Lecture 2 : Euclidean geometry

1. Axioms of Euclid: Playfair's Axiom

Reference : Hartshorne : Section 3

1.1. **Axioms : Definitions and Postulates.**

1.1.1. Euclid's definitions: Definitions of points, lines, surfaces, boundaries, angles appeal to intuition and are not precise. A more modern way would be to define incidence relations (wait till we do Hilbert's axioms) leaving points and line undefined.

1.1.2. For example, in the definition of set, we don't say what a set is, we just assume we already know what inclusion is and write down some axioms about the inclusion, using it to construct new sets. We will do something similar for geometry.

1.1.3. For Euclid,
$$\begin{array}{ll} \text{line} & = \text{any curve,} \\ \text{straight line} & = \text{line as we think.} \end{array}$$

Also for an angle he needs that the two line segments should not line on a straight line. So the angles α are stricltly $0 < \alpha < 180°$.

1.1.4. **The first postulate.** To draw a straight line from any point to any point.

Euclid doesn't talk about uniqueness, though he tacitly assumes it in some of his proofs.

1.1.5. **The second postulate.** To produce a finite straight line continuously in a straight line.

1.1.6. **The third postulate.** To describe a circle with any centre and any distance.

1.1.7. **The fourth postulate.** That all right angles are equal to one another.

1.1.8. **The fifth (parallel) postulate.** That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

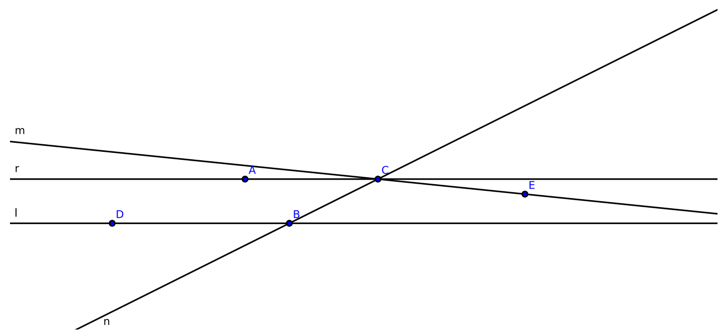
1.1.9. **Common notions.** Euclid also mentions some common notions, which deal with the word "equal". It is not clear what equal means. He does not define it. One can think of it as an equivalence class of objects. For example, one can talk about two (finite) segments being equal to mean that their lengths match. However Euclid avoided using the word length. However he did add, subtract and compare the things, for which he used the notion of "equality".

1.1.10. **Neutral Geometry.** Suppose we were to develop a geometry assuming only the first four postulates, but not the fifth. Then one can deduce Euclid's Book I, results 1 - 28 and 31. (Reference, Thomas Heath, Euclid's Elements.)

One of these results (result 16) states that, given a triangle (say ABC), the exterior angle at A is greater than any of the interior opposite angles (B or C). (I'll make some noise about its proof soon).

1.1.11. Now assume the **Parallel postulate.** In this paragraph, we shall prove that

> (P) Given a line l and a point P, not on l, there is exactly one line passing through P, parallel to l.



{fig:playfift} FIGURE 1. Playfair's Axiom implies the fifth postulate.

This is called Playfair's axiom. Euclid's construction was to draw any line through P meeting l. Construct a line m through P such that the alternate angles match. Claim is that this line is parallel to l. If these lines l and m are not parallel, then they meet at some point Q. Now one of the alternate angles will be exterior to this triangle and the other one will be interior opposite to this triangle, and hence the exterior one will be bigger. This is a contradiction. Therefore, the two line must be parallel.

Hence, given Euclid's first four postulates and the common notions, the fifth postulate implies Playfair's axiom.

1.1.12. Now suppose we assume Euclid's first four axioms, and Playfair's axiom, (P). Suppose l, m and n are three lines such that n meets l and m at B and C respectively (look at figure 1), such that the two interior angles on the same side are less than two right angles. Draw a line r such that the alternate angles with line l are equal. By the above argument, this line r cannot meet the line l, and hence they are parallel to each other. In particular, $\angle ECB + \angle BCA = \angle ECB + \angle CBF$ is less than two right angles. Therefore m and r are two different lines. Now since, there can be at most one parallel line through a point, m must meet l at some point. This is Euclid's fifth postulate.

1.2. **Cavaets: Problems of drawing "rough" pictures.**

1.2.1. Problems with tacitly assuming that curves and lines intersect.
This manifests in construction of the equilateral triangle.

1.2.2. Problems with superposition.
SAS theorem. Rotation, Translation, Reflection. Klein's Erlanger Programm.

1.2.3. Problems with assuming certain things are between certain other things.
Exterior angle greater than any of the interior opposite angles.

1.2.4. Fallacy of W. W. Rouse Ball (1940).

1.3. **Parallels, Area, Pythagoras.**

1.3.1. We say that Playfair is equivalent to Euclid's fifth postulates given all of the other Euclid's axioms.

1.3.2. Congruent figures and equal areas. Parallelograms with same area; triangles with same area.

1.3.3. Euclid's proof of Pythagoras' theorem.

2. Ruler and compass constructions

Reference : Hartshorne : Section 2

2.1. **Legal constructions.**

2.1.1. Given two points, one can draw a line through them.

2.1.2. Given a point and a distance, one can draw a circle with the given point as a center, and radius equal to the distance.

2.1.3. Define new points as intersections of lines and circles drawn using the previous constructions.

2.1.4. Euclid's construction : Given a line segment BC and a point A, drawing a line segment with one end point A, which is congruent to BC.