Lecture 5 : Hilbert and Euclidean planes

1. Rest of the Axioms

1.1. **Recall: Previous axioms.** Here are the axioms we already saw.
I1. For any two distinct points A and B, there exists a unique line l containing A and B.
I2. Every line contains at least two points.
I3. There exists three non-collinear points.
P. For each point A and each line l, there is at most one line containing A that is parallel to l.
B1. If B is between A and C (written $A * B * C$), then A, B and C are three distinct points on a line, and also $C * B * A$.
B2. For two distinct points A, B there exists a point C such that $A * B * C$.
B3. Given three distinct points on a line, one and only one of them is between the other two.
B4. (Pasch) Let A, B and C be three non-collinear points, and let l be a line containing any of A, B and C. If l contains a point D lying between A and B then it must contain *either* a point lying between A and C *or* a point lying between B and C *but* not both.

1.2. **Line segments.** Recall from last time $\overline{AB} = \{C \mid A * C * B\} \cup \{A\} \cup \{B\}$.

{prp:pointsep}

1.2.1. **Proposition.** *Let A be a point on a line l. Then the set of points of l not equal to A can be divided into two nonempty subsets S_1 and S_2, the two sides of A on l, such that*

(1) B, C are on the same side of A if and only if A is not in the segment $\overline{BC}$.
(2) B, D are on opposite sides of A if and only if A belongs to the segment $\overline{BD}$.

We need the following proposition to prove the previous proposition.

1.2.2. **Proposition.** *"A line separates the plane into two sides."*

We shall prove these propositions later. But what we notice is that it immediately helps us write down a bunch of definitions.

1.2.3. **Definition.** (1) Given two distinct points A, B the *ray* $\overrightarrow{AB}$ is the set $\{C \mid C = A \text{ or } C, B \text{ are on the same side of } A\}$. The point A is called the *vertex* or *origin*.
(2) An *angle* is the union of two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ originating from the same point, its vertex, and not lying on the same line.
(3) The *inside* or the *interior* of an angle $\angle BAC$ consists of all points D which lie on the same side AB and C and on the same side of BC as A.
(4) For a triangle $\triangle ABC$, an interior point is a point which is in the interior of all the three angles.

1.2.4. **Theorem** (Crossbar theorem). *Let $\angle BAC$ be an angle, and let D be the point in the interior of the angle. Then the ray $\overrightarrow{AD}$ must meet the segment $\overline{BC}$.*

1.3. **Axioms of Congruence.** This allows us the note down some more axioms.
C1. Given a line segment $\overline{AB}$, and given a ray $\overrightarrow{r}$ originating at a point C, there exists a unique point D on the ray $\overrightarrow{r}$ such that $\overline{AB} \cong \overline{BC}$.
C2. If $AB \cong CD$ and $AB \cong EF$ then $CD \cong EF$. Every line segment is congruent to itself.
C3. (Addition) Given three points A, B, C on a line satisfying $A * B * C$, and three further points D, E, F satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

These axioms are enough to prove that congruence of segments is an equivalence relation and one can take sums of congruence classes of segments (intuitively corresponding to addition of length). Then one can also construct difference of two segments.

1.3.1. **Definition.** Let AB and CD be given line segments. One says that AB is *less than* CD if there exists a point E, $C * E * D$, such that $AB \cong CE$. In that case, we also say that CD is *greater than* AB.

This gives on the congruence classes of line segments, which is transitive and for AB and CD, only one of $AB \cong CD$, $AB < CD$ or $AB > CD$ holds.

Note that coordinate geometry provides a standard model for this axiomatic system.
C4. Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$ on a given side of DF such that $\angle BAC \cong \angle EDF$.
C5. For any three angles α, β and γ, if $\alpha \cong \beta$, $\beta \cong \gamma$, then $\alpha \cong \gamma$. Every angle is congruent to itself.
C6. (SAS) Given triangles ABC and DEF suppose $AB \cong DE$, $AC \cong DF$ and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent.

This allows us to prove that congruence of angles is an equivalence, define supplementary angles and define addition of angles.

One can also define an ordering of angles and we have the same properties as segments. In this case, the Euclid's axiom that two right angles are congruent to each other becomes a proposition.

1.4. **Intersection of two circles.** Definition of a circle.

Center is determined.
E. (Circle-circle intersection property) Given two circles Γ and Δ, if Δ contains at least one point inside Γ, and Δ contaisn at least one point outside Γ, then Γ and Δ meet and that they will meet exactly in two points.

2. Some proofs

2.1. **Line separation property.**

2.2. **Existence of isosceles triangle.**

2.3. **Exterior angle theorem.**