Assignment 3

August 28, 2015

Problem 1. Suppose $F \subset E$ and $E \subset K$ are finite field extensions. Suppose $\{e_1, \ldots, e_m\}$ is a basis of E as a vector space over F and $\{k_1, \ldots, k_n\}$ is a basis of K as a vector space over F. Prove that $\{e_i k_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis of K over F and hence

$$[K : F] = [K : E][E : F].$$

Problem 2. Consider the extension $\mathbb{Q} \subset \mathbb{R}$. Let $\sqrt[3]{2}$ be the real root of the equation $X^3 - 2 = 0$. Prove that

$$\mathbb{Q}(\sqrt[3]{2}) = \left\{ a + b\sqrt[3]{2} + c\left(\sqrt[3]{2}\right)^2 \;\middle|\; a, b, c \in \mathbb{Q} \right\}.$$

Problem 3. Consider $\mathbb{Q} \subset \mathbb{C}$ and $\alpha = \sqrt[3]{2}\omega \in \mathbb{C}$ where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a complex cube root of 1. Then $\alpha^3 - 2 = 0$. Prove that $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\sqrt[3]{2})$.

Problem 4. Show that $p(X) = X^3 + 9X + 6$ is irreducible in $\mathbb{Q}[X]$. Let θ is a root of $p(X)$ in some extension K. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

Problem 5. Prove that the polynomial $(X^{p-1} - 1)/(X - 1)$ is irreducible in $\mathbb{Q}[X]$ for any prime p. (hint[1])

Problem 6. Determine the degree of the extension $\mathbb{Q}\left(\sqrt{3 + 2\sqrt{2}}\right)$ over $\mathbb{Q}$.

Problem 7. What is the irreducible polynomial of $1 + i \in \mathbb{C}$ over $\mathbb{Q}$?

Problem 8. Let $F = \mathbb{Q}(i)$. Prove that $X^3 - 2$ and $X^3 - 3$ are irreducible over F.

Problem 9. Factor $X^9 - X$ in $\frac{\mathbb{Z}}{3\mathbb{Z}}[X]$. Prove that the factors you get are irreducible in $\frac{\mathbb{Z}}{3\mathbb{Z}}[X]$.

Problem 10. Suppose F is a field and α is an algebraic element in some extension, such that α satisfies an irreducible polynomial of degree 5 over F. Prove that $F(\alpha) = F(\alpha^2)$.

[1] *Hint:* Substitute $Y = X - 1$.