SOME LINEAR ALGEBRA

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In this talk all the numbers are real.

1. Linear equations and matrices

Suppose we want to solve the system of linear equations :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

We can write this equation in a matrix form :

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

In short, we shall write this equation as $\mathbf{A}\vec{x} = \vec{b}$.

Here we can think of $\mathbf{A}$ to be a function which takes vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$; i.e., it is the function which takes a vector $\vec{\alpha}$ in $\mathbb{R}^n$ to $\mathbf{A}\vec{\alpha} \in \mathbb{R}^m$. In this language, the above problem reduces to asking, "What are the vectors in $\mathbb{R}^n$, which go to $\vec{b}$ under the function $\mathbf{A}$?"

Now $\mathbf{A}$ has the following properties :

$$\begin{aligned} \mathbf{A}(\vec{x} + \vec{y}) &= \mathbf{A}\vec{x} + \mathbf{A}\vec{y}, \text{ for } x, y \in \mathbb{R}^n; \\ \mathbf{A}(\alpha\vec{x}) &= \alpha\mathbf{A}\vec{x} \text{ for } \vec{x} \in \mathbb{R}^n,\ \alpha \in \mathbb{R}. \end{aligned}$$

Definition 1. Such a function is called a *linear transformation*.

We saw that matrices give rise to linear transformations. The opposite is also true.

Remark 2. Suppose $B : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Let

$$B\underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_1} = \begin{pmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{1m} \end{pmatrix}, \quad B\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_2} = \begin{pmatrix} b_{21} \\ b_{22} \\ \vdots \\ b_{2m} \end{pmatrix}, \quad \cdots, \quad B\underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{e_n} = \begin{pmatrix} b_{n1} \\ b_{n2} \\ \vdots \\ b_{nm} \end{pmatrix}$$

Let $e_1, e_2, \ldots, e_n$ be the vectors in $\mathbb{R}^n$ as marked in the above equation.

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Now for any vector $\vec{x} \in \mathbb{R}^n$, we have

$$B(\vec{x}) = B(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}) = B(x_1 e_1 + \cdots + x_n e_n) = x_1 B(e_1) + \cdots + x_n B(e_n)$$

by linearity,

$$= x_1 \begin{pmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{1m} \end{pmatrix} + \cdots + x_n \begin{pmatrix} b_{n1} \\ b_{n2} \\ \vdots \\ b_{nm} \end{pmatrix} = \mathbf{B} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}.$$

Therefore B corresponds to the matrix $\mathbf{B}$.

2. Examples

2.1. Affine transformations.

Consider some standard motions we know.

Rotation.: All rotations considered are counter-clockwise on $\mathbb{R}^2$ around origin. So the origin is fixed under any of the rotations. Thus a rotation R_θ of angle θ will take the point $(1,0)$ to $(\cos\theta, \sin\theta)$; and will take $(0,1)$ to $(\cos(90^\circ + \theta), \sin(90^\circ + \theta)) = (-\sin\theta, \cos\theta)$.

Thus the matrix for this tranformation is

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Translation: Let $T_{a,b}$ be the translation which takes (x,y) to $(x+a, y+b)$. Translation cannot be a linear transformation if we think of it as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. The reason is a linear transformation always fixes $(0,0)$[1]. However if we think of $\mathbb{R}^2$ sitting in $\mathbb{R}^3$ by identifying (x,y) with $(x,y,1) \in \mathbb{R}^3$, we can code translation by (a,b) to be a linear transformation. Note we want $(x,y,1)$ to go to $(x+a, y+b, 1)$.

- Therefore, $(0,0,1)$ goes to $(a,b,1)$.
- $(1,0,1)$ goes to $(1+a, b, 1)$; and hence by linearity, $(1,0,0) = (1,0,1) - (0,0,1)$ goes to $(1+a,b,1) - (a,b,1) = (1,0,0)$
- Following the same argument, $T_{a,b}(0,1,0) = T_{a,b}((0,1,1) - (0,0,1)) = (a, b+1, 1) - (a,b,1) = (0,1,0)$.

Therefore, the matrix for translation is

$$\mathbf{T_{a,b}} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

[1] $T(\vec{0}) = T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0}) = \vec{0}$

Reflection: Reflection along x-axis is given by $(x, y) \mapsto (x, -y)$. Check that the matrix is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

xercise 1. Prove that the reflection along the line $x = y$ is given by the matrix

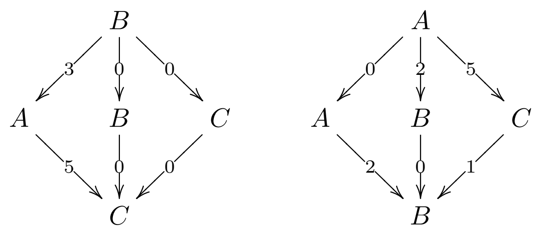
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2.2. **Adjacency matrix.** Suppose A, B and C are three cities. Suppose the number of flights plying between the cities are given by the following table.

		To destination		
		A	B	C
From source	A	0	2	5
	B	3	0	0
	C	3	1	0

This means that there are no flights from B to C, however there are 5 flights from A to C; and so on.

However if you are willing to stop over at exactly one place, then the there are options to go from B to C (left figure):



The right figure gives the options to go from A to B with exactly one stop over. Therefore the total number of options to go from B to C with exactly one stop over is the inner product of the row B and column C. Similarly the total number of options to go from A to B is the product of row A and column B. Thus the number of options for all possible pairs is given by the square of the matrix :

$$\begin{pmatrix} 0 & 2 & 5 \\ 3 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 5 \\ 3 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 21 & 5 & 0 \\ 0 & 6 & 15 \\ 3 & 6 & 15 \end{pmatrix}$$

Note that the number of options to go from B to A with exactly one stop over is 0, though you have 3 ways to go from B to A directly. This is because the stop over cannot be at A since you cannot go from A to A. Similarly you cannot go from B to B. And you cannot go to C from B.

3. Row operations and échelon forms

Elementary row operations are following :

(1) interchanging two rows
(2) multiplying a row by a non-zero scalar
(3) replacing a row by the sum of that row and a scalar multiple of another row.

Note that in the equation $\mathbf{A}\vec{x} = \vec{b}$, suppose we perform row operations on the augmented matrix $(A : b)$ and get another matrix $(U : v)$ where U is an $m \times n$ matrix and and v is an m-dimensional vector. Then, the solutions to $\mathbf{A}\vec{x} = \vec{b}$ are the same as those of $\mathbf{U}\vec{x} = \vec{v}$.

3.1. **Sweeping out a column with respect to a pivot.** Suppose $\mathbf{A}$ is a matrix and $a_{kl} \neq 0$. Sweeping out the l-th column with a_{kl} as a pivot is the following procedure :

(1) Multiply the k-th row by $1/a_{kl}$. (Elementary operation 1)
(2) For each row r, $r \neq k$, do the following :
    (a) Multiply the k-th row by a_{rl} and subtract from the r-th row. (Elementary operation 3)

After this procedure the l-th column will be the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 being at the k-th place.

Example 3. Suppose

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 & 2 \\ 1 & 4 & 5 & 5 \\ 3 & 0 & 2 & 6 \end{pmatrix}$$

and suppose we take a_{23} to be the pivot. The sweeping procedure will then be

Multiply row 2 by $1/a_{23}$**:**

$$\begin{pmatrix} 0 & 2 & 3 & 2 \\ 1/5 & 4/5 & 1 & 1 \\ 3 & 0 & 2 & 6 \end{pmatrix}$$

Sweep row 1: Multiply row 2 by $a_{13} = 3$ and subtract from row 1 (thus replacing row 1) to get

$$\begin{pmatrix} -3/5 & -2/5 & 0 & -1 \\ 1/5 & 4/5 & 1 & 1 \\ 3 & 0 & 2 & 6 \end{pmatrix}$$

Sweep row 3: Multiply row 2 by $a_{33} = 2$ and subtract form row 3 (thus replacing row 3) to get

$$\begin{pmatrix} -3/5 & -2/5 & 0 & -1 \\ 1/5 & 4/5 & 1 & 1 \\ 13/5 & -8/5 & 0 & 4 \end{pmatrix}$$

Note that after the operation the third column is $(0, 1, 0)$ as was expected.

Definition 4. An $m \times n$ matrix with r nonzero rows $(0 \leq r \leq m)$ $\mathbf{A}$ is said to be in *échelon form* if

(1) the first r rows are nonzero (this will force the last $m - r$ rows to be zero);
(2) if p_i is the first column, where the i-th row has its first nonzero entry, $p_1 < p_2 < \cdots < p_r$;
(3) $a_{ip_i} = 1$.

A matrix $\mathbf{A}$ is said to be in *reduced échelon form* if it is in échelon form and $a_{kp_i} = 0$ for all $k \neq i$ (in other words, the p_i-th column has all zeros except at the i-th place).