Transformations and matrices; Circle inversions; complex numbers

1. Affine transformations as matrices

I will come back to this later. Let us look at some other transformations on the Euclidean plane first. This will also help us to study the Poincaré disk model of the hyperbolic plane later on.

2. Circle inversions; cross ratios

2.1. **Geometric constuction.** Let Γ be a circle with center O and radius r and P be any point on the Euclidean plane $P \neq O$.

2.1.1. **Definition.** The *circular inversion* of P is a point P' on the ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. Clearly if $P \in \Gamma$ then $P' = P$.

2.1.2. A ruler and compass construction of the circular inverse is as follows (see figure 1.)

(1) If P lies inside Γ, draw the line passing through OP. Draw a perpendicular OP at P. Suppose it meets Γ at A (take any one of the intersections). The tangent to Γ at A intersects OP at P'.
(2) If P lies outside Γ, draw a tangent from P to Γ. Suppose the tangent meets the circle at A. Drop a perpendicular from A to OP to get P'.

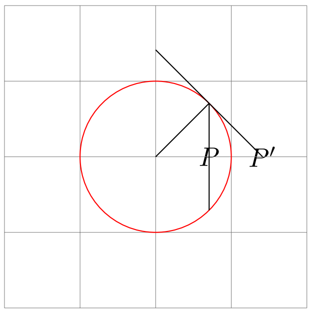


Figure 1. Figure for circular inversion {fig:circinvn}

2.1.3. Now we prove that $OP \cdot OP' = r^2$. Suppose the point the perpendicular to OP intersects the circle is A. Then $\angle OPA = \angle OAP' = 90°$. On the other hand $\angle AOP = \angle P'OA$ is common. Therefore the triangles $\triangle OPA$ and $\triangle OAP'$ are similar. Therefore, the sides are proportional:

$$\frac{OP}{OA} = \frac{OA}{OP'},$$

or in other wordes, $OP \cdot OP' = OA^2 = r^2$ as was to be proved.

2.2. **Some algebra.** Formula for circlular inversion.

2.2.1. Suppose Γ has center (a, b) and radius r. Suppose $P = (p, q)$. Then $OP = \sqrt{(p-a)^2 + (q-b)^2}$. Therefore,

$$OP' = \frac{r^2}{\sqrt{(p-a)^2 + (q-b)^2}}$$

Thus, since P' lies on the ray OP,

$$P' = (a, b) + \frac{OP'}{OP}(p - a, q - b) = (a, b) + \frac{r^2}{(p-a)^2 + (q-b)^2}(p - a, q - b).$$

2.2.2. This has an interesting corollary. Suppose, Γ was the unit circle around origin. Then for $P = (p, q)$,

$$P' = \frac{1}{p^2 + q^2}(p, q) = \frac{1}{\|P\|^2} P.$$

2.2.3. Consider Γ to be the unit circle through origin. Suppose $\gamma(t) = (\cos t + 1, \sin t)$ or

$$(\frac{1 - t^2}{1 + t^2} + 1, \frac{2t}{1 + t^2})$$

be the parametric equation of the circle $(x - 1)^2 + y^2 = 1$ passing through $(0, 0)$. The inversion of $\gamma(t)$ is

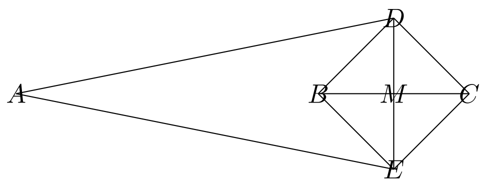
$$\begin{aligned} \frac{1}{(\cos t + 1)^2 + \sin^2 t}(\cos t + 1, \sin t) &= \frac{1}{1 + 2\cos t + 1}(\cos t + 1, \sin t) \\ &= \frac{1}{2}\left(1, \frac{\sin t}{\cos t + 1}\right). \end{aligned}$$

From this we conclude that the circle $\gamma(t)$ goes to the line $x = 1$.

2.3. **Preserves angles.** This is our first example of a conformal map which is not an isometry. Though we can prove that circle inversion takes "circles and lines" to "circle and lines"; and not only that, it also preserves angles between them using purely geometric argument, we shall avoid doing that. We shall introduce steriographic projections and some tricks involving complex numbers to prove this.

2.4. **A real life application.** Before going on to prove something, let us demonstrate a nice application of circle inversion to real world: Paucellier's apparatus.

2.4.1. Consider an apparatus consisting of six shafts arranged as in the picture 2. In this picture, A is fixed and all the other points are free to move. We prove that



{fig:paucelli} FIGURE 2. Pacucellier's apparatus

C is a circular inversion of B and hence if B moves along a circle passing through A, C will move in a straight line.

2.4.2. To see this, suppose the segments DE and BC meet at M. Then, first prove that A, B, M and C are collinear. Then

$$\begin{aligned} AB \cdot AC &= (AM - BM)(AM + MC) = (AM - BM)(AM + BM) \\ &= AM^2 - BM^2 = (AD^2 - DM^2) - (BD^2 - DM^2) = AD^2 - BD^2 \end{aligned}$$

which is a constant.

3. $\mathbb{R}^2$ as the complex plane $\mathbb{C}$, stereographic projection

3.1. **A review of complex numbers.** Addition, multiplication, conjugation, division.

3.2. **Rigid body motions as complex manipulations.**

3.2.1.

Translation	Addition
Rotation around 0	Multiplication by $e^{i\theta}$
Reflection along x-axis	conjugation

3.3. **Equations of lines and circles.**

3.3.1. A general line can be written in the form $ax + by = c/2$. Let $A = a + ib$ and $Z = x + iy$. Then

$$A\bar{Z} + \bar{A}Z = (ax + by) + i(xb - ay) + (ax + by) + i(ay - xb) = 2(ax + by).$$

Therefore the equation takes the form $A\bar{Z} + \bar{A}Z = c$.

3.3.2. For a circle with center $C = (a+ib)$ and radius r, the formula is $\|Z-C\|^2 = r^2$ which is given by $(Z - C)\overline{(Z - C)} = r^2$. That is the equation becomes

$$Z\bar{Z} - Z\bar{C} - \bar{Z}C + C\bar{C} - r^2 = 0.$$

3.3.3. *Exercise.* Check that

(1) Reflection in the line $A\bar{Z} + \bar{A}Z = c$ is given by

$$cA - A^2\bar{Z}.$$

You can assume that $\|A\| = 1$.

(2) If B lies on a circle with centre C and radius r, then show that the equation of the tangent at B is

$$(\bar{B} - \bar{C})Z + (B - C)\bar{Z} = B\bar{B} + C\bar{C} - r^2.$$