Quasihyperbolic Metric and Quasiconformal Mappings

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Outline

THE QUASIHYPERBOLIC DISTANCE

1. Given $x, y \in G \subset \mathbb{R}^n$, $\Gamma(x, y)$ stands for the collection of all rectifiable paths $\gamma \subset G$ joining x and y.
2. The quasihyperbolic distance [GP76] between $x, y \in G \subsetneq \mathbb{R}^n$ is defined by
$$k_G(x, y) := \inf_{\gamma \in \Gamma(x,y)} \int_\gamma \frac{1}{\operatorname{dist}(z, \partial G)} |dz| .$$
3. For a given pair of points $x, y \in G$, the infimum is always attained [GO79], i.e. there always exists a quasihyperbolic geodesic γ_k which minimizes the above integral.

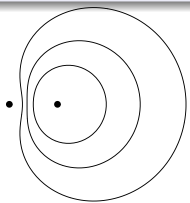
F.W. GEHRING AND B.P. PALKA, Quasiconformally homogeneous domains, *J. Anal. Math.* **30** (1976), 172–199.

F.W. GEHRING AND B.G. OSGOOD, Uniform domains and the quasihyperbolic metric, *J. Anal. Math.* **36** (1979), 50–74.

EXAMPLES

- It is monotone: $k_{G_1}(x,y) \le k_{G_2}(x,y)$ whenever $x, y \in G_2 \subset G_1$.
- In the punctured space $\mathbb{R}^n \setminus \{0\}$ the quasihyperbolic distance is given by the formula [MO86]

  $k_{\mathbb{R}^n\setminus\{0\}}(x,y) = \sqrt{\theta^2 + \log^2 \frac{|x|}{|y|}}$, where θ is the angle between the segments $[0,x]$ and $[0,y]$, $0 < \theta < \pi$.



G. Martin and B.G. Osgood, The quasihyperbolic metric and the associated estimates on the hyperbolic metric, *J. Anal. Math.* **47** (1986), 37–53.

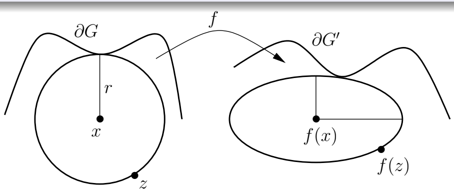
Quasiconformal Mappings

- Let $f: G \to G'$ be a homeomorphism, where $G, G' \subset \mathbb{R}^n$. The linear dilatation of f at a point $x \in G \setminus \partial G$ is defined by

$$H(x,f) = \lim_{r \to 0^+} \frac{\max_z \{|f(z)-f(x)| : |z-x| = r\}}{\min_z \{|f(z)-f(x)| : |z-x| = r\}}$$

  for $0 < r \leq \text{dist}(x, \partial G)$.

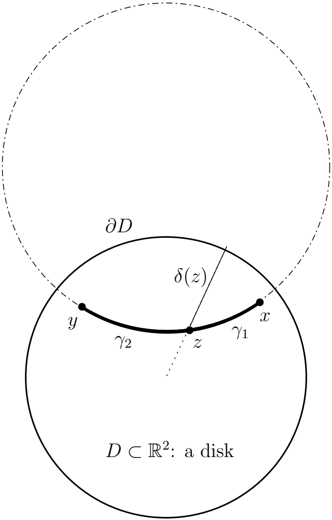
- A homeomorphism $f: G \to G'$ is said to be K-quasiconformal (K-qc), $K \geq 1$, if there exists a constant $K < \infty$ such that $H(x,f) \leq K$ for all $x \in G \setminus \partial G$. Observe that conformal maps are 1-qc.



Examples

1. A homeomorphism $f:\ G \to G'$ satisfying
$$|x-y|/L \le |f(x)-f(y)| \le L|x-y|$$
for all $x, y \in G$ is called L-bilipschitz map.
Note: L-bilipschitz maps are L^2-quasiconformal.
2. For $K \ge 1$, the function
$$f(x) = |x|^{\frac{1}{K}-1}x,\ f(0)=0, \quad x \in \mathbb{R}^n,$$
is K-qc.
3. For $|\alpha| < 1$, the function
$$f(z) = \begin{cases} z^{1+\alpha} & \text{for } \mathrm{Im}\, z \ge 0 \\ z\overline{z}^{\alpha} & \text{for } \mathrm{Im}\, z < 0 \end{cases}$$
defines a $K(\alpha)$-qc automorphism of $\mathbb{C}$, where
$K(\alpha) = (1+|\alpha|)/(1-|\alpha|)$.

Arc-length property



$$\gamma := \gamma[x,y]$$
$$\gamma_1 := \gamma[x,z]$$
$$\gamma_2 := \gamma[z,y]$$
$$\gamma = \gamma_1 \cup \gamma_2$$
$$\delta(z) := \text{dist}\,(z, \partial D)$$

$$\ell(\gamma) \leq \frac{\pi}{2}|x-y|$$

$$\min_{j=1,2} \ell(\gamma_j) \leq \frac{\pi}{2}\delta(z) \quad \forall\ z \in \gamma$$

LEMMA [GEH99]

Let $D \subset \mathbb{R}^2$ be a disk. For every $x, y \in D$, there exists a constant $c > 0$ and a rectifiable path $\gamma \subset D$ joining x and y such that

1. $\ell(\gamma) \leq c\,|x - y|$
2. $\min_{j=1,2} \ell(\gamma_i) \leq c \operatorname{dist}(z, \partial D)$, for all $z \in \gamma$, where γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$.

F.W. GEHRING, Characterizations of quasidisks, *Quasiconformal geometry and dynamics*, Banach center publications, Vol **48**, Polish Academy of Sciences, Warszawa 1999.

EXAMPLES

There are many cases, where the above two properties do hold and do not hold. For example,

- **Do hold:** half-plane, square, triangle, annulus, any bounded convex domain, snowflake region.
- **Do not hold:** parallel strip, disk with a radial slit, apple shaped region, complement of half-strip, onion shaped region, the complex plane with two distinct slits aligned in a line.

UNIFORM DOMAINS AND CHARACTERIZATIONS

UNIFORM DOMAINS [MS79]

A domain $G \subset \mathbb{R}^n$ is said to be uniform if any pair of points $x, y \in G$ can be joined by a rectifiable path $\gamma \subset G$ such that γ satisfies

1. $\ell(\gamma) \leq c\,|x - y|$;
2. $\min_{j=1,2} \ell(\gamma_i) \leq c\,\mathrm{dist}(z, \partial G)$, for all $z \in \gamma$;

for some constant $c > 0$, where γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$.

O. MARTIO AND J. SARVAS, Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Math.* **4** (1979), 384–401.

We need characterizations.

Characterizations of uniform domains

Lemma [Ahl63, MS79]. A simply connected Jordan domain (bounded) $G \subset \mathbb{R}^2$ is uniform if and only if there exists a constant $c > 0$ such that for each pair of points $a, b \in \partial G$ we have

$$\min_{j=1,2} \operatorname{diam}(\gamma_j) \leq c\,|a - b|$$

where γ_1, γ_2 are the components of $\partial G \setminus \{a, b\}$.

L. V. Ahlfors, Quasiconformal reflections, *Acta Math.* **109** (1963), 291–301.

In terms of the quasihyperbolic metric [GO79]

A domain $G \subsetneq \mathbb{R}^n$ is uniform if and only if there exist some positive constants c and d such that

$$k_G(x,y) \leq c \log\left(1 + \frac{|x-y|}{\min\{\delta(x),\delta(y)\}}\right) + d$$

for all $x, y \in G$, where $\delta(x) = \mathrm{dist}(x, \partial G)$.

F.W. Gehring and B.G. Osgood, Uniform domains and the quasihyperbolic metric, *J. Anal. Math.* **36** (1979), 50–74.

In terms of the quasihyperbolic metric [Vu85]

A domain $G \subsetneq \mathbb{R}^n$ is uniform if and only if there exists a positive constant c such that

$$k_G(x, y) \leq c \log \left(1 + \frac{|x - y|}{\min\{\delta(x), \delta(y)\}} \right)$$

for all $x, y \in G$.

M. Vuorinen, Conformal invariants and quasiregular mappings, *J. Anal. Math.* **45** (1985), 69–115.

φ-UNIFORM DOMAINS

φ-UNIFORM DOMAINS [VU85]

- Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $\varphi(0) = 0$. A domain $G \subsetneq \mathbb{R}^n$ is said to be φ-*uniform* if
$$k_G(x,y) \le \varphi(|x-y|/\min\{\delta(x),\delta(y)\})$$
for all $x, y \in G$.
- Consider domains G satisfying the following property: there exists a constant $C \ge 1$ such that each pair of points $x, y \in G$ can be joined by a rectifiable path $\gamma \in G$ with
$$\ell(\gamma) \le C\,|x-y| \text{ and } d(\gamma,\partial G) \ge \min\{\delta(x),\delta(y)\}/C\,.$$
Then G is φ-uniform with $\varphi(t) = C^2 t$.
- In particular, every convex domain is φ-uniform with $\varphi(t) = t$.

Uniform continuity

Uniform continuity

Let (X_j, d_j), $j = 1, 2$, be metric spaces. A function $f : X_1 \to X_2$ is said to be *uniformly continuous* if there exists a function, *modulus of continuity* $\omega : [0, r_1) \to [0, r_2)$ such that $\omega(0) = 0$ and $\omega(t) \to 0$, as $t \to 0$, and for all $x, y \in X_1$ with $d_1(x, y) < r_1$ we have $d_2(f(x), f(y)) \le \omega(d_1(x, y)) < r_2$.

Example

For $x, y \in B^n$ let $t = \sqrt{(1-|x|^2)(1-|y|^2)}$. Then

$$|x-y| \leqslant 2\tanh\frac{\rho_{B^n}(x,y)}{4} = \frac{2|x-y|}{\sqrt{|x-y|^2+t^2}+t},$$

where equality holds for $x = -y$. In particular, the identity map $id : (B^n, \rho_{B^n}) \to (B^n, |.|)$ is uniformly continuous with the modulus of continuity $\omega(t) = 2\tanh(t/4)$.

QUASIHYPERBOLIC COUNTERPART [KSV]

If $x, y \in B^n$ are arbitrary and $w = |x-y|\, e_1/2$, then

$$k_{B^n}(x,y) \geq k_{B^n}(-w,w) = 2\log \frac{2}{2-|x-y|} \geq |x-y|\,,$$

where the first inequality becomes equality when $y = -x$.

The modulus of continuity: $\omega(t) = 2(1-e^{-t/2})$.

R. KLÉN, S.K. SAHOO, AND M. VUORINEN, Uniform continuity and φ-uniform domains, *arXiv:0812.4369v3 [math.MG]*, preprint.

Generalization to bounded domains in $\mathbb{R}^n$ [KSV]

Let $G \subsetneq \mathbb{R}^n$ be a domain with $\operatorname{diam} G < \infty$ and $r = \sqrt{n/(2n+2)}\operatorname{diam} G$. Then we have

$$k_G(x,y) \geq 2\log\left(\frac{2}{2-t}\right) \geq t = |x-y|/r\,,$$

for all distinct $x, y \in G$ with equality in the first step when $G = B^n(z,r)$ and $z = (x+y)/2$.

For the proof, we use Jung's Theorem [Ber87]

Let $G \subset \mathbb{R}^n$ be a domain with $\operatorname{diam} G < \infty$. Then there exists $z \in \mathbb{R}^n$ such that $G \subset B^n(z,r)$, where $r \leq \sqrt{n/(2n+2)}\operatorname{diam} G$.

M. Berger, *Geometry I*, Springer-Verlag, Berlin, 1987.

Theorem (Characterization of φ-uniform domains [KSV])

The identity mapping id : $(G, j_G) \to (G, k_G)$ is uniformly continuous if and only if G is φ-uniform, where j_G is defined by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{\delta(x),\delta(y)\}}\right).$$

Proof.

Sufficiency part is trivial. Indeed, for $x, y \in G$ we have

$$k_G(x,y) \le \varphi(\exp(j_G(x,y)) - 1) = \omega(j_G(x,y)), \quad \omega(t) = \varphi(e^t - 1).$$

For the necessary part, we define

$$\varphi(t) = \sup\{k_G(x,y) : j_G(x,y) \le t\} \quad t \ge 0.$$

Quasisymmetric maps and φ-uniform domains

Quasisymmetric maps

Let $\eta : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $\eta(0) = 0$. A homeomorphism $f : (G, d_G) \to (G', d_{G'})$ is said to be η-quasisymmetric (η-QS) if

$$\frac{d_{G'}(f(x), f(y))}{d_{G'}(f(y), f(z))} \le \eta\left(\frac{d_G(x,y)}{d_G(y,z)}\right)$$

for all $x, y, z \in G$ with $y \neq z$.

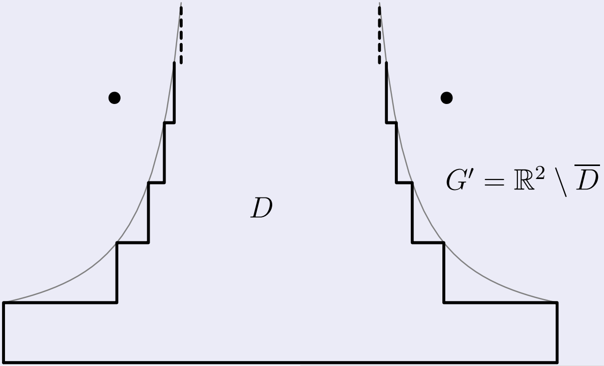
Question!

Can we say that the identity map $id : (G, j_G) \to (G, k_G)$ is η-QS if and only if G is φ-uniform for some φ depending on η ?

Complement of φ-uniform domains [KSV]

- Complimentary domains of simply connected uniform domains are uniform in the complex plane [Ahl63, MS79].

An unbounded φ_1-uniform domain $D \subset \mathbb{R}^2$ whose complement $G' = \mathbb{R}^2 \setminus \overline{D}$ is not φ-uniform for any φ.



QUESTION!

Are there any bounded φ_1-uniform domains whose complements are not φ-uniform ?

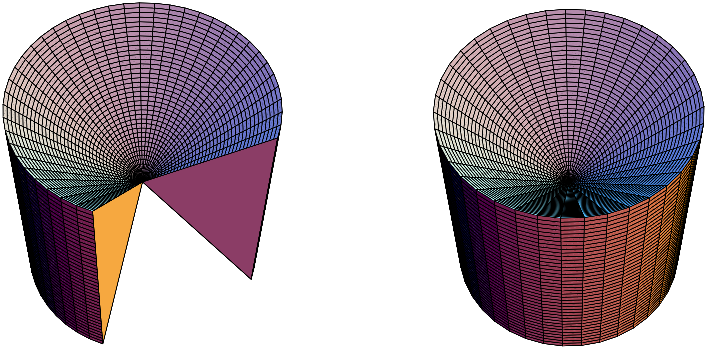


Image of φ-uniform domains under bilipschitz mappings [KSV]

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an L-bilipschitz mapping, that is

$$|x - y|/L \le |f(x) - f(y)| \le L|x - y|$$

for all $x, y \in \mathbb{R}^n$. If $G \subsetneq \mathbb{R}^n$ is φ-uniform, then $f(G)$ is φ_1-uniform with $\varphi_1(t) = L^2\varphi(L^2 t)$.

Question!

Can we ask a similar question for quasiconformal mappings?

Remark

Conformal images of the unit disk need not be uniform in general !

Quasi-invariance property of the quasihyperbolic metric [GO79]

If f is a K-quasiconformal map of $G \subsetneq \mathbb{R}^n$ onto $G' \subsetneq \mathbb{R}^n$, then there exists a constant c depending only on n and K such that

$$k_{G'}(f(x), f(y)) \leq c \max\{k_G(x,y), k_G(x,y)^\alpha\}$$

for all $x, y \in G$, where $\alpha = K^{1/(1-n)}$.

Quasi-invariance property of the j_G-metric [GO79]

If f is a K-quasiconformal map of $\mathbb{R}^n$ which maps $G \subsetneq \mathbb{R}^n$ onto $G' \subsetneq \mathbb{R}^n$, then there exist constants c and d depending only on n and K such that

$$j_{G'}(f(x), f(y)) \leq c\, j_G(x,y) + d$$

for all $x, y \in G$.

Quasi-invariance property of the j-metric [HKSV]

If f is a K-quasiconformal map of $\mathbb{R}^n$ which maps G onto G', then there exists a constant C depending only on n and K such that

$$j_{G'}(f(x), f(y)) \leq C \max\{j_G(x,y), j_G(x,y)^\alpha\}$$

for all $x, y \in G$, where $\alpha = K^{1/(1-n)}$.

Invariance property of φ-uniform domains [HKSV]

Suppose that $G \subsetneq R^n$ is a φ-uniform domain and f is a quasiconformal map of R^n which maps G onto $G' \subsetneq R^n$. Then G' is φ_1-uniform for some φ_1 as well.

P. Hästö, R. Klén, S.K. Sahoo, and M. Vuorinen, Geometric properties of φ-uniform domains, *In preparation*.

Open problem

- In simply connected planar domains uniform domains are associated with the quasiconformal maps of the whole plane $\mathbb{R}^2$, because uniform domains are nothing but quasidisks. What can we say about φ-uniform domains ?
- If a domain $G \subsetneq \mathbb{R}^n$ and its complement $\mathbb{R}^n \setminus \overline{G}$ are φ-uniform, is it true that G is uniform ?
- In the case of φ-uniform domains, are there any conditions on φ which imply that a geometric condition similar to uniformity condition holds ?

References

- L.V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, Princeton, 1966.
- A.F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, Vol. 91, Springer-Verlag, Berlin and New York, 1983.
- O. Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, New York, 1973.
- J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, 229, Springer-Verlag, 1971.
- M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Mathematics 1319, Springer-Verlag, Berlin–Heidelberg–New York, 1988.

THANK YOU