Assignment 4

September 4, 2015

Problem 1. Are the fields
$$\frac{\mathbb{F}_2[X]}{(X^3+X^2+1)} \quad \text{and} \quad \frac{\mathbb{F}_2[Y]}{(Y^3+Y+1)}$$
isomorphic? Can you construct such an isomorphism? (hint[1])

Problem 2. Find the degree of the splitting field of $(X^2-2)(X^2-3)$ over $\mathbb{Q}$.

Problem 3. Write down a polynomial over $\mathbb{Q}$ whose splitting field is $\mathbb{Q}(\sqrt{2},\sqrt{3})$. What can you say about $\mathrm{Gal}(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q})$?

Problem 4. Prove that $\alpha_1,\dots\alpha_n \in E$ for some field extension E/F. Prove that
$$F(\alpha_1,\dots,\alpha_n) = \left\{ \frac{f(\alpha_1,\dots,\alpha_n)}{g(\alpha_1,\dots,\alpha_n)} \;\middle|\; f,g \in F[X_1,\dots,X_n] \text{ and } g(\alpha_1,\dots,\alpha_n) \neq 0 \in E \right\}.$$

Problem 5. Prove that if $f(X) \in F[X]$ is a polynomial over a field F, and E/F is a splitting field of f, and if $\alpha_1,\dots,\alpha_n$ are the roots of f in E, then $E = F(\alpha_1,\dots,\alpha_n)$.

Problem 6. If R is a ring and if $c \in R$, prove that the map
$$\psi_c : R[X] \to R[X]$$
given by $(\psi_c(f))(X) = f(X+c)$ is an isomorphism and conclude that in the case R is a field, $p(X)$ is irreducible iff $\psi_c(p)$ is irreducible.

Problem 7. Let $p(X)$ be a monic polynomial in $\mathbb{Z}[X]$. Prove that any rational root of p has to be an integer. (hint[2])

Problem 8. Suppose $f(X) \in F[X]$ be an irreducible polynomial of degree n. Let E/F be the splitting field of $f(X)$. Prove that $n|[E:F]$. If f is separable, prove that $n|\,\mathrm{Gal}(E/F)$.

[1] *Hint:* $X \mapsto Y+1$.

[2] *Hint:* Prove that if r/s is a root of $p(X) = a_0 + a_1X + \cdots + a_kX^k$, then $r|a_0$ and $s|a_k$.