T] + [H^n, T] H$

$= H^n \; 0 + 0 \; H = 0$

$\therefore [H^n, T] = 0$ for all n. Hence $[\hat{U}, \hat{T}] = 0$.

<u>Another method:</u> we need to show $[U, T] = e^{-iHt/\hbar} T - T e^{-iHt/\hbar} = 0$

i.e $e^A B = \sum_n \frac{A^n}{n!} B$. If $[A, B] = 0$, then

$e^A B = B e^A$.

Hence, $[U, T] = 0$.

7. $\hat{U} = I - \frac{i\varepsilon}{\hbar} L_z$

$$U^{+} X U = X - \varepsilon Y$$

$$\left(I + \frac{i\varepsilon}{\hbar} L_z\right) X \left(I - \frac{i\varepsilon}{\hbar} L_z\right) = X - \varepsilon Y$$

$$X - \frac{i\varepsilon}{\hbar} X L_z + \frac{i\varepsilon}{\hbar} L_z X = X - \varepsilon Y$$

$$\frac{i\varepsilon}{\hbar}\left(L_z X - X L_z\right) = -\varepsilon Y$$

$$\frac{i\varepsilon}{\hbar}\left(X L_z - L_z X\right) = \varepsilon Y$$

$$\therefore \quad [X, L_z] = \frac{\hbar}{i} Y \quad \text{or} \quad [X, L_z] = -i\hbar Y$$

$$\text{or} \quad [L_z, X] = i\hbar Y$$