Orientation class and Poincaré Duality

1. Orientable topological manifolds

For us a manifold is a Hausdorff topological space which is locally Euclidean.

Let $j_{x,A} : H_n(M, M - A; G) \to H_n(M, M - \{x\}; G)$ be the map induced by inclusion.

Definition 1. Let $\Theta_x \otimes G := H_n(M, M - \{x\}; G) \cong G$. For $G = \mathbb{Z}$, we shall denote $\Theta_x \otimes \mathbb{Z}$ by just Θ_x. Let $\Theta \otimes G$ (or just Θ when $G = \mathbb{Z}$), be the disjoint union of $\Theta_x \otimes G$ for $x \in M$. Let $p : \Theta \otimes G \to M$ be the function which takes the elements of $\Theta_x \otimes G$ to $x \in M$. We give a topology on $\Theta \otimes G$ by defining the basic open sets to be $U_\alpha = \left\{ j_{x,\overline{U}}(\alpha) \mid x \in U \right\}$ where U and α vary over $U \subset M$ open and $\alpha \in H_n(M, M - \overline{U}; G)$.

Definition 2. For a closed subset A of M, one defines the group of sections to be

$$\Gamma(A, \Theta \otimes G) := \{ s : A \to \Theta \otimes G \mid s \text{ is continuous, and } p \circ s = 1 \}.$$

Let $\Gamma_c(A, \Theta \otimes G)$ be the group of sections whose supports are compact, i.e., $s(x) = 0$ for x outside a compact set.

Definition 3. Suppose $A \subset M$ is closed. Then M is said to be *orientable* along A, if there exists

$$\vartheta \in \Gamma(A, \Theta) := \{ s : A \to \Theta \mid s \text{ is continuous, and } p \circ s = 1 \}.$$

such that for $x \in A$, $s(x)$ generates Θ_x.

We shall associate a homology class to ϑ. For that we refer to the following theorem.

Theorem 4. *The homomorphism*

$$J_A : H_n(M, M - A; G) \to \Gamma_c(A, \Theta \otimes G)$$
$$J_A(\alpha)(x) = j_{x,A}(\alpha)$$

is an isomorphism for all manifolds M with $A \subset M$ closed.

Now taking $A = M$ and M to be orientable and compact; we see that ϑ will correspond to an element of $[M] \in H_n(M; \mathbb{Z})$.

Theorem 5. *The map $\cap[M] : H^p(K, L; G) \to H_{n-p}(M - L, M - K; G)$ is an isomorphism for compact subsets $L \subset K \subset M$. In particular; by taking $K = M$ and $L = \varnothing$, we get $\cap[M]$ gives us an isomorphism $H^p(M; G) \cong H_{n-p}(M; G)$.*