Lecture 6 : Conic sections

For this lecture, the main reference is "Roger Fenn : Geometry".

1. Ellipse, Parabola, Hyperbola

Consider a line ℓ making an angle α with the z-axis passing through the origin O. A cone is the two dimensional surface obtained by *rotating* ℓ around z-axis.

[Picture]

The formula for the cone will be

$$x^2 + y^2 = z^2 \tan^2 \alpha$$

1.1. **Conic sections.**

1.1.1. Conic sections are curves determined by the intersection of a cone with a plane. Suppose the plane is given by the equation $ax + by + cz + d = 0$, such that $(a, b, c) \neq (0, 0, 0)$.

Suppose $c \neq 0$. Then $z = px + qy + r$ where $p = -a/c$, $q = -b/c$ and $r = -d/c$. Now substituting back in the equation of a cone, we get a quadratic equation in x and y. Similar analysis can be done when $a \neq 0$ and $b \neq 0$. Thus an equation of a conic is always of the form

(1.1.2) {eqn:genconic} $$ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0.$$

1.1.3. The above equation can be rewritten in the form

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

1.1.4. *Exercise.* Prove that a conic given by the above equation is *degenerate*, that is, it is a union of two lines if and only if

$$\det \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix} = 0.$$

1.2. **Geometric definiton of conics.** There is a description of conics which restricts the definition only to a plane and would make sense on Euclidean plane.

Let ℓ be a line and P be a point *not* on ℓ. Let e be a positive real number. For a point A let $d(A, \ell)$ be the length of the perpendicular segment from A to ℓ. Then a conic is the set of all points A on the plane such that $d(P, A) = ed(A, \ell)$. e is called the *eccentricity* of the conic.

2. School examples

2.1. **Ellipse.** Circle falls in this class. The general equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are non-zero real numbers.

One can compute the eccentricity of the ellipse to be $\sqrt{1 - b^2/a^2}$ when $a > b$. Therefore, for an ellipse the eccentricity is less than one. This actually determines the ellipse. Note that a circle has $b = a$ and has eccentricity 0.

2.2. **Parabola.** The standard formula is

$$y^2 = 4ax.$$

In this case, the eccentricity is exactly 1.

2.3. **Hyperbola.** The standard equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

In this case, the eccentricity is $\sqrt{1 + b^2/a^2} > 1$.

3. Classification

3.1. **Transformations preserving "shape".** We work on coordinate geometry.

3.1.1. It is easy to see that the following procedure preserves distances and angles.

Translations: Given $a, b \in \mathbb{R}$, one can consider the transformation $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ sending $(x, y) \mapsto (x + a, y + b)$. It clearly preserves distances. It is easy to see that it preserves angles too.

Rotations: Given an angle θ, one can consider the rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ sending $(x, y) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. This is a rotation around the origin. For rotations around other points, one can compose translations and rotations. So rotations around (a, b) by an angle θ will be given by $T_{a,b} \circ R_\theta \circ T_{-a,-b}$.

Reflections: The reflection along the x-axis is easy to describe. It is given by $(x, y) \mapsto (x, -y)$. More general reflections can be realized by composing with rotations and translations. It is instructive to write down a formula for reflection along the line $ax + by + c = 0$.

3.1.2. Now the aim is to use these transformations to show that a general quadratic equation like (1.1.2) can be reduced to one of the standard forms we saw in the previous subsection.

3.2. **Using transformations to reduce the quadratic to a standard form.**

3.2.1. The easy case is when $h = 0$, that is the equation of the conic is of the form $ax^2 + by^2 + 2fx + 2gy + c = 0$. In this case, we can complete the square and use translation to reduce to the standard form. For example when a and b both are non-zero, we can write

$$\begin{aligned}
&a(x^2 + 2xf/a + (f/a)^2) + b(y^2 + 2y(g/b) + (g/b)^2) + c - f^2/a - g^2/b \\
=&a(x + f/a)^2 + b(y + g/b)^2 + (c - f^2/a - g^2/b) \\
=&0.
\end{aligned}$$

Now send $x \mapsto x - f/a$ and $y \mapsto y - g/b$.

3.2.2. *Exercise.* What happens when one of a or b is non-zero? Is there a case when cannot do the above reduction? Explain what happens in that case?

3.2.3. If the xy term is there (or equivalently $h \neq 0$), one can use rotations to get rid of it. Note that doing a rotation changes the equation to a form like

$$(\cdots)x^2 + ((b - a)\sin 2\theta + 2h \cos 2\theta)xy + (\cdots)y^2 + \text{lower degree terms}.$$

Thus one can choose θ such that one reduces to the previous case.

3.2.4. *Exercise.* Can this fail?