Calculus on Manifolds (Jan–Apr 2020)

Assignment 1

January 2, 2020

(Will not be graded)

1. Suppose $A \subset \mathbb{R}^m$ and $\vec{a} \in A$ is such that there is a neighbourhood U of $\vec{a}$. Let $f\colon A \to \mathbb{R}^n$ be a function which is given to be differentiable at $\vec{a}$. Prove that f is continuous at $\vec{a}$.

2. Let $A \subset \mathbb{R}^m$ and $f\colon A \to \mathbb{R}^n$ be a function differentiable at $\vec{a}$. Then prove that all the directional derivatives of f at $\vec{a}$ exist and
$$f'(\vec{a}, \vec{u}) = \mathbf{D}f(\vec{a})\vec{u}.$$

3. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the map defined by
$$f(x,y) = \begin{cases} (0,0) & \text{if } (x,y) = (0,0) \\ \frac{x^2 y}{x^4+y^2} & \text{otherwise.} \end{cases}$$
Show that all the directional derivatives of f exist at $(0,0)$ but f is *not* differentiable at $(0,0)$.

4. Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable functions. Then
$$\mathbf{D}(fg) = f\,\mathbf{D}g + g\,\mathbf{D}f \tag{1}$$
$$\mathbf{D}\left(\frac{1}{f}\right) = -\frac{1}{f^2}\mathbf{D}f \text{ whenever } f \neq 0. \tag{2}$$

5. Let $f\colon \mathbb{R} \to \mathbb{R}^n$ be differentiable. If $\|f(t)\| = 1$ for all $t \in \mathbb{R}$, prove that $f(t) \cdot \mathbf{D}f(t) = 0$.

6. Define
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Prove that $D_1 f$ and $D_2 f$ exist at every point of $\mathbb{R}^2$. Is the function differentiable everywhere?

7. Define
$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
   (a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in a neighbourhood of $(0,0)$.

(b) Using this prove that f is continuous everywhere.

(c) Let $\vec{u}$ be any unit vector in $\mathbb{R}^2$. Prove that $D_{\vec{u}} f(0,0)$ exists and $|D_{\vec{u}} f(0,0)| \leq 1$.

(d) Let $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ be any differentiable map such that $\gamma(0) = (0,0))$ and $\|\gamma'(0)\| > 0$. Prove that $g(t) = f(\gamma(t))$ is differentiable for all $t \in \mathbb{R}$. If γ is of the class C^1, prove that so is g.

(e) Prove that f is *not* differentiable at $(0,0)$.

8. Suppose $A \subset \mathbb{R}^n$ be an open and connect subset. Let $f\colon A \to \mathbb{R}$ be a differentiable function with $\mathbf{D}f(\vec{x}) = 0$ for all $\vec{x} \in A$. Prove that f is constant on A.

9. In the following cases, find the points around which an inverse function exists

   (a) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = t^2$.

   (b) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, where

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^3 + xy + y^3 \\ x^2 - y^2 \end{pmatrix}.$$

   In the second problem can you find the approximate value of

$$f^{-1}\begin{pmatrix} 3.01 \\ -0.01 \end{pmatrix} \text{ given the fact that } f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

10. Let $L = \left\{ \vec{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \in \mathbb{R}^3 \,\middle|\, x_1 + x_2 + x_3 + 1 = 0 \right\}$. Consider the function $f\colon \mathbb{R}^3 \setminus L \to \mathbb{R}^3$ defined by

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{1 + x_1 + x_2 + x_3}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

   Find $f(\mathbb{R}^3 \setminus L)$ and compute $\det \mathbf{D}f(\vec{x})$ for $\vec{x} \in \mathbb{R}^3 \setminus L$. Compute f^{-1} explicitly.

11. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a function of the class C^1 and $g_1, \ldots, g_n\colon \mathbb{R} \to \mathbb{R}$ be n real functions of class C^1. Let $h(x_1, \ldots, x_n) = f(g_1(x_1), \ldots, g_n(x_n))$. Prove that

$$\det \mathbf{D}h(\vec{x}) = \det \mathbf{D}f(g_1(x_1), \ldots, g_n(x_n))g_1'(x_1) \cdots g_n'(x_n).$$