Hyperbolic Geometry

1. Introduction

We shall now look at a geometry where, given any line there are more than one parallel line through a point.

1.1. **Hilbert Plane.** Hilbert plane was a geometry, that is a collection of points and lines, with the notion of length and angle which satisfy the following axioms. Axioms of incidence, axioms of betweenness, axioms of commensurability.
I1. For any two distinct points A and B, there exists a unique line l containing A and B.
I2. Every line contains at least two points.
I3. There exists three non-collinear points.
B1. If B is between A and C (written $A * B * C$), then A, B and C are three distinct points on a line, and also $C * B * A$.
B2. For two distinct points A, B there exists a point C such that $A * B * C$.
B3. Given three distinct points on a line, one and only one of them is between the other two.
B4. (Pasch) Let A, B and C be three non-collinear points, and let l be a line containing any of A, B and C. If l contains a point D lying between A and B then it must contain *either* a point lying between A and C *or* a point lying between B and C *but* not both.
C1. Given a line segment $\overline{AB}$, and given a ray $\overrightarrow{r}$ originating at a point C, there exists a unique point D on the ray $\overrightarrow{r}$ such that $\overline{AB} \cong \overline{BC}$.
C2. If $AB \cong CD$ and $AB \cong EF$ then $CD \cong EF$. Every line segment is congruent to itself.
C3. (Addition) Given three points A, B, C on a line satisfying $A * B * C$, and three further points D, E, F satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.
C4. Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$ on a given side of DF such that $\angle BAC \cong \angle EDF$.
C5. For any three angles α, β and γ, if $\alpha \cong \beta$, $\beta \cong \gamma$, then $\alpha \cong \gamma$. Every angle is congruent to itself.
C6. (SAS) Given triangles ABC and DEF suppose $AB \cong DE$, $AC \cong DF$ and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent.

To get the Euclidean plane we had added the parallel axiom.
P. For each point A and each line l, there is at most one line containing A that is parallel to l.

However, we intend to study geometries other than Euclidean plane. We start with all the axioms of a Hilbert plane, and replace the parallel axiom by the following axiom of limited parallels.

1.2. **Limiting parallel.** We shall consider geometries which satisfy all the axioms of a Hilbert plane, and does not satisfy the parallel axiom. We replace the parallel axiom by the following axiom of Hilbert.
L. For each line ℓ and each point A not on ℓ, there are two rays Aa and Aa' from A and not lying on the same line, and not meeting ℓ such that any ray An in the interior of the angle $\angle aAa'$ meets ℓ.

1.2.1. **Definition.** Two rays $\overrightarrow{Aa}$ and $\overrightarrow{Bb}$ are said to be *coterminal* if they lie on the same line and one is a subset of the other ("go in the same direction."). $\overrightarrow{Aa}$ is a *limiting parallel* to $\overrightarrow{Bb}$ if **either** they are coterminal, **or** if they lie on distinct lines not equal to the line AB, they do not meet, and every ray in the interior of the angle $\angle BAa$ meets the ray $\overrightarrow{Bb}$. The notation for this is $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb}$.

1.2.2. **Theorem.** *Being limiting parallels is an equivalence relation.*

Proof. We assume this result and the following corollary. □

1.2.3. **Corollary.** *If $\overrightarrow{Aa}$ is a limiting parallel to the ray $\overrightarrow{Bb}$, and if C is a point between A and B, and if $\overrightarrow{Cc}$ is a ray entirely in the interior of the angles $\angle BAa$ and $\angle ABb$, then $\overrightarrow{Cc}$ is also a limiting parallel to $\overrightarrow{Aa}$ and $\overrightarrow{Bb}$.*

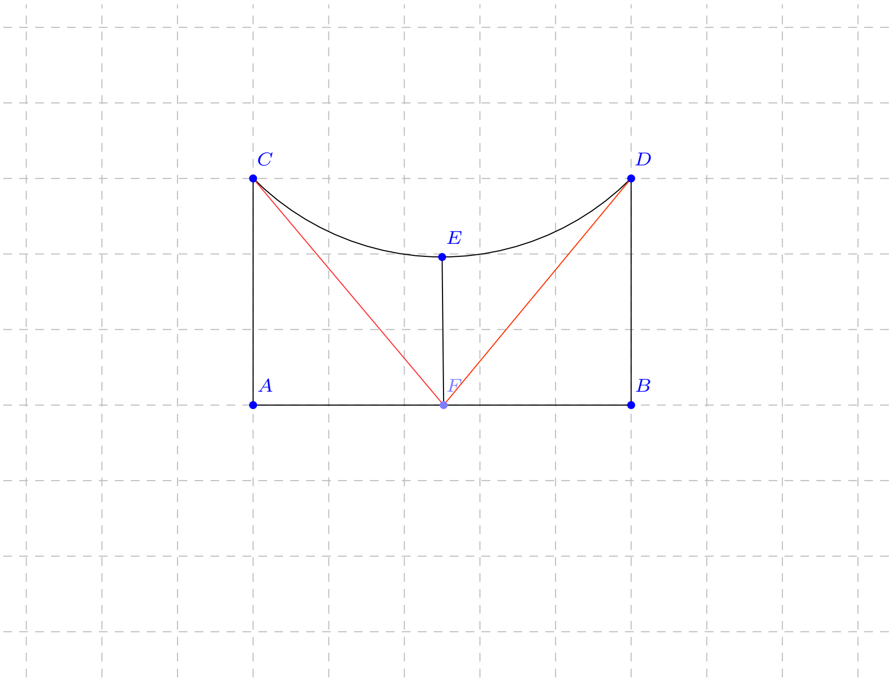
1.3. **Saccheri quadrilateral.**

1.3.1. **Proposition.** *In a Hilbert plane, suppose that two equal perpendicular AC and BD stand at the ends of an interval AB. Join CD with a line. (*Such a quadrilateral is called Saccheri*). Then the angles C and D are equal. Furthermore the line joining the midpoints of AB and CD (*called the midline*) is perpendicular to both.*

Proof. Refer to figure 1.

Let $ABCD$ be as in the hypothesis, and let E be the midpoint of AB. Let l be the perpendicular to AB at E. By Euclid's I.27, if n interesects l and m with alternate angles equal, then $l \parallel m$. Therefore, l is parallel to both AC and BD. Therefore, A and C lie on the same side of l; and so does B and D. But A and B lie on opposite sides by construction.

Now by construction, $AF \cong BF$, $\angle CAF \cong \angle DBF$ and $AC \cong BD$. Therefore $\triangle CAF \cong \triangle DBF$. In particular, $\angle CFA \cong \angle DFB$ and $CF \cong DF$. Therefore, $\angle CFE \cong \angle DFE$ and again by SAS, $\triangle CEF \cong \triangle DEF$ and hence $CE \cong DE$, $\angle FEC \cong \angle FDE$ equal to a right angle, and $\angle FCE \cong \angle FDE$. Therefore $\angle ACE \cong \angle ACF + \angle FCE \cong \angle BDF + \angle FDE \cong \angle BDE$. This completes the proof. □



{fig:sachquad}

FIGURE 1. Saccheri quadrilateral