Lecture 1

* Let's recall a few definitions:

   - Let $\Omega \subset_{open} \mathbb{C}$ be a domain. We say that $f : \Omega \to \mathbb{C}$ is <u>holomorphic</u> ($\Leftrightarrow$ complex analytic) on Ω if

$$f'(z) := \cancel{\lim_{\Omega \setminus \{z\} \to}} \lim_{\mathbb{C}^* \ni \zeta \to 0} \frac{f(z+\zeta) - f(z)}{\zeta}$$

   exists at <u>each</u> $z \in \Omega$, and $f' : \Omega \to \mathbb{C}$ is cont.

   - We need the "f' is cont." part of the definition to derive the <u>CR-equations</u>:

$$\underset{u+iv}{\overset{\|}{f}} \in \mathcal{O}(\Omega) \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

   - However (by Goursat), the existence of f' implies that $f' \in \mathcal{C}(\Omega)$ automatically, so

$$f \in \mathcal{O}(\Omega) \Leftrightarrow f'(z) \text{ exists at each } z \in \Omega$$

$\Leftrightarrow f$ ~~satisfies C~~ is such that 1st partial-derivatives exist & satisfy CR-equations.

   - DEFINE $\partial/\partial\bar{z}$.

* DEFINE "POLYDISC".

* <u>3 possible generalizations of holomorphicity</u>

<u>Option 1</u>: $f : \Omega \to \mathbb{C}$ is said to be holomorphic if, at each $z \in \Omega$, $\exists$ a complex linear functional $\lambda_z \in (\mathbb{C}^n)^*$ s.t.

$$\lim_{\mathbb{C}^* \ni \zeta \to 0} \frac{f(z+\zeta V) - f(z) - \lambda_z(V)\zeta}{\zeta} = 0 \quad \forall\ V \in \mathbb{C}^n.$$

<u>Option 2</u>: $f : \Omega \to \mathbb{C}$ is said to be holomorphic if all 1st-order partial derivatives exist and

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \text{ on } \Omega \text{ for } j = 1, 2, \ldots, n.$$

Option 3 : $f: \Omega \to \mathbb{C}$ is holomorphic if, ~~##~~ at each $z_0 \in \Omega$, $\exists$ a polydisc $\Delta(z_0;\vec{R})$, with $\Delta(z_0;\vec{R}) \subset \Omega$, ~~such that ther~~ and a power-series representation

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z-z_0)^\alpha, \quad \forall z \in \Delta(z_0;\vec{R})$$

that is absolutely convergent, and cvgs. unif. on compacts $\subsetneq \Delta(z_0;\vec{R})$.

Which is the "correct" definition of holomorphicity?

Ans. All defn's. are equivalent!

* SHOW A PROOF OF THE "IDENTITY PRINCIPLE"

* Theorem : Let $\varepsilon, \delta : 0<\varepsilon, \delta<1$. Every function that is holo. in

$$\Omega := \cancel{\{\underbrace{(z_1, z_2, \dots, z_n)}_{z^*} \in \mathbb{C}^n \mid}$$

$$\Delta(0;(1,\delta,\dots,\delta,\delta)) \cup \mathrm{Ann}(0;1\pm\varepsilon)\times\mathbb{D}^{n-1}$$

extends to a holomorphic fn. in $\mathbb{D}^n$.

* This is most certainly NOT true in one variable !!

s.t. $\partial\Omega$ contains no isolated points.

- Fact 1: Let $\Omega \subset\subset \mathbb{C}$ be a bounded domain. Then, $\exists f \in \mathcal{O}(\Omega)$ s.t. there is no point $\zeta \in \partial\Omega$ that admits a disc $\mathbb{D}_\zeta$ centered at ζ and an $F \in \mathcal{O}(\mathbb{D}_\zeta)$ s.t. $F|_{\mathbb{D}_\zeta \cap \Omega} = f$.

Proof : Let $\mathcal{S} :=$ a countable dense subset of $\partial\Omega$. Let $A_0 = \phi$, and let A_j be defined recursively as follows:

$$\left.\begin{array}{l} A_j \supsetneq A_{j-1}, \\ A_j \setminus A_{j-1} \text{ is a finite set } F^{(j)} \text{ s.t.,} \\ \bigcup_{a \in F^{(j)}} D(a;1/j) \supseteq \partial\Omega. \end{array}\right\} \; j \geq 1.$$

Construct a sequence $\{z_j\}_{j \in \mathbb{N}}$ by choosing its first $\#(F^{(1)})$ elements — one from each of the balls $D(a;1)$, $a \in F^{(1)}$ — and not lying in $\bigcup_{a \in F^{(2)}} D(a;1/2)$; its next $\#(F^{(2)})$ elements — one from each of the balls $D(a;1/2)$, $a \in F^{(2)}$ — and not lying in $\bigcup_{a \in F^{(3)}} D(a;1/3)$; etc.

By construction, $\{z_j : j\in\mathbb{N}\}$ is discrete & $\overline{\{z_j : j\in\mathbb{N}\}} = \partial\Omega$.

By Weierstrass' Interpolation Theorem, $\exists\, h\in\mathcal{O}(\Omega)$ s.t. $h(z_j)=0\ \forall j\in\mathbb{N}$ AND HAS NO OTHER ZEROS. Suppose $\exists\, \zeta\in\partial\Omega$, ~~and a~~ an $r>0$, and a function $H : D(\zeta;r) \xrightarrow[\text{holo.}]{} \mathbb{C}$ s.t. $H|_{D(\zeta;r)\cap\Omega} = h$. By continuity

$$H(p) = \lim_{z_\nu^p \to p} h(z_\nu^p) = 0 \text{ for some subseq. } \{z_\nu^p\} \subsetneq \{z_j\}_{j\in\mathbb{N}}.$$

Thus $H\equiv 0$, which is a contradiction.

[Proved]

Lecture 2

* RECAPITULATE FACT 1 BEFORE PROCEEDING.

* All this raises the following question: What is the maximal domain of existence of the collection $\mathcal{O}(\Omega)$ if Ω was as in the previous theorem?

- This is a difficult question as we have to make sense of the word "maximal".
- Consider the following problem:

$$\Omega := D(1; 1/2)$$
$$f : z \longmapsto \log|z| + i\,\mathrm{Arg}(z)$$

where $\mathrm{Arg}(\cdot)$ is measured CCW from the +ve real axis.

f extends to a $f_1 \in \mathcal{O}(\widetilde{\Omega}_1)$, $\widetilde{\Omega}_1 := \mathbb{C} \setminus \{te^{2\pi i/3} \mid t \in [0,\infty)\}$

—— " —— $f_2 \in \mathcal{O}(\widetilde{\Omega}_2)$, $\widetilde{\Omega}_2 := \mathbb{C} \setminus \{te^{-2\pi i/3} \mid t \in [0,\infty)\}$

f_j does not extend past # $\partial\widetilde{\Omega}_j$, $j = 1,2$ & yet, as $\widetilde{\Omega}_1 \not\subset \widetilde{\Omega}_2$ & $\widetilde{\Omega}_2 \not\subset \widetilde{\Omega}_1$, neither $\widetilde{\Omega}_1$ nor $\widetilde{\Omega}_2$ are maximal!

- So we will pass upon the question of defining the "maximal of domain of existence" and simply state that — in a well-defined sense — it exists & is called the envelope of holomorphy.

* We have shown that $\mathrm{Env}[\Delta(0; 1, \delta, \dots, \delta) \bigcup \mathrm{Ann}(0; 1 \pm \varepsilon) \times \mathbb{D}^{n-1}] = \mathbb{D}^n$.

Reason: Let $\varphi_0 \in \mathcal{O}(\mathbb{D})$ be such that it does not continue holomorphically past $\partial\mathbb{D}$. Then $F : (z_1, z_2, \dots, z_n) \longmapsto \sum_{j=1}^{n} \varphi_0(z_j)$ is a function for which $\mathbb{D}^n$ is the maximal domain of existence.

* However, $\Delta(0; 1, \delta, \dots, \delta) \bigcup \mathrm{Ann}(0; 1 \pm \varepsilon) \times \mathbb{D}^{n-1}$ has a GREAT DEGREE of rotational symmetry. Have we any hope of exactly determining $\mathrm{Env}(\Omega)$ when Ω has no rotational symmetry?

* A classical theorem: Let $\varphi : \overline{\mathbb{D}} \to \mathbb{C}^{n-1}$ be a non-const. fn. such that $\varphi \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$. Suppose $\max_{\partial\mathbb{D}} |\varphi_j| < 1$. ~~Write~~ Let $\mathscr{D}$ be ~~a~~ a connected open set s.t.

$$\mathrm{gr}(\varphi) \bigcup \partial\mathbb{D} \times \mathbb{D}^{n-1} \subset\subset \mathscr{D}. \quad \text{Now set } \Omega := \mathscr{D} \cap \Delta(0; (1+\varepsilon, 1, \dots, 1)).$$

Then, every $f \in \mathcal{O}(\Omega)$ extends to a holo. fn. in $\mathbb{D}^n$.

Proof: Let $\tilde{\varepsilon}>0$ be so small that $\mathrm{Ann}(0;1\pm\tilde{\varepsilon})\times\mathbb{D}^{n-1}\subset\Omega$. If we write

$$f_{z^*}: z_1 \longmapsto f(z_1, \underbrace{z_2,\dots,z_n}_{z^*}) \quad \text{for } z^*\in\mathbb{D}^{n-1}, \underline{\text{fixed}},$$

$f_{z^*}\in\mathcal{O}(\mathrm{Ann}(0;1\pm\tilde{\varepsilon}))$. I.e.

f_{z^*} admits a Laurent expansion:

$$f(z_1)=\sum_{j\in\mathbb{Z}} A_j(z^*)\, z_1^{j} \quad \forall z_1\in \mathrm{Ann}(0;1\pm\tilde{\varepsilon}), \text{ and}$$

$$A_j(z^*)=\frac{1}{2\pi i}\oint_{\partial\mathbb{D}}\frac{f(\zeta,z^*)}{\zeta^{j+1}}\,d\zeta \qquad \text{——— } \underline{(1)}$$

As in the earlier proof, the integrand is sufficiently well behaved as to admit differentiation under the integral sign. So

$$\frac{\partial A_j(z^*)}{\partial z_k}=\frac{1}{2\pi i}\oint_{\partial\mathbb{D}}\frac{1}{\zeta^{j+1}}\frac{\partial}{\partial z_k}f(\zeta,z^*)\,d\zeta=0 \quad \forall k=2,\dots,n, \ \forall z^*\in\mathbb{D}^{n-1}.$$

whence, by Option 2, $\forall j$ $A_j\in\mathcal{O}(\mathbb{D}^{n-1})$.

Let $\delta<<1$ be so small that:

$$* \ \bigcup_{z_1\in\mathbb{D}}\{z_1\}\times\left[D(0;\delta)^{n-1}+\varphi(z_1)\right]\subsetneq\Omega$$

$$* \ \zeta\varphi(z_1)+D(0;\delta)^{n-1}\subsetneq\Omega \quad \forall\, \cancel{z_1\in\partial\mathbb{D}}\ z\in\mathrm{Ann}(0;1\pm\tilde{\varepsilon}/2) \ \& \ \forall\,\zeta\in D(0;1+\delta) \quad (*)$$

Arguing as before,

$$a_j^W(x):=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{f(\zeta,\varphi(\zeta)x+W)}{\zeta^{j+1}}\,d\zeta, \quad x\in D(0;1+\delta),\ W\in D(0;\delta)^{n-1}, \qquad \text{—} \underline{(2)}$$

is holomorphic in $D(0;1+\delta)$. Now let us <u>temporarily fix</u> $x\in$ ~~$D(0;1+\delta)$~~, $D(1;\delta/2)$, $W\in D(0;\delta)^{n-1}$. Then, by $(*)$:

$$g_W^x:\zeta\longmapsto f(\zeta,\varphi(\zeta)x+W) \text{ is well-defined } \forall\,\zeta\in D(0;1+\tilde{\varepsilon}/2);$$

is holomorphic in fact in $D(0,\tilde{\varepsilon}/2+1)$ & $a_j^W(x)$ is its j^{th} Laurent coefficient, with the proviso, of course, that $a_j^W(x)=0$ $\forall j<0$, $(x,W)\in D(0;1+\delta)\times D(0,\delta)^{n-1}$.

I.e.

$$a_j^W\big|_{D(1;\delta/2)}\equiv 0 \quad \forall\, W\in D(0;\delta)^{n-1} \ \& \ \forall j<0.$$

Thus $a_j^W\equiv 0$ $\forall\, W\in D(0;\delta)^{n-1}$ and $\forall j<0$

Compare (1) & (2)

$$A_j(w) = \frac{1}{2\pi i} \oint \frac{f(\zeta, w)}{\zeta^{j+1}} d\zeta = a_j^{w}(0) = 0 \quad \forall\, w \in \mathbb{D}(0;\delta)^{n-1}, \quad \forall\, j < 0.$$

In other words, the negative Laurent coefficients of f_{z^*} are zero when $z^* \in \mathbb{D}(0;\delta)^{n-1}$. Thus, f has an extension that is holomorphic on

$$\omega := \Delta(0;(1,\delta,\dots,\delta)) \bigcup \mathrm{Ann}(0;1\pm\tilde{\varepsilon}) \times \mathbb{D}^{n-1}.$$

& ~~hec~~ hence we can apply the previous theorem.

[Proved]

Lecture 3

* BEGIN BY TALKING ABOUT CHIRKA'S THEOREM. THIS MOTIVATES:

* Theorem: Let $\phi : \overline{\mathbb{D}} \xrightarrow[\text{cont.}]{} \mathbb{C}^{n-1}$ and suppose, for each ϕ_j — where $\phi = (\phi_1, \dots, \phi_n)$,

$$\phi_j \in \mathcal{C}(\overline{\mathbb{D}}) \cap \{ z \mapsto \psi(z, \bar{z}) \mid \psi \in \mathcal{O}(\mathbb{D}^2) \ \& \ \sup_{\mathbb{D}^2} |\psi| < 1 \}.$$

Let $\mathcal{D}$ be a domain such that

$$gr(\phi) \cup \partial\mathbb{D} \times \mathbb{D}^{n-1} \subset\subset \mathcal{D} \ \cancel{\subsetneq \mathbb{D}(0;1+\varepsilon)^{n-1}, \ 0<\varepsilon<<1.}$$

Write $\Omega := \mathcal{D} \cap D(0;1+\varepsilon) \times \mathbb{D}^{n-1}$ (where $\varepsilon > 0$). Then, every $f \in \mathcal{O}(\Omega)$ extends to a holo. fn. in $\mathbb{D}^n$.

Steps in proof

- Construct the sets & fix an approximant $P := (P_1, \dots, P_{n-1}) \Big|_{(z_1, \bar{z}_1)} \approx \phi$ with $P_j \in \mathbb{C}[z_1]$ s.t.

$$R_{\tilde{\varepsilon}} := \mathrm{Ann}(0; 1 \pm \tilde{\varepsilon}) \times \mathbb{D}^{n-1} \subset \Omega,$$

$$T_\delta := \bigcup_{z_1 \in \overline{\mathbb{D}}} \{z_1\} \times [D(0;\delta)^{n-1} + P(z_1, \bar{z}_1)] \subsetneq \Omega,$$

as before ~~& choose approximants~~ (i.e. in previous proof).

- We can find $\varepsilon_0 \in (0, \delta/2)$ s.t.

$$\left.\begin{array}{l} |z_1| \le 1 + \tilde{\varepsilon} \\ |\zeta - \bar{z}_1| < \varepsilon_0 \end{array}\right\} \Rightarrow \begin{array}{l} |w_k + P_k(z_1, \zeta) - P_k(z_1, \bar{z}_1)| < \delta \ \forall\, k = 1, \dots, n-1 \ \& \ \forall\, w \in D(0;\delta/2); \\ \cancel{\text{AND } z_1 + P} \ |w + P(z, \zeta)| < 1 \ \forall\, w \in D(0;\delta/2). \end{array}$$

Set

$$g_w : (z, \zeta) \longmapsto f(z, w + P(z, \zeta)), \quad w \in D(0;\delta/2)^{n-1},$$

which is well-defined — indeed holomorphic — on

$$\omega := \left[\bigcup_{z \in \overline{\mathbb{D}}} \{z\} \times [\bar{z} + D(0;\varepsilon_0)] \right] \cup \mathrm{Ann}(0; 1 \pm \tilde{\varepsilon}) \times \mathbb{D}.$$

- If we could extend g_w to be holomorphic in a nbhd. of $\mathbb{D}_z \times \{\zeta = 0\}$, then the same idea that follows showing $a_j^w(x=0)$, $j<0$, in the previous proof, [if we define $g_w^z := g_w(z, \cdot)$, then g_w^z plays a similar role as a_1^w] would complete the prove. But we can use Chirka's theorem, with $\varphi : z \longmapsto \bar{z}$ as our choice of the continuous function, to extend g_w to $\mathbb{D}^2$.

[Proved]