Stereographic Projection and Circular Inversion

1. Extended complex line: Riemann sphere

1.1. Adding ∞.

1.1.1. Consider the extended complex number system $\mathbb{C} \cup \{\infty\}$, with the following convention :

$$Z + \infty = \infty, \qquad \frac{Z}{\infty} = 0,$$
$$W \cdot \infty = \infty, \qquad \frac{W}{0} = \infty$$

for any complex number Z and any *non-zero* complex number W. and we say that the following operatons are undefined :

$$\infty\infty, \quad \infty + \infty, \quad \infty 0, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}, \quad \frac{\infty}{0}.$$

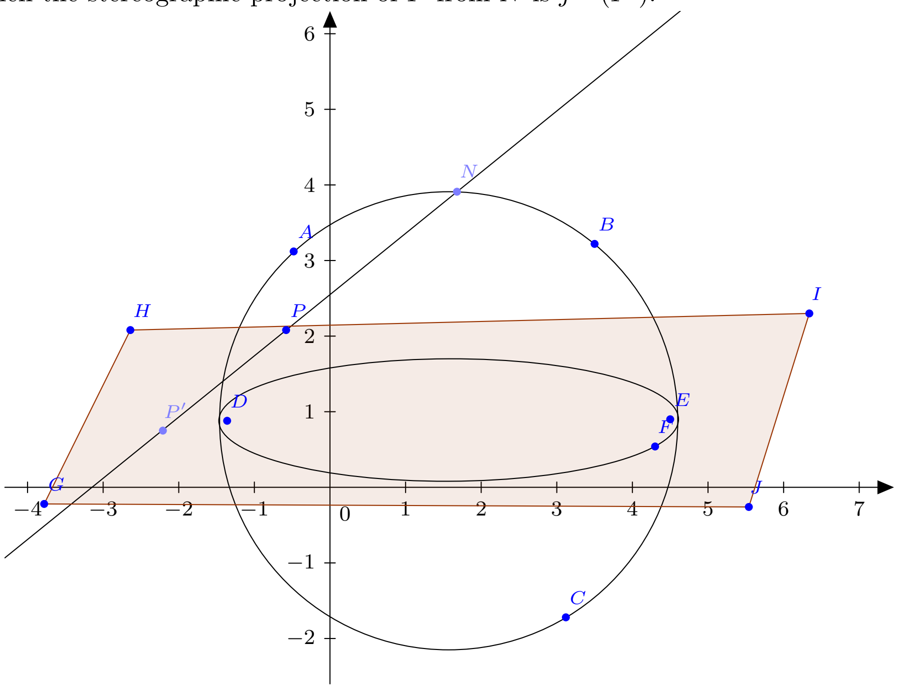
We denote the extended complex plane by $\mathbb{C}^+$.

1.2. Viewing the extended line as a sphere : Stereographic projection.

1.2.1. Think of the complex plane as being embedded in $\mathbb{R}^3$ as the plane $z = 0$: $j : \mathbb{C} \hookrightarrow \mathbb{R}^3$ where $j(x + iy) = (x, y, 0)$. Consider the unit sphere

$$S^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = 1 \right\} \subset \mathbb{R}^3.$$

Let us define the north pole, $N = (0, 0, 1)$. Now we construct map $\psi : S^2 \setminus \{N\} \to \mathbb{C}$ as follows. For any point $P \in S^2$, $P \neq N$, consider the intersection of the line passing through N and P and the plane $z = 0$. Suppose the intersection is P'. Then the stereographic projection of P from N is $j^{-1}(P')$.



1.2.2. Let us compute some formulas. Suppose $P = (u, v, w)$ and $Z = j^{-1}(P') = x+iy$. Then the distance of P from the vertical line ON is $\rho = \sqrt{u^2+v^2}$. Suppose the perpendicular from P meets ON at O'. $r = \sqrt{x^2+y^2}$ is the distance of Z from the origin O. Comparing the sides of the similar triangles $\triangle PO'N$ and $\triangle ZON$, we get

$$\frac{r}{1} = \frac{\rho}{1-w}.$$

Therefore,

$$\frac{\rho}{r} = \frac{1-w}{1} = \frac{u}{x} = \frac{v}{y}.$$

Therefore, $x = u/(1-w)$ and $y = v/(1-w)$. Thus,

$$Z = \frac{u+iv}{1-w}; \qquad \bar{Z} = \frac{u-iv}{1-w}.$$

Thus, $\psi(u, v, w) = (u + iv)/(1 - w)$.

1.2.3. *Exercise.* Construct the inverse of the above map: Write u, v, w in terms of Z where $Z = (u + iv)/(1 - w)$. You should get the following :

$$u = \frac{Z+\bar{Z}}{Z\bar{Z}+1} \qquad\qquad v = \frac{i(\bar{Z}-Z)}{Z\bar{Z}+1}$$

$$w = \frac{Z\bar{Z}-1}{Z\bar{Z}+1}.$$

1.2.4. Now we prove that the stereographic projection takes circles on the unit sphere not passing through N to circles on the complex plane and vice versa. All the circles passing through N are mapped to lines and the lines on the complex plane are mapped back to circles passing through N.

1.2.5. Consider the plane $au + bv + cw = d$, or $(a, b, c) \cdot (u, v, w) = d$. Since $(a, b, c) \cdot (u, v, w) \le \sqrt{a^2+b^2+c^2}\sqrt{u^2+v^2+w^2}$,

$$\sqrt{u^2+v^2+w^2} \ge \frac{(a,b,c)\cdot(u,v,w)}{\sqrt{a^2+b^2+c^2}} = \frac{d}{\sqrt{a^2+b^2+c^2}}.$$

Note that the bound on the right is achieved for $(u, v, w) = (da, db, dc)/\sqrt{a^2+b^2+c^2}$.

Thus the shortest distance from the origin is $d/\sqrt{a^2+b^2+c^2}$. Thus, the plane will intersect the sphere if and only if $d^2 < a^2 + b^2 + c^2$.

Suppose this condition holds for the plane $au + bv + cw = d$ and suppose the intersection, which is a circle, does not pass through N. The necessary and sufficient condition for the circle to pass through N is $d = c$.

1.2.6. *Exercise.* Check that for P on such an intersection (with $c \ne d$), Z satisfies the equation

$$Z\bar{Z} - \bar{C}Z - C\bar{Z} + C\bar{C} = \frac{a^2+b^2+c^2-d^2}{c-d}, \text{ where } C = \frac{a+ib}{c-d}.$$

which is a circle with center C and radius $\sqrt{(a^2+b^2+c^2-d^2)/(c-d)}$.

1.2.7. *Exercise.* When the plane passes through N (that is, $d = c$), prove that the image of the intersection circle ($\setminus \{N\}$) is the line

$$(a-ib)Z + (a+ib)\bar{Z} = 2d.$$

Thus if we think of a line as a circle of infinite radius, the stereographic projection takes circles to circles.

2. Circular Inversion again

2.1. **Reflection of the sphere.** Recall that the circular inversion along the unit circle centered at origin corresponds to

$$I(Z) = \frac{1}{\bar{Z}}.$$

2.1.1. In general inversion around a circle of radius r and center C is given by

$$I(Z) = C + \frac{r^2}{\overline{Z - C}} = \frac{C\bar{Z} - C\bar{C} + r^2}{\bar{Z} - \bar{C}}.$$