Assignment 8

October 30, 2015

Problem 1. Do Exercises 92, 93 and 94 from Rotman, Galois Theory, page 90.

Problem 2. Let E/F be a Galois extension, where F is a field of characteristic $p > 0$. Let $\sigma \in \mathrm{Gal}(E/F)$ have order p. View σ as a F-linear transformation on E and define the F-linear transformation

$$\tau = \sigma - \mathrm{id}\,.$$

(1) Prove that $\tau^p = 0$.
(2) Prove that if $\alpha \in \ker\tau + \operatorname{im}\tau$, then $\tau^{p-1}(\alpha) = 0$. Conclude $\tau(\alpha) = 0$.
(3) Prove that $\ker\tau = F$ and $\operatorname{im}\tau \cap \ker\tau \neq \{0\}$.
(4) By proving $\operatorname{im}\tau \cap \ker\tau = F$, conclude that $1 \in \operatorname{im}\tau$. In other words, there exists $\alpha \in E$ such that $\sigma(\alpha) - \alpha = 1$.
(5) Prove that if E/F has degree p and $\mathrm{Gal}(E/F) = \langle\sigma\rangle$, $E = F(\alpha)$ where α is the root of the irreducible polynomial of the form $X^p - X - c \in F[X]$.

Problem 3. Determine the Galois groups of the following polynomials in $\mathbb{Q}[X]$:

(1) $X^3 - X^2 - 4$,
(2) $X^3 - 2X + 4$,
(3) $X^3 - X + 1$, and
(4) $X^3 + X^2 - 2X - 1$.