Assignment 5

September 11, 2015

Problem 1. If $K = \mathbb{F}_p(t)$, prove that $X^p - t \in K[X]$ is irreducible.

Problem 2. A field F of characteristic p is perfect iff every element of F has a p-th root in F.

Problem 3. Let F be a field and $f(X) \in F[X]$. Suppose E/F is the splitting field of f and $G = \mathrm{Gal}(E/F)$ is the Galois group.

(1) Prove that if $f(X)$ is irreducible, then G acts transitively on the set of all roots of $f(X)$.
(2) If $f(X)$ has no repeated roots and G acts transitively on all the roots, prove that $f(X)$ is irreducible in $F[X]$.

Problem 4. Prove that

$$X^{p^n-1} - 1 = \prod_{\alpha \in \mathbb{F}_{p^n}^\times} (X - \alpha).$$

Conclude that $\prod_{\alpha \in \mathbb{F}_{p^n}^\times} = (-1)^{p^n}$. Taking p odd and $n = 1$ derive Wilson's theorem: $(p-1)! = -1 \pmod{p}$.

Problem 5. If $n = p^k m$ where p is a prime and $(m, p) = 1$, then there exists precisely m distinct n-th roots of unity over a field of characteristic p.

Problem 6. Prove that there are only a finite number of roots of unity in any finite extension K of $\mathbb{Q}$.

Problem 7. Find the splitting field E of $X^8 - 2 \in \mathbb{Q}[X]$. Prove that $\mathrm{Gal}(E/\mathbb{Q}) \cong D_8 := \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$.