Hyperbolic geometry: sum of angles of a triangle

1. Saccheri Quadrilaterals

1.1. **Some preliminary results.**

1.1.1. **Theorem.** *Given a triangle $\triangle ABC$, there is a Saccheri quadrilateral for which the sum of two top angles is equal to the sum of three angles of the triangle.*

Proof. Refer to figure 1.

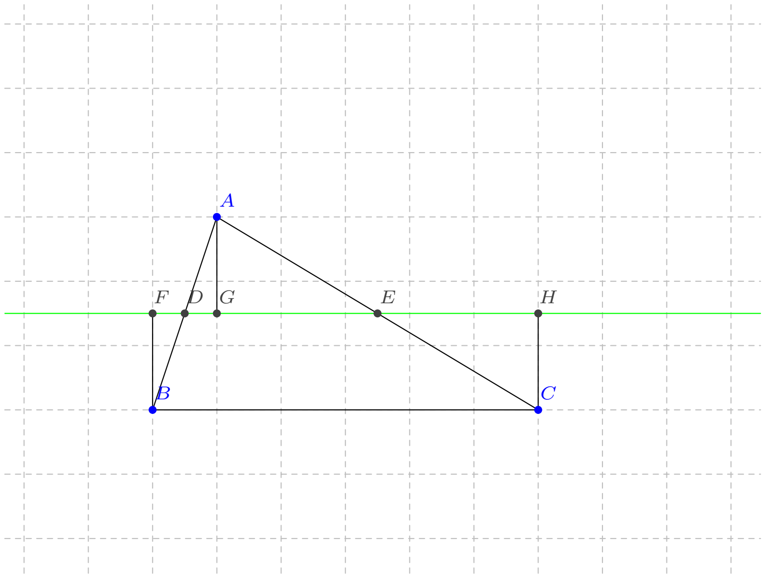


Figure 1. Saccheri quadrilateral and triangle {fig:sacctrng}

Let $\triangle ABC$ be given. D and E are the mid-points of AB and AC respectively. Join D and E be a line and drop perpendiculars from A, B and C. Using the facts that $AD = DB$, $AE = EC$ and opposite angles are equal, we conclude that $\triangle ADG \cong \triangle BDF$ and $\triangle AGE \cong \triangle CHE$. Therefore, $\angle FBC + \angle HCB = \angle FBD + \angle DBC + \angle ECB + \angle ECH = \angle DAG + \angle ABC + \angle ACB + \angle GAE = \angle ABC + \angle ACB + \angle BAC$ as was to be proved.

1.1.2. *Exercise.* This is not all. Find out the missing case and provide a proof for that.

□

1.1.3. **Theorem** (ASAL)**.** *Given four rays $\overrightarrow{Aa}$, $\overrightarrow{Bb}$, $\overrightarrow{A'a'}$ and $\overrightarrow{B'b'}$, assume that $\angle BAa = \angle B'A'a'$, $AB = A'B'$ and $\angle ABb = \angle A'B'b'$. Then $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb}$ if and only if $\overrightarrow{A'a'} \,|||\, \overrightarrow{B'b'}$*

Proof. We prove by contradiction. Suppose $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb}$ but $\overrightarrow{A'a'} \,\not|||\, \overrightarrow{B'b'}$. Then two things can happen:

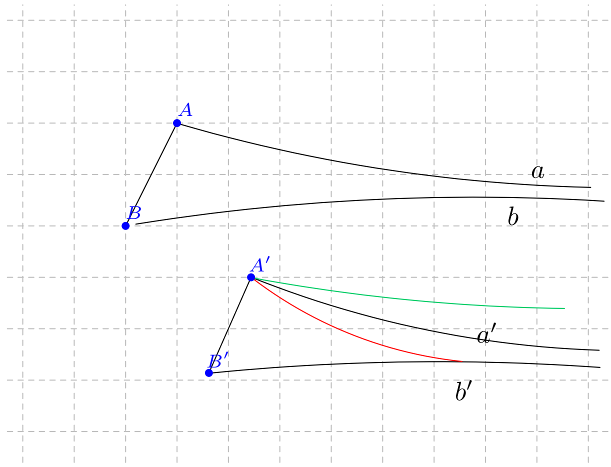
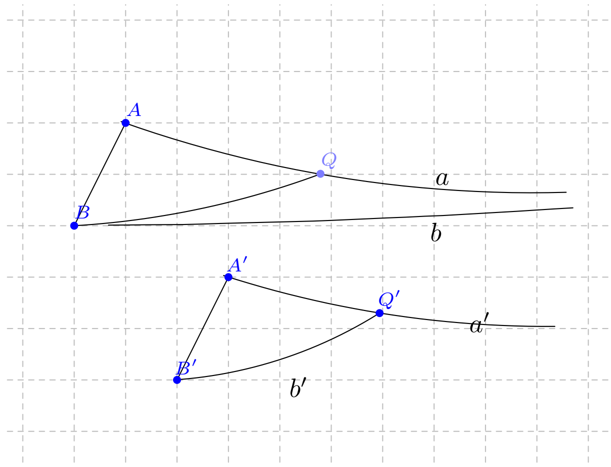


FIGURE 2. Proof of ASAL {fig:asalposs}

(1) $\overrightarrow{B'b'}$ meets $\overrightarrow{A'a'}$ (the red line); or
(2) The limiting parallel to $\overrightarrow{B'b'}$ through A' lies between $A'a'$ and $B'b'$. (This would be the case of $\overrightarrow{B'b'}$ were the green ray.)

We shall prove the first case and leave the second case as an exercise.

Suppose Q' be the point where $A'a'$ meets $B'b'$. Cut of a point Q on the ray $\overrightarrow{Aa}$ at a distance $A'Q'$ from A. Join A and Q.



{fig:asal-prf} FIGURE 3. Proof of ASAL

Now $\angle BAa \cong \angle B'A'a'$, $AB \cong A'B'$ (given) and $AQ \cong A'Q'$ (by construction) implies that $\triangle ABQ \cong \triangle A'B'Q'$. In particular $\angle ABQ \cong \angle A'B'Q' \cong \angle ABb$ implies that $Q \in \overrightarrow{Bb}$ by (C4) which says that there exists a unique ray which attends a given angle on a given side of a ray. Therefore, $\overrightarrow{Aa}$ cannot be a limiting parallel to $\overrightarrow{Bb}$ which is a contradiction.

Thus $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb} \implies \overrightarrow{A'a'} \,|||\, \overrightarrow{B'b'}$. The reverse inclusion follows by reversing the roles of the primed letters and the unprimed letters. □

1.1.4. **Theorem** (ASL)**.** *Suppose we are given rays* $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb}$ *and* $\overrightarrow{A'a'} \,|||\, \overrightarrow{B'b'}$. *Also assume* $AB \cong A'B'$ *and* $\angle BAa \cong \angle B'A'a'$. *Then* $\angle ABb \cong \angle A'B'b'$.

Proof. Exercise. □

1.2. Hyperbolic Plane.

1.2.1. **Definition.** A Hilbert plane satisfying the axiom L is called a *Hyperbolic plane*.

1.2.2. **Definition.** For any segment AB, let b be the line perpendicular to AB at B. Chose one ray $\overrightarrow{Bb}$ on b. Let $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb}$. Then $\alpha(AB) := \angle BAa$ is called the *angle of parallelism* of the segment AB.

1.2.3. *Remark.* Because of the axiom (L), the angle of parallelism is always acute ($< 90°$).