Some remaining bits and some introduction to hyperbolic geometry

1. Remaining circular inversion

1.1. **Cross ratio.** Before we go into hyperbolic geometry, let us introduce a concept : *cross ratio*. We shall deal with them in detail later, but right now we just note that circular inversion respects cross ratios.

1.1.1. **Definition.** Suppose A, B, P and Q be four points on $\mathbb{R}^2$. Then the cross ratio is defined to be the number

$$(AB,\ PQ) = \frac{AP}{AQ} \div \frac{BP}{BQ}.$$

1.1.2. **Proposition.** *Let Γ be a circle with centre O. If A, B, P and Q are four distinct points all different from the center O, then the circular inversion of these points A', B', C' and D' satisfy*

$$(AB,\ CD) = (A'B',\ C'D')\,.$$

Proof. First consider O, A and P. Since $OA \cdot OA' = OP \cdot OP' = r^2$,

$$\frac{OA}{OP} = \frac{OP'}{OA'}$$

First suppose O, A and P are *not* collinear. Since the angle O is common the triangles OAP and $OP'A'$ are similar and hence

$$\frac{AP}{A'P'} = \frac{OA}{OP'}.$$

Even if the three points are collinear, we have

$$\frac{OA}{OP'} = \frac{OP}{OA'} = \frac{OA - OP}{OP' - OA'} = \frac{AP}{A'P'}.$$

Using the same argument for A and Q we get

$$\frac{AQ}{A'Q'} = \frac{OA}{OQ'}.$$

Therefore

$$\frac{AP}{A'P'} \div \frac{AQ}{A'Q'} = \frac{OQ'}{OP'}.$$

Doing this for B, P and Q we get

$$\frac{BP}{B'P'} \div \frac{BQ}{B'Q'} = \frac{OQ'}{OP'}.$$

Comparing the last two equations we get the result. □

Thus circular inversion preserves cross ratios. We shall revisit cross ratios when we do projective geometry.