Hyperbolic geometry: sum of angles and Poincaré model

1. Sum of angles

1.1. **Angle of parallelism.**

1.1.1. **Theorem.** *The angle of parallelism varies inversely with the segment.*

(1) $AB < A'B' \iff \alpha(AB) > \alpha(A'B')$*;*
(2) $AB \cong A'B' \iff \alpha(AB) = \alpha(A'B')$*.*

Proof. We shall just prove that $AB \cong A'B' \implies \alpha(AB) = \alpha(A'B')$ and $AB < A'B' \implies \alpha(AB) > \alpha(A'B')$. Once we have these, reversing the roles of AB and $A'B'$, we get the other implication about inequalities. And putting all of these together we get the implications in the other direction.

Thus first assume that $AB \cong A'B'$ (figure 1). Then by ASL, $\angle BAa \cong \angle B'A'a'$. Therefore, $\alpha(AB) = \alpha(A'B')$.

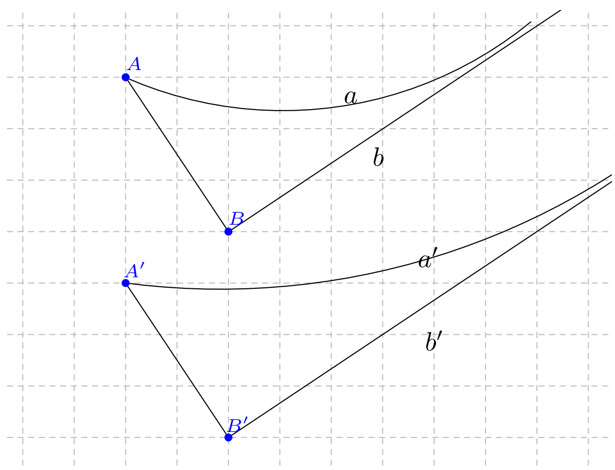


FIGURE 1. Angles of parallelism {fig:angparll}

Now suppose that $AB < A'B'$ (figure 2).

- Mark C on $\overrightarrow{AB}$ such that $AC \cong A'B'$.
- Let $\overrightarrow{Cc}$ be prependicular to AB and on the same side as $\overrightarrow{Bb}$.
- Let a'' be such that $\overrightarrow{Aa''} \,|||\, \overrightarrow{Cc}$.

Then by what we have already proved, $\alpha(A'B') = \alpha(AC) \cong \angle CAa''$.

I claim that $\overrightarrow{Bb}$ *intersects the ray* $\overrightarrow{Aa''}$. Suppose $\overrightarrow{Bb}$ does not intersect the ray $\overrightarrow{Aa''}$. Then $\overrightarrow{Bb}$ is a limiting parallel to $\overrightarrow{Cc}$.

Fact we assume : $\overrightarrow{Bb}$ is a limiting parallel to both $\overrightarrow{Aa''}$ and $\overrightarrow{Cc}$. This follows from the proof of the fact that $|||$ is an equivalence relation.

Therefore, $\alpha(BC) = 90°$. Which cannot be by (L). Therefore, $\overrightarrow{Bb}$ intersects the ray Aa''. Therefore $\alpha(AB) > \angle CAa'' = \alpha(A'B')$. This completes the proof by what we discussed in the first paragraph. □

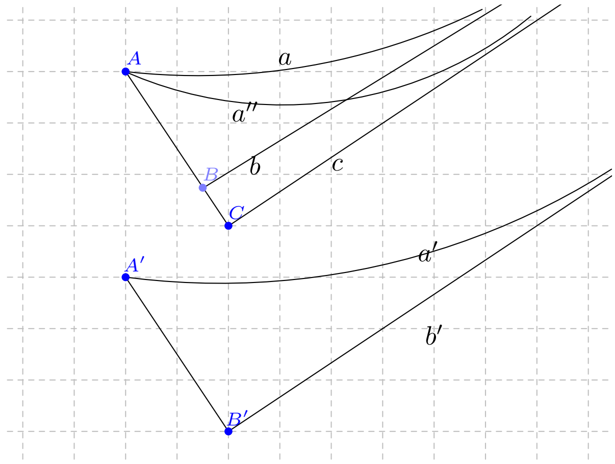


FIGURE 2. Angle of parallelism - II {fig:angpartw}

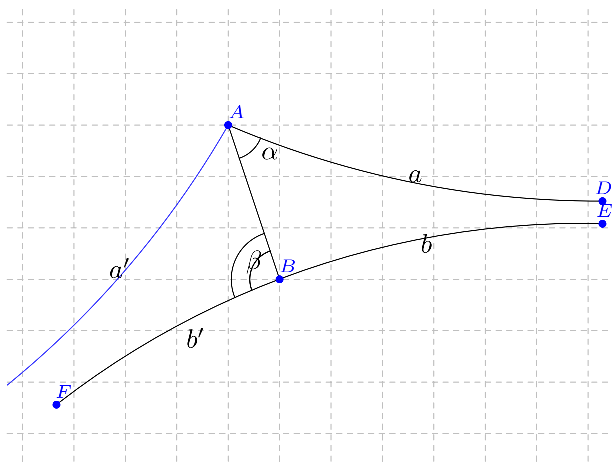
Our aim is to prove the following result on a hyperbolic plane.

{thm:sumanghp}

1.1.2. **Theorem.** *In a hyperbolic plane, the sum of angles of any triangle is less than two right angles.*

Before that we prove the following proposition.

1.1.3. **Proposition.** *If AB is a segment, with limiting parallel rays $\overrightarrow{Aa}$ and $\overrightarrow{Bb}$ emanating from A and B, then the exterior angle β at B is strictly greater than the interior angle α at A.*



{fig:extangle} FIGURE 3. External angle

Proof. Let $\overrightarrow{Ac}$ be the ray through A on the same side of A as $\overrightarrow{Aa}$ and making an angle β with AB. Then $\overrightarrow{Ac} \parallel \overrightarrow{Bb}$. Therefore $\alpha \leq \beta$.

Thus we have to rule out the case $\alpha = \beta$. Suppose that happens. Then let $\overrightarrow{Aa'}$ and $\overrightarrow{Bb'}$ be the opposite rays as in figure 3. Then $\angle BAa' \cong \angle ABb$. Therefore by ASAL (top angles are the same as the opposite lower angles), $\overrightarrow{Aa'} \mid\mid\mid \overrightarrow{Bb'}$. Now the sum of the top angles is the same as the sum of the bottom angles is the same as $180°$. This contradicts (L) as the two limiting parallels cannot lie on the same straight line. $\square$

Proof of theorem 1.1.2. We know that there exists a Saccheri quadrilateral such that the sum of the top two angles is the same of the angles of the given triangle.

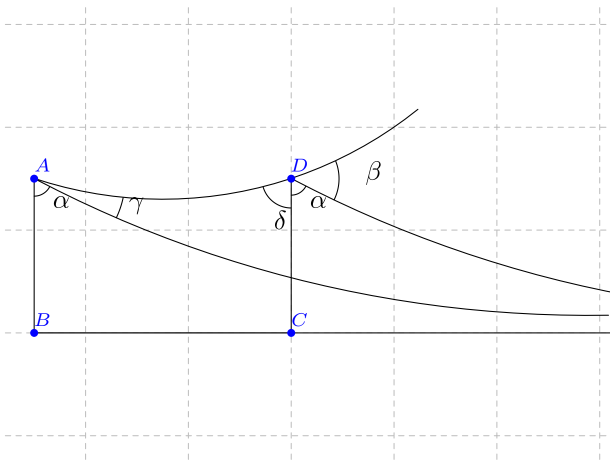


FIGURE 4. Saccheri Quadrilateral corresponding to a triangle {fig:sactrngl}

Let $ABCD$ be such a quadrilateral (figure 4). We want to prove that $\delta < 90°$. Note that the limiting parallels from C and D define the angles γ, β. $\alpha = \alpha(AC) = \alpha(BD)$. Note $\delta + \alpha + \beta = 180°$. On the other hand the exterior angle theorem says that $\beta > \gamma$. Therefore, $2\delta = \delta + \alpha + \gamma < \delta + \alpha + \beta = 180°$. Therefore, $\delta < 90°$. This completes the proof. $\square$

2. Hyperbolic geometry: Poincaré model

Next we shall see a model for the above geometry.

2.1. **The Poincaré model.**

2.1.1. Just described the points and lines. The rest will be done after mid-sem.