Circular Inversion and Hyperbolic geometry

1. Circular Inversion

1.1. **Stereographic projection is conformal.**

1.1.1. Clircles are circles and lines.

1.1.2. Now the result follows from the fact that reflections take circles to circles. For conformality, we just have to check that stereographic projection is conformal. Which is what we do the next.

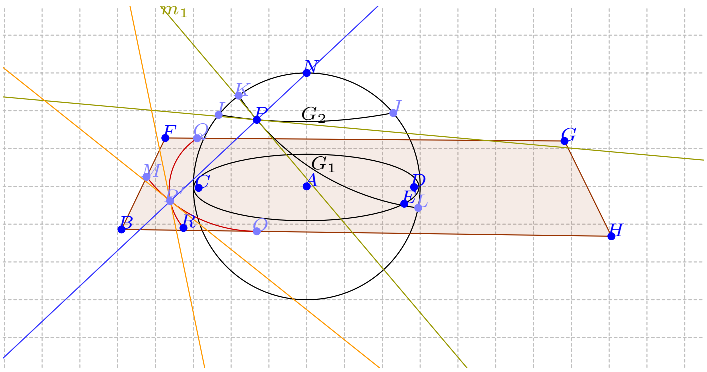


Figure 1. Stereographic projection {fig:stereocn}

1.1.3. Suppose Γ_1 and Γ_2 are two circles which intersect at a point P. Consider the line ℓ passing through N and P. At P, let m_1 be the tangent to Γ_1 and m_2 be the tangent to Γ_2. Let Q_i be the plane containing ℓ and m_i for $i = 1, 2$. Suppose Q_i intersect the $\mathbb{C} = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; z = 0\right\}$ plane at the lines n_1 and n_2 repectively. Note that the steregraphic projection of P, $\psi(P)$ lies on ℓ and on $\mathbb{C}$ and hence n_1 and n_2 intersect at $\psi(P)$.

1.1.4. Let set up some notation:

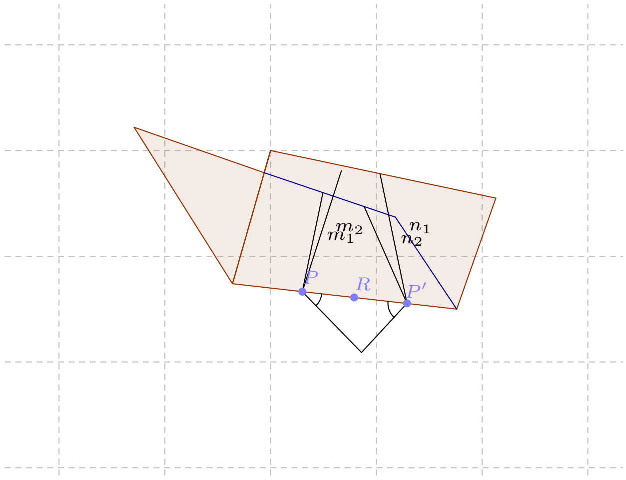
$\langle A, B\rangle$	The unique plane containing the object/objects A, B etc.,
Q_i	$\langle \ell, m_i\rangle$,
Π_i	$\langle \Gamma_i\rangle \ni m_i$,
T_p	$\langle m_1, m_2\rangle$,
$\mathbb{C}$	the X-Y plane,
$\angle lm$	Angle between the curves l and m,
$\angle ABC$	$\angle\overleftrightarrow{AB}\overleftrightarrow{BC}$.

1.1.5. Note that there are maps $p_i : \Pi_i \to \mathbb{C}$ given by $p_i(A) = \overleftrightarrow{NA} \cap \mathbb{C}$.

1.1.6. *Exercise.* Prove that p_i is a bijection for $i = 1, 2$.

1.1.7. Note that $p_i(\Gamma_i) = \psi(\Gamma_i)$ and $p_i(m_i) = n_i$ as is evident from the definitions. Therefore n_i are tangent to $\psi(\Gamma_i)$. Thus we only have to prove that $\angle m_1 m_2 = \angle n_1 n_2$

1.1.8. **Lemma.** *Suppose Ω_1 and Ω_2 be two planes intersecting at the line μ. Suppose Σ and Λ be two planes intersecting the line μ at points S and L, say. If the perpendiculars σ and λ to these planes from their points of intersection (S and L) with μ meet at a point Q such that $\triangle QSL$ is isosceles with $QS \cong QL$, then $\angle(\Omega_1 \cap \Sigma)(\Omega_2 \cap \Sigma) = \angle(\Omega_1 \cap \Lambda)(\Omega_2 \cap \Lambda)$.*



{fig:anglplne} FIGURE 2. Angles between planes

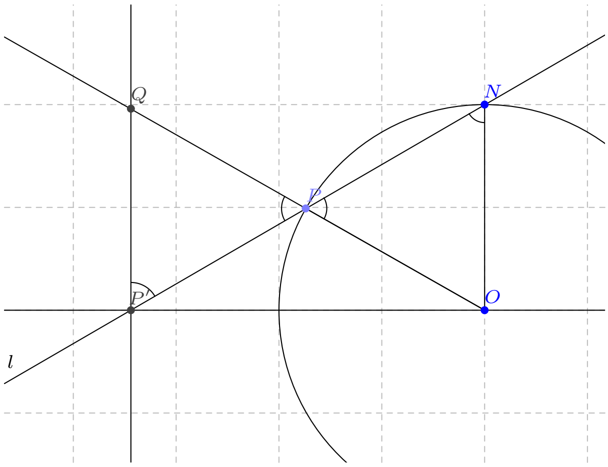
Proof. This is easy. Consider the midpoint of SL, say R and consider the plane Ξ perpendicular to SL at R. Then the reflection along Ξ keeps Ω_i intact, interchanges S and T and interchanges QS with QL. Therefore it also interchanges Σ and Λ. As reflection preserves angles we have the result. $\square$

1.1.9. For us, $\Omega_i = Q_i$, $\Sigma = T_P$ and $\Lambda = \mathbb{C}$. Thus we have to show that the prependicular λ_P of T_P at P intersects the prependicular $\lambda_{\psi(P)}$ of $\mathbb{C}$ at $\psi(P)$ at, say, Q and we have to show that $QP \cong Q\psi(P)$. For that consider the plane $\Delta = \langle N, O, P\rangle$ where O is the centre of the sphere. Since $P' = \psi(P)$ lies on the line $\overleftrightarrow{NP}$, it lies on Δ. Furthermore, $\lambda_{\psi(P)} = \overleftrightarrow{PQ}$ is parallel to ON and passes through $P' = \psi(P)$ and hence lies on Δ. Thus the whole picture above lies on Δ. Since $ON = OP$ is the radius of the sphere, $\angle OPN = \angle ONP$. This means that the opposite angle $\angle QP\psi(P) = \angle OPN = \angle ONP = \angle Q\psi(P)P$, where the last equality follows as they are alternate angles on parallel lines ON and $Q\psi(P)$. This proves that $QP\psi(P)$ is an isosceles triangle with $QP = Q\psi(P)$ as was to be proved. Thus the hypothesis of the above lemma holds. Therefore,

$$\angle m_1 m_2 = \angle(Q_1 \cap T_P)(Q_2 \cap T_P) = \angle(Q_1 \cap \mathbb{C})(Q_2 \cap \mathbb{C}) = \angle n_1 n_2.$$

1.2. **Circular inversion is conformal.**

1.2.1. Since we already checked conformality of the stereographic projection, let us first we give a proof using that. For that first we need to see that if ρ denotes a reflection of the sphere along $z = 0$, then $\psi \circ \rho \circ \psi^{-1}$ corresponds to circular inversion along the unit circle around origin. Checking conformality of this is enough as any



{fig:applnlem} FIGURE 3. Figure for the application of the lemma

other circular inversion can be obtained by this by a composition of translation and scaling, both of which are conformal. Since ρ and ψ are conformal, this is all we have to check.

1.2.2. Note that the reflection of the sphere S^2 along the complex plane corresponds to the transformation $(u, v, w) \mapsto (u, v, -w)$. Now $\psi(u, v, w) = (u+iv)/(1-w)$ and $\psi(u, v, -w) = (u + iv)(1 + w)$. Suppose Z be a complex number. We had written down formulas for u, v and w in terms of Z last time. $Z = (u + iv)/(1 - w)$.

$$\begin{aligned} \psi \circ \rho \circ \psi^{-1}(Z) = \psi \circ \rho(u, v, w) = \\ \psi(u, v, -w) = \frac{u+iv}{1+w} = \frac{(u+iv)(1-w)}{1-w^2} \\ = \frac{(u+iv)(1-w)}{u^2+v^2} = \frac{1-w}{u-iv} = \frac{1}{\bar{Z}}. \end{aligned}$$

Thus stereographic projection of the reflection is the circular inversion along the unit circle centered at origin.