Assignment 1

August 14, 2015

Problem 1. If R is a ring and S is a set. Let
$$R^S = \{f \mid |f \text{ is a function } f : S \to R\}.$$
Prove that R^S is a ring with respect to pointwise addition and multiplication: $(f+g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$.

Problem 2. Show that there is a factorization $x = f(x)g(x)$ in $R = \frac{\mathbb{Z}}{4\mathbb{Z}}[x]$ where neither $f(x)$ and $g(x)$ is a constant (hint[1]). Give an example of a non-constant polynomial in R which is a unit.

Problem 3. Can you construct an infinite field containing $\frac{\mathbb{Z}}{p\mathbb{Z}}$ as a subfield?

Problem 4. Suppose $\sigma : R \to S$ is a ring map, prove that $\sigma^* : R[x] \to S[x]$ defined by
$$\sigma^*(\sum r_i x^i) = \sum \sigma(r_i) x^i$$
is also a ring map. If $\tau : S \to T$ is a ring map prove that $(\tau\sigma)^* = \tau^*\sigma^*$. If σ is an isomorphism, then so is σ^*.

Problem 5. Prove that if $u \in R$ is a unit, and $f \in R$ is any element, then
$$(f) = (uf).$$
If R is a domain and if $r, s \in R$ are such that $(r) = (s)$, then $s = ur$ for some unit $u \in R$.

Problem 6. Find the gcd of $f(x) = x^3 - x$ and $g(x) = x^2 + 2x + 1$ in $\mathbb{Q}[x]$ using Euclidean algorithm. Can you write the gcd as a $\mathbb{Q}[x]$-linear combination of $f(x)$ and $g(x)$?

Problem 7. Consider the map $\varphi : \mathbb{Z}[x] \to \mathbb{R}$ defined by $\varphi(f(x)) = f(1+\sqrt{2})$. Is this a ring homomorphism? What is the kernel?

Problem 8. Let R be a ring such that $\underbrace{1 + \cdots + 1}_{p \text{ times}} = 0$. Prove that the map $\varphi : R \to R$ defined by $\varphi(x) = x^p$ is a ring homomorphism. Prove that if $a \in R$ satisfies $a^n = 0$ for some $n > 0 \in Z$, then there exists an interger $m > 0$ such that $(1+a)^m = 1$.

[1] *Hint:* What is $(2x+1)^2$?

Problem 9. There is a bijective correspondence between the maximal ideals of $\mathbb{R}[x]$ and points of upper half plane.

Problem 10. Is $\{a + b\epsilon \mid |a, b \in \mathbb{Q}, \epsilon^2 = 0\}$ a ring? What are its ideals? Maximal ideals? Prime ideals?