Cohomology cross product

Recall that we had defined a cross product between homologies. Today, we shall define a cross product between cohomologies. But before we do that, we need to fix a sign convention for the coboundary.

1. Sign convention for coboundary

Recall that the product $\times : \Delta_p(X) \otimes \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$, satisfies the following property

$$\partial(a \times b) = \partial a \times b + (-1)^p a \times \partial b.$$

And this was crucial in the proof that $\times$ induced a cross product on homologies. So to get a reasonable cross product on cohomology, we should have a sign convention for δ such that a similar thing holds for $\times : \Delta^p(X;G) \otimes \Delta^q(Y;G) \to \Delta^{p+q}(X \times Y;G)$; which we still have to define.

However, we consider the cosimplices $\sigma \in \Delta^p(X;G)$, $\sigma : \Delta_p(X) \to G$ to be maps of graded groups $\sigma_* : \Delta_*(X) \to G_*$, where

$$G_i = \begin{cases} G & \text{when } i = p \\ 0 & \text{otherwise.} \end{cases}$$

and $\sigma_p = \sigma$. This says that σ_* is a graded group homomorphism, which is homogeneous of degree $-p$.

Now consider the map of graded groups

$$\lambda : \Delta^*(X;G) \otimes \Delta_*(X) \to G_*$$

given by $\lambda(f \otimes \tau) = f(\tau)$. One way to remember the sign condition is to pretend that this is a homomorphism of chain complexes and formally write down an equation

$$0 = \partial f(\tau) = (\delta f)(\tau) + (-1)^p f(\partial \tau).$$

Definition 1. $(\delta f)(\tau) = (-1)^{p+1} f(\partial \tau)$, where $f \in \Delta^p(X;G)$ and $\tau \in \Delta_p(X)$.

Remark 2. This definition differs from the previous definiton of coboundary by a sign. However, almost everything still follows since the definition of cocycles and coboundaries are not affected by this change of definition. However some chain maps we considered before (like the one in deRham's theorem) will no longer be a chain map, but they will commute up to a sign.

2. The cross product map on cochains

Till now we had been using cohomology with coefficients in a group G. For defining cross product, we need to have coefficients in a ring Λ with a unit 1. Note that, to define cohomology we only use the additive group structure of Λ and the product structure on Λ plays no role there. However we shall use it to define the cross product on chains. We want to define

$$\times : \Delta^p(X, \Lambda) \otimes \Delta^q(X, \Lambda) \to \Delta^{p+q}(X \times Y, \Lambda).$$

In other words, for $f \in \Delta^p(X, \Lambda)$ or $f : \Delta_p(X) \to \Lambda$ and for $g : \Delta_q(Y) \to \Lambda$ we want a map $f \times g : \Delta_{p+q}(X \times Y) \to \Lambda$. For that I define $f \times g$ to be the composition

$$\Delta_{p+q}(X \times Y) \xrightarrow{\Theta} \Delta_p(X) \otimes \Delta_q(Y) \xrightarrow{f \otimes g} \Lambda \otimes \Lambda \xrightarrow[\lambda_1 \otimes \lambda_2 \mapsto \lambda_1 \lambda_2]{m} \Lambda.$$

Note that, this maps depends on the choice of Θ, which if you remember involved a lot of choices. However, any two such choices of Θ are homotopic by a proposition we proved earlier.

We finally would want to say that $\times$ induces a map

$$\times : H^p(X;\Lambda) \otimes H^q(Y;\Lambda) \to H^{p+q}(X \times Y;\Lambda).$$

For that we prove the following lemma.

Lemma 3. *Let $a \in \Delta^p(X,\Lambda)$ and $b \in \Delta^q(Y,\Lambda)$. Then*

$$\delta(a \times b) = \delta a \times b + (-1)^p a \times \delta b.$$

Proof. This follows just be following the definition and using a similary property of the boundary.

$$\begin{aligned}
\delta(a \times b) &= (-1)^{p+q+1}(a \times b) \circ \partial \\
&= (-1)^{p+q+1} m \circ (a \otimes b) \circ \Theta \circ \partial \\
&= (-1)^{p+q+1} m \circ (a \otimes b) \circ \partial^{\Delta_*(X)\otimes\Delta_*(Y)} \circ \Theta \\
&= (-1)^{p+q+1} m \circ (a \otimes b) \circ \partial^{\Delta_*(X)\otimes\Delta_*(Y)} \circ \Theta \\
&= (-1)^{p+q+1} m \circ (a \otimes b) \circ (\partial^X \otimes 1^Y + 1^X \otimes \partial^Y) \circ \Theta \\
&= (-1)^{p+q+1} m \circ \big((-1)^q((a \circ \partial^X) \otimes b) + a \otimes (b \circ \partial^Y)\big) \circ \Theta \\
&= (-1)^{p+q+1} m \circ \big((-1)^q(-1)^{p+1}(\delta a \otimes b) + (-1)^{q+1}(a \otimes \delta b)\big) \circ \Theta \\
&= m \circ (\delta a \otimes b) \circ \Theta + m \circ ((-1)^p a \otimes \delta b) \circ \Theta \\
&= \delta a \times b + (-1)^p a \times \delta b.
\end{aligned}$$

□

Lemma 4. *If $a \in Z^p(X;\Lambda)$, the group of p-cocycles in X and if $b \in Z^q(Y;\Lambda)$, then $a \times b \in Z^{p+q}(X \times Y)$.*

Proof. Immediate. □

Lemma 5. *$a' \in B^p(X;\Lambda)$, $b' \in B^q(Y;\Lambda)$, $a \in Z^p(X;\Lambda)$ and $b \in Z^q(Y;\Lambda)$, then $(a + a') \times (b + b') = a \times b +$ a coboundary.*

Proof. Let $a' = \delta c$ and $b' = \delta d$. Then

$$\begin{aligned}
(a \times \delta c) \times (b \times \delta d) &= a \times b + a \times \delta d + \delta c \times b + \delta c \times \delta d \\
&= a \times b \pm \delta(a \times d) \pm (\delta a \times d) + \delta(c \times b) \pm c \times \delta b + \delta(c \times \delta d) \pm c \times \delta\delta d \\
&= a \times b \pm \delta(a \times d) + \delta(c \times b) + \delta(c \times \delta d)
\end{aligned}$$

which is the result we want. □

Therefore the $\times$ product we defined gives us a product on cohomologies as mentioned in (2).

Now we claim that the induced map on cohomologies is independent of the Θ chosen. But that follows from the fact that any two such Θ and Θ' are homotopic and that would imply that $m \circ (f \otimes g) \circ \Theta \simeq m \circ (f \otimes g) \circ \Theta'$ and therefore the induced map on cohomologies are the same.