Name: **Roll Number:**

IISER Pune; PH-3214; End-semester exam

23.4.2026 *Time: 2 hours.* *Maximum Marks : 50.*

NOTE 1: Among questions 9 to 11, answer ANY TWO questions. Use the same symbols/notation given in the questions. For MCQs 1-5, choose only one correct answer.
NOTE 2: No partial marks for questions 1 to 5. Show all steps of calculations for questions 6 to 11. Write clearly and legibly.

1. A particle is in the angular momentum state $|l, m = l\rangle$, which is a simultaneous eigenstate of L^2 and L_z. In this state, what is the product of uncertainties $\Delta L_x \Delta L_y$? (3)

   (a) $\frac{\hbar^2}{2}\sqrt{l(l+1)}$
   (b) $\hbar^2 l(l+1)$
   (c) $\hbar^2 l$
   (d) 0
   (e) $\frac{\hbar^2 l}{2}$

2. For ground state of hydrogen-like system, the radial probability density is $P(r) = A\, r^2 e^{-4r/a_0}$. Let $r_{\max}$ be the radius where $P(r)$ is maximum, and let $\langle r\rangle$ be the expectation value of r. Which of the following is correct? (3)

   (a) $r_{\max} = \frac{a_0}{2}$, $\langle r\rangle = \frac{3a_0}{8}$
   (b) $r_{\max} = 0$, $\langle r\rangle = \frac{3a_0}{4}$
   (c) $r_{\max} = \frac{a_0}{2}$, $\langle r\rangle = \frac{3a_0}{4}$
   (d) $r_{\max} = a_0$, $\langle r\rangle = \frac{3a_0}{2}$
   (e) $r_{\max} = \frac{3a_0}{4}$, $\langle r\rangle = \frac{a_0}{2}$

3. A particle is in a normalized state $|\psi\rangle$ with $\langle x\rangle = x_0, \langle p\rangle = p_0, (\Delta x)^2 = \sigma_x^2$. It is translated using $\hat{T}(\epsilon) = \exp\left(-\frac{i\epsilon\hat{p}}{\hbar}\right)$ where $\epsilon > 0$ is real. The translated state is $|\psi'\rangle = \hat{T}(\epsilon)|\psi\rangle$. Choose the correct mean position $\langle x\rangle'$ and variance $(\Delta x')^2$ after translation ? (3)

   (a) $\langle x\rangle' = x_0 + \epsilon, \quad (\Delta x')^2 = \sigma_x^2 + \epsilon^2$
   (b) $\langle x\rangle' = x_0 - \epsilon, \quad (\Delta x')^2 = \sigma_x^2 + p_0\epsilon^2$
   (c) $\langle x\rangle' = x_0 + \epsilon p_0, \quad (\Delta x')^2 = \sigma_x^2 + p_0\epsilon^2$
   (d) $\langle x\rangle' = x_0 + \epsilon p_0, \quad (\Delta x')^2 = \sigma_x^2 + p_0\epsilon^2$
   (e) $\langle x\rangle' = x_0 + \epsilon, \quad (\Delta x')^2 = \sigma_x^2$

4. The normalized trial wavefunction $\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$, with $\alpha > 0$, is used to estimate the ground-state energy via the variational principle for two one-dimensional systems:

System I with $V_1(x) = \frac{1}{2}m\omega^2x^2$, and System II with $V_2(x) = \frac{1}{2}k|x|$, $k > 0$. Which of the following statements is correct? (3)

(a) $\psi_\alpha(x)$ gives the exact ground-state energy for System I, but only an upper bound for System II.
(b) $\psi_\alpha(x)$ gives the exact ground-state energy for both System I and System II.
(c) $\psi_\alpha(x)$ cannot be used for System II because $|x|$ is not differentiable at $x = 0$.
(d) $\psi_\alpha(x)$ gives the exact ground-state energy for System II, but only an upper bound for System I.
(e) $\psi_\alpha(x)$ gives only upper bounds for both System I and System II, and is never exact for either system.

5. For the hydrogen atom, consider all bound states with principal quantum number $n = 4$. How many linearly independent eigenstates simultaneously have magnetic quantum number $m = 0$, electron spin projection $m_s = +\frac{1}{2}$, and odd parity? (3)

(a) 4
(b) 1
(c) 3
(d) 2
(e) 0

For the MCQs 1-5 given above, write your answer only inside the boxes here:

1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐

For questions 6 to 11 given below, show all the steps of the calculations.

6. Obtain a general expression for the matrix element $\langle n|\hat{X}|m\rangle$, where n and m are harmonic oscillator eigenstates. (5)

7. Consider the Hamiltonian of two spins:

$$H = -\left(\gamma_1\vec{S}_1 + \gamma_2\vec{S}_2\right).\vec{B}, \quad \text{and} \quad \vec{B} = B_0\hat{k}.$$

In this, γ_1, γ_2 and B_0 are constants. Find all the eigenvalues of H in direct product basis. (5)

8. Consider a particle without spin in a state given by

$$\psi(x, y, z) = \sqrt{\frac{1}{8\pi}}(x + y + 2z)\ e^{-\alpha r}, \quad \text{with} \quad r = \sqrt{x^2 + y^2 + z^2}.$$

In this, $\alpha > 0$. Find the total angular momentum quantum number l of the particle. (7)

9. Consider a bosonic system of two identical particles in an infinite square well of width L. One particle is in ground state $\phi_1(x)$ and the other is in an excited state $\phi_2(x)$.
(a) Assuming the positions of particles to be x_1 and x_2, write down the normalised symmetric state $\Psi(x_1, x_2)$ for this system.

(b) Now, let $x_1 = x_2 = x$. Show that the probability density for the two particles to be in the same position x is of the form,

$$P_S = C\ \sin^2(\theta)\ \sin^2(2\theta),$$

and explicitly obtain the values of C and θ in terms of L and x.
(c) Find P_S if $x = L/3$. (2+5+2)

10. A simple harmonic oscillator is initially in its ground state $|0\rangle$ at $t = -\infty$. It is perturbed by a small, time-dependent potential $V(t) = -\epsilon x e^{-t/\tau}$, where $\epsilon > 0, \tau > 0$ are constants. What is the probability of finding the oscillator in the first excited state at $t = \infty$. (9)

11. Consider the system given the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{V_0}{a}|x|,$$

where V_0 and a are constants. WKB method estimtes energy as

$$E_n = K\left(n + \frac{1}{2}\right)^{\alpha}.$$

(a) Determine K and α.
(b) If $a = 1/4, m = 1/2$ and $V_0\hbar = \frac{1}{3\pi}$, then what is the value of K.
(c) For the choice of parameters given in (b), how many discrete energy levels are there below energy $E = 4$. (5+2+2)

Some useful information:

First few spherical harmonics:

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\ e^{\pm i\phi}, \quad Y_1^0 = \mp\sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\ e^{\pm i\phi}$$

Hydrogen atom wavefunction : $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\ Y_l^m(\theta,\phi)$.

Ladder operators acting on $|lm\rangle$: $\quad L_\pm|lm\rangle = \hbar\left[(l \mp m)(l \pm m + 1)\right]^{1/2}|l, m \pm 1\rangle$

Creation and annihilation operators : $a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

Spherical-polar coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,-1\rangle$$

Clebsch-Gordan Coefficients:

J, M	(m_1, m_2)	CG Coefficient
$\frac{3}{2},\frac{3}{2}$	$(\frac{1}{2},1)$	1
$\frac{3}{2},\frac{1}{2}$	$(\frac{1}{2},0)$	$\sqrt{\frac{2}{3}}$
	$(-\frac{1}{2},1)$	$\sqrt{\frac{1}{3}}$
$\frac{3}{2},-\frac{1}{2}$	$(-\frac{1}{2},0)$	$\sqrt{\frac{2}{3}}$
	$(\frac{1}{2},-1)$	$\sqrt{\frac{1}{3}}$
$\frac{3}{2},-\frac{3}{2}$	$(-\frac{1}{2},-1)$	1
$\frac{1}{2},\frac{1}{2}$	$(\frac{1}{2},0)$	$\sqrt{\frac{1}{3}}$
	$(-\frac{1}{2},1)$	$-\sqrt{\frac{2}{3}}$
$\frac{1}{2},-\frac{1}{2}$	$(-\frac{1}{2},0)$	$\sqrt{\frac{1}{3}}$
	$(\frac{1}{2},-1)$	$-\sqrt{\frac{2}{3}}$

Q7

~~Q3~~ If $H = -\left(\gamma_1\vec{S}_1 + \gamma_2\vec{S}_2\right)\cdot\vec{B}$ and $\vec{B} = B_0\hat{k}$, then find the eigenvalues of H using direct product basis.

Soln: Since $\vec{B}$ is in the z-direction, only the S_z components contribute:

$$H = -B_0\left(\gamma_1 S_{1z} + \gamma_2 S_{2z}\right).$$

The spin states of each particle are given by:

$$S_{1z}|s_1,m_1\rangle = m_1|s_1,m_1\rangle, \quad S_{2z}|s_2,m_2\rangle = m_2|s_2,m_2\rangle.$$

Thus in product basis $|m_1,m_2\rangle$ the Hamiltonian acts as:

$$H|m_1,m_2\rangle = -B_0(\gamma_1 m_1 + \gamma_2 m_2)|m_1,m_2\rangle.$$

Product Basis: For two spin-$\frac{1}{2}$ particles, the possible values of m_1 and m_2 are:

$$m_1, m_2 \in \left\{\frac{1}{2}, -\frac{1}{2}\right\}.$$

$$|++\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle, \quad |+-\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle, \quad |-+\rangle = \left|-\frac{1}{2},\frac{1}{2}\right\rangle, \quad |--\rangle = \left|-\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Applying the Hamiltonian to each basis state

$$H|++\rangle = -B_0\left(\gamma_1\cdot\frac{1}{2} + \gamma_2\cdot\frac{1}{2}\right)|++\rangle$$

$$= -\frac{B_0}{2}(\gamma_1+\gamma_2)|++\rangle.$$

$$E_{++} = -\frac{B_0}{2}(\gamma_1+\gamma_2), \quad \implies \text{Eigenvalue of } |++\rangle.$$

Similarly

$$H|+-\rangle = -B_0\left(\gamma_1 \cdot \frac{1}{2} + \gamma_2 \cdot \left(-\frac{1}{2}\right)\right)|+-\rangle$$
$$= -\frac{B_0}{2}(\gamma_1 - \gamma_2)|+-\rangle.$$

$$E_{+-} = -\frac{B_0}{2}(\gamma_1 - \gamma_2) \quad \Longrightarrow \text{ Eigenvalue of } |+-\rangle.$$

For $|-+\rangle$:

$$H|-+\rangle = -B_0\left(\gamma_1 \cdot \left(-\frac{1}{2}\right) + \gamma_2 \cdot \frac{1}{2}\right)|-+\rangle$$
$$= -\frac{B_0}{2}(-\gamma_1 + \gamma_2)|-+\rangle.$$

Eigenvalue for $|-+\rangle$ is:

$$E_{-+} = -\frac{B_0}{2}(-\gamma_1 + \gamma_2) = \frac{B_0}{2}(\gamma_1 - \gamma_2).$$

For $|--\rangle$:

$$H|--\rangle = -B_0\left(\gamma_1 \cdot \left(-\frac{1}{2}\right) + \gamma_2 \cdot \left(-\frac{1}{2}\right)\right)|--\rangle$$
$$= -\frac{B_0}{2}(-\gamma_1 - \gamma_2)|--\rangle.$$

Eigenvalue for $|--\rangle$ is:

$$E_{--} = \frac{B_0}{2}(\gamma_1 + \gamma_2).$$

Basis State	Eigenvalue
$\|++\rangle$	$-\frac{B_0}{2}(\gamma_1 + \gamma_2)$
$\|+-\rangle$	$-\frac{B_0}{2}(\gamma_1 - \gamma_2)$
$\|-+\rangle$	$\frac{B_0}{2}(\gamma_1 - \gamma_2)$
$\|--\rangle$	$\frac{B_0}{2}(\gamma_1 + \gamma_2)$

Q4) The Hamiltonian describing the hyperfine interactions is $H_{hf} = A\vec{S}_1 \cdot \vec{S}_2$ $(A > 0)$. The total Hamiltonian $H = H_{\text{coloumb}} + H_{hf}$ Show that H_{hf} splits the ground state into two levels

$$E_+ = -R_y + \frac{\hbar^2 A}{4} \text{ and } E_- = -R_y - \frac{3\hbar^2 A}{4}$$

soln:

The Hamiltonian describing the hyperfine interaction is:

$$H_{hf} = A\vec{S}_1 \cdot \vec{S}_2, \quad A > 0.$$

The total Hamiltonian is given by:

$$H = H_{\text{Coulomb}} + H_{hf}.$$

Expressing H_{hf} in Terms of Total Spin $\vec{S}$

Q8 Consider particle w/o spin with $\psi = K(x+y+2z)e^{-\alpha r}$

$r = \sqrt{x^2+y^2+z^2}$, K & α are real constants. $K = \sqrt{\frac{1}{8\pi}}$

(a) Find total ang mom. of the particle.

(b) If z-component of ang. mom. L_z is measured, what is the prob. that the result would be $L_z = +\hbar$?

check { You may need: $Y_0^0 = \sqrt{\frac{1}{4\pi}}$ $\quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$

$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \quad Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$

$$\psi(\theta,\varphi) = \sqrt{\frac{1}{8\pi}}\left(\cos\varphi\sin\theta + \sin\varphi\sin\theta + 2\cos\theta\right) r e^{-\alpha r}$$

$\| \; x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\theta$

$$\psi(\theta,\varphi) = \sqrt{\frac{1}{8\pi}}\left(\sin\theta\,\frac{e^{i\varphi}+e^{-i\varphi}}{2} + \sin\theta\,\frac{e^{i\varphi}-e^{-i\varphi}}{2i} + 2\cos\theta\right)$$

$$= \sqrt{\frac{1}{8\pi}}\left(\frac{\sin\theta\, e^{i\varphi}}{2}\left(1+\frac{1}{i}\right) + \frac{\sin\theta\, e^{-i\varphi}}{2}\left(1-\frac{1}{i}\right) + 2\cos\theta\right)$$

$$= \sqrt{\frac{1}{8\pi}}\left(\frac{\sin\theta\, e^{i\varphi}}{2}(1-i) + \frac{\sin\theta\, e^{-i\varphi}}{2}(1+i) + 2\cos\theta\right)$$

$$= -\frac{1-i}{2\sqrt{3}}\left(\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}\right) + \frac{1+i}{2\sqrt{3}}\left(\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}\right) +$$

$$\sqrt{\frac{1}{8\pi}}\, 2\sqrt{\frac{4\pi}{3}}\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)$$

$$\psi(\theta,\varphi) = \left(\frac{-(1-i)}{2\sqrt{3}}Y_1^1 + \frac{1+i}{2\sqrt{3}}Y_1^{-1} + \sqrt{\frac{2}{3}}Y_1^0\right) r e^{-\alpha r}$$

$\sqrt{L^2} = \sqrt{l(l+1)}\,\hbar = \sqrt{2}\,\hbar$ Since wave fn corresponds to $l=1$.

(10) ~~$\dot{d}(t) = -\frac{i}{\hbar} H_{ab}\, e^{-i\omega_0 t}$~~

$$d_f(t) = \frac{-i}{\hbar} \int_0^\infty \langle f^0 | V | i^0 \rangle\, e^{i(\omega_{fi})t'}\, dt'$$

$$V(t) = -\varepsilon\, x\, e^{-t/\tau}$$

$$= -\frac{i}{\hbar} \langle 0 | \hat{x} | 1 \rangle \int_0^\infty -\varepsilon\, e^{-t'/\tau}\, e^{i(\omega_{fi}\, t')}\, dt'$$

$$= \frac{-i}{\hbar} \langle 0 | x | 1 \rangle (-\varepsilon) \int_0^\infty e^{i\omega_{fi} t'}\, e^{-t/\tau}\, dt'$$

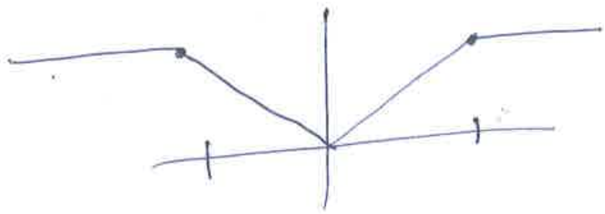
$$= \frac{i\varepsilon}{\hbar} \langle 0 | x | 1 \rangle \frac{1}{\frac{1}{\tau} + \omega_{fi}\, i}$$ $t \to \infty$

$$d_f(t) = \frac{i\varepsilon}{\hbar} \langle 0 | x | 1 \rangle \frac{1}{(1/\tau + i\omega_{fi})} \times \frac{(1/\tau - i\omega_{fi})}{(1/\tau - i\omega_{fi})}$$

$$P_f(t) = \frac{\varepsilon^2}{\hbar^2} |\langle 0 | x | 1 \rangle|^2 \frac{\left(\frac{1}{\tau^2} + \omega_{fi}\right)}{\left(\frac{1}{\tau^2} + \omega_{fi}^2\right)^2}$$

$$\boxed{P_f(t) = \frac{\varepsilon^2}{\hbar^2} |\langle 0 | x | 1 \rangle|^2 \frac{1}{\left(\frac{1}{\tau^2} + \omega_{fi}\right)^2}}$$

(11)



~~E = n + ½~~ $\oint p\,dq = \left(n+\frac{1}{2}\right)\hbar\omega$

$$H = \frac{p^2}{2m} + \frac{V_0}{a}|x| = E$$

$$\oint p\,dq = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$x_c = \frac{aE}{V_0}$$

$$4\int_0^{x_c} p\,dq = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$4\int_0^{x_c} \sqrt{2mE}\sqrt{1-\frac{V_0}{aE}x}\;dx$$

put $x = u\,\frac{aE}{V_0}$

$$4\int_0^1 \sqrt{2mE}\sqrt{1-u}\;du\,\frac{aE}{V_0}$$

$$4\sqrt{2m}\,\frac{a}{V_0}\,E^{3/2}\underbrace{\int_0^1\sqrt{1-u}\,du}_{\frac{2}{3}}$$

$$\frac{8}{3}\sqrt{2}\,\frac{\sqrt{m}\,a}{V_0}\,E^{3/2} = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$\therefore\quad E^{3/2} = \frac{3V_0\hbar\omega}{8\sqrt{2m}\,a}\left(n+\frac{1}{2}\right)$$

$$E_n = \left(\frac{3V_0\hbar\omega}{8\sqrt{2m}\,a}\right)^{2/3}\left(n+\frac{1}{2}\right)^{2/3}$$

Ask n_{max} for given set of parameters/

$$E_n = K\left(n+\frac{1}{2}\right)^{2/3}$$