Some remarks on conics on a projective plane

I want to prove that under projective transformations, any non-degenerate conic can be mapped to any other non-degenerate conic.

1. Conics in $\mathbb{RP}^2$

As discussed in class, the conics in $\mathbb{RP}^2$ are defined to be the zeroes of a second degree equation of the form

$$q(X,Y,Z) := aX^2 + 2bXY + cY^2 + 2dXZ + 2eYZ + fZ^2 = 0$$

1.1. **Quadratic forms and bilinear forms.** We begin by observing that $q(X,Y,Z)$ can be written as

$$q(X,Y,Z) = \begin{pmatrix} X & Y & Z \end{pmatrix} \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{1.1.1}$$

1.1.2. **Notation.** For the sake of brevity, let us denote

$$v = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad\qquad A = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}.$$

1.1.3. Therefore, $q(X,Y,Z) = v^t A v$.

1.1.4. Note that this is a quadratic form corresponding to the bilinear form

$$(v,w) \mapsto v^t A w, \qquad v,w \in \mathbb{R}^3.$$

Now we recall a result from linear algebra.

1.1.5. **Proposition.** *Suppose for v and w in $\mathbb{R}^n$ if $(v,w) \mapsto B(v,w)$ is a bilinear form, then there exists a basis for $\mathbb{R}^n$, with respect to which B is represented by a matrix of the form*

$$\begin{pmatrix} I_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

A very brief sketch of the proof. First note that if a bilinear form satifies $B(v,w) = 0$ for all $v, w \in V$, the respective matrix is forced to be the zero matrix, in which case we are done.

Suppose there exist v and w such that $B(v,w) \neq 0$. We shall construct a u such that $B(u,u) \neq 0$. If $B(v,v) \neq 0$, take $u = v$. Similarly, if $B(w,w) \neq 0$ take $u = w$. If $B(v,v) = B(w,w) = 0$, note that $B(v+w, v+w) = 2B(v,w) \neq 0$. Therefore take $u = v + w$.

If $B(u,u) > 0$, let $b_1 = u/\sqrt{B(u,u)}$. In this case $B(b_1,b_1) = +1$. In case $B(u,u) < 0$, take $b_1 = u/\sqrt{-B(u,u)}$ in which case $B(b_1,b_1) = -1$.

Let $b_1^{\perp}$ be the vector subspace

$$\{v \in V \mid B(b_1, v) = 0\}.$$

For any basis $e_2, \ldots, e_n$ of $b_1^{\perp}$, the matrix for B will look like

$$\begin{pmatrix} \pm 1 & \vec{0}^t \\ \vec{0} & B' \end{pmatrix}$$

where the ± 1 stands for $B(b_1,b_1)$. Now restricting the bilinear form to $b_1^{\perp}$, our result follows by induction. □

1.1.6. Note that the bilinear form we are considering is symmetric. (One can associate a symmetric bilinear form to any quadratic form, since $x^t Q x = x^t((Q + Q^t)/2)x$ for any matrix Q.)

1.2. **Effect of projective transformations on the formula of a conic.** We know that a projective transformation is given by an invertible linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose

$$\begin{aligned} T(e_1) &= v_1 \\ T(e_2) &= v_2 \\ T(e_3) &= v_3 \end{aligned}$$

where e_i is the standard basis element with 1 at the i-th place and zero everywhere else.

1.2.1. Therefore, the matrix of T would look like $(v_1\ v_2\ v_3)$. Now for a vector

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3,$$

$v = \alpha v_1 + \beta v_2 + \gamma v_3$ where

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = T^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

One way to see this is to note that,

$$ae_1 + be_2 + ce_3 = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = I_3 T T^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} T^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Thus if $v = (a, b, c)^t$ is a point on the conic, i.e. $v^t A v = 0$, in the new coordinates, we shall have $(\alpha\ \beta\ \gamma) T^t A T (\alpha\ \beta\ \gamma)^t$.

1.2.2. Thus starting with a symmetric bilinear form $(v, w) \mapsto v^t A w$, suppose we find a basis x_1, x_2, x_3 such that $x_i^t A x_j = 0$ for $i \neq j$ and is ± 1 for $i = j$. Then the transformation $(x_1\ x_2\ x_3)$ will reduce the equation of the conic to a standard form.

2. Example

2.1. **Finding the basis.**

2.1.1. Suppose we want to find the projective transformation which transforms $Y^2 + 2XZ + Z^2$ into a circle ($U^2 + V^2 - W^2$ for some U, V and W).

2.1.2. Note that

$$Y^2 + 2XZ + Z^2 = \begin{pmatrix} X & Y & Z \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

2.1.3. Consider the bilinear form $B(v, w)$ defined by

$$B(v, w) = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \text{ for } v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},\ w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

In short, $B(v, w) = v_3 w_1 + v_2 w_2 + v_1 w_3 + v_3 w_3$.

2.1.4. Note that $B((0,0,1)^t,(0,0,1)^t)=1$. Set

$$U=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Note that in this case $B(U,U)$ is already of the form ± 1. In case, it was nonzero but not of the form ± 1, we would have to replace U by $U/\sqrt{B(U,U)}$ in case $B(U,U)>0$ or by $U/\sqrt{-B(U,U)}$ in case $B(U,U)<0$.

2.1.5. Now we do the same procedure on the space orthogonal[1] to $\langle U\rangle$. The orthogonal space is

$$\begin{aligned}U^\perp &= \{v\in\mathbb{R}^3 \mid B(U,v)=0\} = \{(v_1,v_2,v_3)^t\in\mathbb{R}^3 \mid v_1+v_3=0\}\\ &= \{(a,b,-a)^t \mid a,b\in\mathbb{R}\}.\end{aligned}$$

Note that $(0,1,0)^t\in U^\perp$ and $B((0,1,0)^t,(0,1,0)^t)=1$ and hence we set

$$V=\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Now look at the orthogonal space

$$\begin{aligned}V^\perp &= \{(a,b,-a)^t \mid B((0,1,0)^t,(a,b,-a)^t=0\}\\ &= \{(a,b,-a)^t \mid b=0\} = \{(a,0,-a)^t \mid a\in\mathbb{R}\}.\end{aligned}$$

For the sake of complicating the example let us look at $W'=(2,0{-}2)^t$. $B(W',W')=(-2)2+2(-2)+(-2)(-2)=-4$. Therefore, we take $W=W'/\sqrt{4}$, that is

$$W=\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

2.1.6. Therefore, the basis we want is given by U, V and W.

2.2. **The transformation.**

2.2.1. By the discussion in 1.2.2, the transformation $T=\begin{pmatrix}U & V & W\end{pmatrix}$ would satisfy

$$T^t\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}T=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

2.2.2. Hence the formula of the conic after transforming by T is

$$\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\alpha^2+\beta^2-\gamma^2=0$$

which is nothing but the equation of a circle.

2.2.3. Now suppose you want to find the transformation which takes a non-degenerate conic given by the equation, say $f(X,Y,Z)=0$ to another non-degenerate conic given by the equation, say $g(X,Y,Z)=0$. One way to do this will be to find a transformation, say T, which takes $f=0$ to the circle and find another transformation, say S, which takes $g=0$ to a circle. Then $S^{-1}\circ T$ will take $f=0$ to $g=0$.

[1] with respect to B