Lecture 3 : Hilbert's Axioms

We shall now try to define geometry purely in terms of set theory. We start with existence of two sets,

(1) $\mathscr{P}$, whose elements, we shall call points; and
(2) $\mathscr{L}$ a collection of subsets of P. We call the sets in $\mathscr{L}$ as lines.

We want $\mathscr{P}$ and $\mathscr{L}$ to satisfy certain axioms, which we shall introduce over the next few lectures.

1. Axioms of Incidence

First I list the axioms.

1.1. The axioms of incidence.

{itm:linthpts}

I1. For any two *distinct points* A, B, there exists a *unique* line l containing both A and B. {itm:existwpt}
I2. Every line contains at least *two* points. {itm:notlineg}
I3. There exist three noncollinear points (that is, three points which do not line in a single line).

1.1.1. **Definition.** If l is a line and $P \in l$ is a point, we say that P *lies on* l, or that l passes through P.

1.1.2. **Definition.** An *incidence geometry* consists of a set $\mathscr{P}$ of points and a a collection of subsets of $\mathscr{P}$, $\mathscr{L}$ called lines, satisfying axioms of incidence 1, 2 and 3.

1.1.3. **Proposition.** *Two distinct lines can have* at most *one point in common.*

Proof. Suppose $l, m \in \mathscr{L}$ be two distinct lines. Suppose there are two distinct points A and B common to both the lines. Axiom of incidence 1 says there is a unique line passing through these two points and hence $l = m$. ☐

1.2. The models.
A *model* of an axiom system is a realization of undefined terms in some particular context, such that the axioms are satisfied.

1.2.1. **Model 1 : The real Cartesian plane**

Let $\mathscr{P} = \mathbb{R}^2$, the set of ordered pairs of real numbers.

$$\mathscr{L} = \{L \mid L = \{(x, y) \mid x, y \in \mathbb{R};\ ax + by + c = 0\};\ a, b, c \in \mathbb{R}\},$$

the set of all lines defined by linear equations of the form $ax + by + c = 0$ for variables a, b and c in $\mathbb{R}$.

1.2.2. *Exercise.* Check that all the axioms of incidence are satisfied.

1.2.3. **Model 2 : 3 points and 3 lines.**

Let $\mathscr{P} = \{A, B, C\}$, a set of three elements. Suppose the set of lines are $\mathscr{L} = \{\{A, B\}, \{B, C\}, \{A, C\}\}$.

1.2.4. *Exercise.* Check that this also forms an incidence geometry.

1.2.5. **Definition.** Two distinct lines are *parallel* if they have no points in common. We also say that any line is parallel to itself.

We also have Playfair's axiom which we *do not* include in incidence geometry.

P. For each point A and each line l, there is *at most* one line containing A that is parallel to L.

Note that both the above models satisfy parallel axiom. *Why does the second one satisfy?*

1.2.6. **Definition.** An *automorphism* of an incidence geometry is an isomorphism of the geometry with itself, that is, it is a 1-1 mapping of the set of points to itself, preserving lines.

1.2.7. There are six automorphisms of the second model!

1.3. **Some examples.**

1.3.1. *Example.*

$$\mathscr{P} = \{A, B, C, D, E\}$$
$$\mathscr{L} = \{\{P, Q\} \mid P \neq Q;\ P, Q \in \mathscr{P}\}$$

This is an example of a geometry in which (P.) fails. Note both $\{A, C\}$ and $\{A, B\}$ are parallel to $\{D, E\}$.

2. Axioms of Betweenness

Now we shall try to make the proof of exterior angle being greater than interior opposite angles rigorous. For that, we need the following axioms.

2.1. **The axioms.** Betweenness is an undefined relation between sets of three points. We say B is in between A and C, and the short hand notation is $A * B * C$.

{itm:betsymmt}

{itm:extenson} B1. $A * B * C$ implies that A, B and C are collinear and $C * B * A$.

{itm:betstlns} B2. For any two distinct points A and B, there exists a point C such that $A*B*C$.

B3. Given any three distinct points on a line, one and only one is between the other

{itm:betPasch} two.

B4. Let A, B and C be three non-collinear points. Let l be a line *not containing* any of A, B or C. If l contains a point D lying between A and B, then it must also contain a point lying between A and C, or a point lying between B and C, but not both.

2.2. **Models.**

2.2.1. A model will again the the real Cartesian plane. Suppose a, b and c be three *distinct* real numbers. We say that b lies between a and c (denoted $a * b * c$) if $a < b < c$ or $c < b < a$. Now for three points $A = (a_1, a_2)$, $B = (b_1, b_2)$ and $C = (c_1, c_2)$, we say that $A * B * C$, *if A, B and C are three distinct points on a line, and either $a_1 * b_1 * c_1$ or $a_2 * b_2 * c_2$ or both.*

2.2.2. One first checks that linear operations form $\mathbb{R}^2 \to \mathbb{R}$ preserves betweenness. Then uses them to check that the above notion of betweenness satisfies all the axioms.

2.3. **Some propositions.**

2.3.1. **Definition.** If A and B are distinct points, one defines the *line segment* $\overline{AB}$ to be the set

$$\overline{AB} = \{A\} \cup \{B\} \cup \{C \in \mathscr{P} \mid A * C * B\}$$

Given three *noncollinear* points A, B and C the *triangle* $\triangle ABC$ is defined to be

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}.$$

A, B and C are said to be the *vertices* of the triangle and $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are said to be the *sides* of the triangle.

2.3.2. Note that $\overline{AB} = \overline{BA}$ by the first betweenness axiom.

2.3.3. **Proposition** (Plane separation)**.** *Let l be any line. Then the set of points not lying on l can be divided into two non-empty subsets S_1 and S_2 with the following properties:*

(1) *Two points A and B not on l belong to the same set (S_1 or S_2) if and only if $\overline{AB}$ does not intersect l.*
(2) *Two points A, C not on l belong to the opposite sets, if and only if $\overline{AC}$ intersects l in a point.*

One refers to S_1 and S_2 as two sides of l and talks about points being on the same side of l or being on the opposite sides of l.

Proof. Let $S = \mathscr{P} \setminus l$. Let $\sim$ be a relation on S defined by $A \sim B$, $A, B \in S$ if and only if $A = B$ or $\overline{AB} \cap l = \varnothing$. We claim that this is an equivalence relation.

(1) $A \sim A$ follows from definition of $\sim$.
(2) $A \sim B \iff \overline{AB} \cap l = \varnothing \iff \overline{BA} \cap l = \varnothing \iff B \sim A$.
(3) Suppose $A \sim B$ and $B \sim C$. We want to show that $A \sim C$. If any two of A, B and C are equal, then there is nothing to prove. Assume that they are distinct. This also has two cases.
   (a) Suppose A, B and C are not collinear. Then consider the triangle $\triangle ABC$. Since $A \sim B$ and $B \sim C$, l does not meet $\overline{AB}$ and $\overline{BC}$. By the fourth axiom, l cannot meet $\overline{AC}$ either.
   (b) This is reduced to the first case by a construction which starts with choosing a point on l which is not on the line passing through A, B and C.

Now that we have an equivalence relation, one shows that there are at least two equivalence classes using the extension axiom.

To show that there are exactly two classes, one proves that $A \nsim B$ and $A \nsim C$ implies that $B \sim C$. This is also done in two cases as above, using B4 for the non-collinear case and reducing to the noncollinear case by choosing a point on l which is not on the line passing through A, B and C. $\square$