Laplacian operator in cylindrical coordinates in 2D

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Rightarrow \nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}$$

Let $x = \rho \cos\varphi$, $y = \rho \sin\varphi$

$$\rho = \sqrt{x^2+y^2} \qquad \cos\varphi = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$$

STEP-1: Cartesian to cylindrical operators

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} \quad \text{AND} \quad \frac{\partial}{\partial y} = \frac{\partial \rho}{\partial y}\frac{\partial}{\partial \rho} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi}$$

STEP-2: Calculate the derivatives – $\frac{\partial \rho}{\partial x}$ etc.

$$\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \cos\varphi = \frac{x}{\rho}$$

$$\frac{\partial \rho}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}} = \sin\varphi = \frac{y}{\rho}$$

With implicit derivatives:

i.e $\quad d(\cos\varphi) = d\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

$$\Rightarrow -\sin\varphi \, d\varphi = \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}}\right)dx + (\quad)\,dy$$

But $dy = 0$ since we only want $\frac{\partial \varphi}{\partial x}$ i.e, y is held constant. ($dy = 0$).

$$-\sin\varphi \frac{\partial \varphi}{\partial x} = \frac{1}{\rho}\left(\frac{y^2}{x^2+y^2}\right) = \frac{1}{\rho}\left(\frac{y^2}{\rho^2}\right) = \frac{1}{\rho}\sin^2\varphi$$

$$\therefore \frac{\partial \varphi}{\partial x} = -\frac{1}{\rho}\sin\varphi$$

Similarly, use $d(\sin\varphi) = d\left(\frac{y}{\sqrt{x^2+y^2}}\right)$ to get

$$\frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{\rho}$$

STEP-3 ②

$$\frac{\partial^2}{\partial x^2} = \left(\cos\varphi \frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$\frac{\partial^2}{\partial y^2} = \left(\sin\varphi \frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$

Schrodinger equation for Rotationally invariant problem $V(\rho,\varphi) = V(\rho)$

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y)\right]\psi(x,y) = E\,\psi(x,y)$$

In (ρ,φ) coordinate system:

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right) + V(\rho)\right]\psi(\rho,\varphi) = E\,\psi(\rho,\varphi)$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) + V(\rho)\,\psi - \frac{\hbar^2}{2\mu}\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} = E\,\psi$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) + \left(V(\rho) - E\right)\psi = \frac{\hbar^2}{2\mu}\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}$$

$$-\rho^2\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) + \frac{2\mu}{\hbar^2}\rho^2\left(V(\rho) - E\right)\psi = \frac{\partial^2\psi}{\partial\varphi^2}$$

$$\psi(\rho,\varphi) = R(\rho)\,\Phi_m(\varphi) = R(\rho)\,\frac{e^{im\varphi}}{\sqrt{2\pi}} \quad ---- ①$$

$$---- ②$$

NOTE: $$-\frac{\partial^2}{\partial\varphi^2}\Phi_m(\varphi) = m^2\,\Phi_m(\varphi)$$

where $L_z = -i\hbar\frac{\partial}{\partial\varphi}$ and $\Phi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$

$$\rho^2\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\psi(\rho,\varphi) - \frac{2\mu}{\hbar^2}\rho^2\left(V(\rho)-E\right)\psi(\rho,\varphi) = -\frac{\partial^2\psi(\rho,\varphi)}{\partial\varphi^2}$$

Now, substitute from Eq. (1):

$$\rho^2\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)R(\rho)\,\Phi_m(\varphi) - \frac{2\mu}{\hbar^2}\rho^2\left(V(\rho)-E\right)R(\rho)\,\Phi_m(\varphi) = -\frac{\partial^2}{\partial\varphi^2}R(\rho)\,\Phi(\varphi)$$

$$\Phi_m(\varphi)\,\rho^2\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)R(\rho) - \frac{2\mu}{\hbar^2}\Phi_m(\varphi)\,\rho^2\left(V(\rho)-E\right)R(\rho) = -R(\rho)\frac{\partial^2}{\partial\varphi^2}\Phi_m(\varphi)$$

Substitute from Eq (2) for $-\frac{\partial^2}{\partial\varphi^2}\Phi_m(\varphi)$ & divide by ρ^2 throughout

$$\cancel{\Phi_m(\varphi)}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)R(\rho) - \frac{2\mu}{\hbar^2}\cancel{\Phi_m(\varphi)}\left(V(\rho)-E\right)R(\rho) = \frac{m^2}{\rho^2}\cancel{\Phi_m(\varphi)}\,R(\rho)$$

This gives

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)R(\rho) + V(\rho)\,R(\rho) = E\,R(\rho)$$

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right) + V(\rho)\right]R(\rho) = E\,R(\rho)$$

This eigenvalue equation is called Radial Schrödinger equation.

$$\left(\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} - \frac{m^2}{\rho^2}\psi\right) - \frac{2\mu}{\hbar^2}\left(V(\rho) - E\right)\psi = 0$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{\partial \rho} - \frac{m^2}{\rho^2}\right)\psi + V(\rho)\,\psi = E\,\psi$$

Radial part of the Schrodinger equation.