## Künneth Formula

In the last lecture, we reduced the computation of homologies of product of two spaces $X$ and $Y$ to computing homologies of $\Delta_*(X) \otimes \Delta_*(Y)$. Künneth formula gives a way(?) to do it.

The algebraic version of the theorem states:

**Theorem 1** (Algebraic Künneth formula)**.** *Let $K_*$ and $L_*$ be free chain complexes. Then there is a natural exact sequence*

$$0 \to (H_*(K_*) \otimes H_*(L_*))_n \xrightarrow{\times} H_n(K_* \otimes L_*) \to \operatorname{Tor}_1\left(H_*(K_*), H_*(L_*)\right)_{n-1} \to 0$$

*which splits, but not naturally.*

Before we can prove the theorem, we fix some algebraic lemmas.

**Lemma 2.** $0 \to Z_* \otimes L_* \to K_* \otimes L_* \to (B_* \otimes L_*)[-1] \to 0$ *is a split exact sequence of chain complexes, where $B_{n-1} = \operatorname{im} \partial_{n-1}$ and $Z_n = \ker \partial_n$.*

*Proof.* Note that for any pair of integers $k$ and $l$ we have an exact sequence

$$0 \to Z_k \otimes L_l \to K_k \otimes L_l \to B_{k-1} \otimes L_l \to 0$$

We also have the following two commutative diagrams for all values of $l$ and $k$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_k \otimes L_l & \longrightarrow & K_k \otimes L_l & \longrightarrow & B_{k-1} \otimes L_l & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0\otimes\mathrm{id}} & & \downarrow{\scriptstyle \partial\otimes\mathrm{id}} & & \downarrow{\scriptstyle 0\otimes\mathrm{id}} & & \\
0 & \longrightarrow & Z_{k-1} \otimes L_l & \longrightarrow & K_{k-1} \otimes L_l & \longrightarrow & B_{k-2} \otimes L_l & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_k \otimes L_l & \longrightarrow & K_k \otimes L_l & \longrightarrow & B_{k-1} \otimes L_l & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes\partial} & & \downarrow{\scriptstyle \mathrm{id}\otimes\partial} & & \downarrow{\scriptstyle \mathrm{id}\otimes\partial} & & \\
0 & \longrightarrow & Z_k \otimes L_{l-1} & \longrightarrow & K_k \otimes L_{l-1} & \longrightarrow & B_{k-1} \otimes L_{l-1} & \longrightarrow & 0
\end{array}$$

Thus we can build a short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} Z_p \otimes L_q \longrightarrow \bigoplus_{p+q=n} K_p \otimes L_q \longrightarrow \bigoplus_{p+q=n-1} B_p \otimes L_q \longrightarrow 0$$

which we rewrite as

$$0 \longrightarrow (Z_* \otimes L_*)_n \longrightarrow (K_* \otimes L_*)_n \longrightarrow (B_* \otimes L_*)_{n-1} \longrightarrow 0.$$

The commutativity of the following diagram will give us the lemma.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (Z_* \otimes L_*)_n & \longrightarrow & (K_* \otimes L_*)_n & \longrightarrow & (B_* \otimes L_*)_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & (Z_* \otimes L_*)_{n-1} & \longrightarrow & (K_* \otimes L_*)_{n-1} & \longrightarrow & (B_* \otimes L_*)_{n-2} & \longrightarrow & 0.
\end{array}$$

**Exercise:** Check that this diagram commutes. □

**Lemma 3.** *Suppose $F_*$ is a complex of free objects (or even projective, or flat) objects with zero differentials. Then $F_* \otimes \_$ is exact.*

*Proof.* We have to show that if $0 \to A_* \xrightarrow{i} B_* \to C_* \to 0$ is a short exact sequence of complexes then $0 \to A_* \otimes F_* \xrightarrow{i \otimes \mathrm{id}} B_* \otimes F_* \to C_* \otimes F_* \to 0$ is also a short exact sequence.

We shall only show injectiveness of $i \otimes \mathrm{id}$ and leave the rest as an exercise.

For the sake of brevity, let us denote $i \otimes \mathrm{id}$ by $\alpha$. Note that we have to show that $\alpha_n : (A_* \otimes F_*)_n \to (B_* \otimes F_*)_n$ is injective. Since $i_p$ maps $A_p$ to $B_p$, we have that $\alpha_n = \oplus \alpha_n^p$ where $\alpha_n^p : A_p \otimes F_{n-p} \to B_p \otimes F_{n-p}$. If we show each $\alpha_n^p = i_p \otimes \mathrm{id}$ is injective. Now $\alpha_n^p(\sum_{r=1}^m a_p^r \otimes f_p^r) = 0$ implies $\sum_{r=1}^m i(a_p^r) \otimes f_p^r = 0$.

Now consider the short exact sequence $0 \to A_p \to B_p \to C_p \to 0$. Since $F_{n-p}$ is free, $0 \to A_p \otimes F_{n-p} \xrightarrow{i_p \otimes \mathrm{id}} B_p \otimes F_{n-p} \to C_p \otimes F_{n-p} \to 0$ is exact too. Therefore, $\sum_{r=1}^m i(a_p^r) \otimes f_p^r = 0$ implies that $\sum_{r=1}^m a_p^r \otimes f_p^r = 0$ proving injectivity of $\alpha_n^p$ and hence of $\alpha$. □

**Lemma 4.** *Suppose $0 \to A_* \to B_* \to C_* \to 0$ is a short exace sequence of complexes. Let $L_*$ be another complex. Then we have a long exact sequence of graded groups*

$$\begin{aligned} \cdots \to \mathrm{Tor}_i\,(A_*,\, L_*) \to \mathrm{Tor}_i\,(B_*,\, L_*) \to \mathrm{Tor}_i\,(C_*,\, L_*) \to \mathrm{Tor}_{i-1}\,(A_*,\, L_*) \to \cdots \\ \to \mathrm{Tor}_1\,(A_*,\, L_*) \to \mathrm{Tor}_1\,(B_*,\, L_*) \to \mathrm{Tor}_1\,(C_*,\, L_*) \to \\ A_* \otimes L_* \to B_* \otimes L_* \to C_* \otimes L_* \to 0 \end{aligned}$$

*where $\otimes$ is the tensor product of complexes and*

$$(\mathrm{Tor}_i\,(M_*,\, N_*))_k = \bigoplus_{p=0}^{k} \mathrm{Tor}_i\,(M_p,\, N_{k-p})$$

*Remark* 5. Note that, in the above sequence we just consider all the entries as graded groups. That is enough for our needs.

*Proof.* For every $k$ and for every $p$, $p \leq k$, we do have a sequence

$$\begin{aligned} \cdots \to \mathrm{Tor}_i\,(A_p,\, L_{k-p}) \to \mathrm{Tor}_i\,(B_p,\, L_{k-p}) \to \mathrm{Tor}_i\,(C_p,\, L_{k-p}) \to \\ \mathrm{Tor}_{i-1}\,(A_p,\, L_{k-p}) \to \cdots \\ \to \mathrm{Tor}_1\,(A_p,\, L_{k-p}) \to \mathrm{Tor}_1\,(B_p,\, L_{k-p}) \to \mathrm{Tor}_1\,(C_p,\, L_{k-p}) \to \\ A_p \otimes L_{k-p} \to B_p \otimes L_{k-p} \to C_p \otimes L_{k-p} \to 0 \end{aligned}$$

and the sequence, which we are looking for, is just a direct sum of these sequences over $p$ for $0 \leq p \leq k$. □

**Lemma 6.** *If $A_*$ is a complex, and if $L_*$ is a complex of free objects with boundaries all 0, then $H_*(A_* \otimes L_*) = H_*(A_*) \otimes L_*$ as graded groups.*

*Proof.* Consider the sequence $0 \to B_*^A \to Z_*^A \to H_*^A \to 0$ where $B_n^A = \mathrm{im}\,\partial_{n+1}^A$, and $Z_n^A = \ker \partial_n^A$. Then we have an exact sequence $0 \to B_*^A \otimes L_* \to Z_*^A \otimes L_* \to H_*^A \otimes L_* \to 0$.

Now $\partial(a_p \otimes l_q) = \partial a_p \otimes l_q + (-1)^p a_p \otimes \partial l_q = \partial a_p \otimes l_q$. Therefore $\mathrm{im}\,\partial_n^{A_* \otimes L_*} \cong ((\mathrm{im}\,\partial^A)_* \otimes L_*)_n = (B_*^A \otimes L_*)_n$. Similarly $\ker \partial_n^{A_* \otimes L_*} = ((\ker \partial^A)_* \otimes L_*)_n = (Z_*^A \otimes L_*)_n$. This, and the short exact sequence in the previous paragraph, immediately give us the lemma. □

*Remark* 7. Before we embark upon the proof of Künneth formula, convince yourself that $H_n(A_*[-1]) \cong H_{n-1}(A_*)$.

*Proof of Künneth formula.* From what we proved we have a short exact sequence

$$0 \to Z_* \otimes L_* \to K_* \otimes L_* \to (B_* \otimes L_*)[-1] \to 0$$

is an exact sequence of chain complexes. Thus we have a long exact sequence of homologies[1]

$$0 \leftarrow H_0((B_* \otimes L_*)[-1]) \leftarrow H_0(K_* \otimes L_*) \leftarrow H_0(Z_* \otimes L_*) \leftarrow H_1(B_* \otimes L_*) \cdots$$

Let us look at the sequence around $H_n(K_* \otimes L_*)$.

$$\cdots \to H_{n+1}(B_* \otimes L_*[-1]) \xrightarrow{\delta_n} H_n(Z_* \otimes L_*) \to H_n(K_* \otimes L_*) \to$$

$$H_n(B_* \otimes L_*[-1]) \xrightarrow{\delta_{n-1}} H_{n-1}(Z_* \otimes L_*) \to \cdots$$

Thus we have a short exact sequence

$$(1) \qquad 0 \to \operatorname{coker} \delta_n \to H_n(K_* \otimes L_*) \to \ker \delta_{n-1} \to 0.$$

Note that by the lemmas above we have $H_*(B_* \otimes L_*) \cong B_* \otimes H_*(L_*)$ and $H_*(Z_* \otimes L_*) \cong Z_* \otimes H_*(L_*)$. The short exact sequence of graded groups, or of complexes with differential 0,

$$0 \to B_* \to Z_* \to H_*(K_*) \to 0$$

induces the long exact Tor sequence (with $H_*(L_*)$ and 0 differentials)

$$\operatorname{Tor}_1(Z_*, H_*(L_*)) \to \operatorname{Tor}_1(H_*(K_*), H_*(L_*)) \to B_* \otimes H_*(L_*)$$

$$\xrightarrow{\delta_*} Z_* \otimes H_*(L_*) \to H_*(K_*) \otimes H_*(L_*) \to 0$$

From this, if we read the $n$-th component, we get

$$0 \overset{\text{Why?}}{=} \operatorname{Tor}_1(Z_*, H_*(L_*))_n \to \operatorname{Tor}_1(H_*(K_*), H_*(L_*))_n \to (B_* \otimes H_*(L_*))_n \xrightarrow{\delta_n}$$

$$(Z_* \otimes H_*(L_*))_n \to (H_*(K_*) \otimes H_*(L_*))_n \to 0$$

This immediately gives $\ker \delta_{n-1} = \operatorname{Tor}_1(Z_*, H_*(L_*))_{n-1}$ and also $\operatorname{coker} \delta_n = (H_*(K_*) \otimes H_*(L_*))_n$; plugging this in (1) leads us to the Künneth formula:

$$0 \to (H_*(K_*) \otimes H_*(L_*))_n \xrightarrow{\times} H_n(K_* \otimes L_*) \to \operatorname{Tor}_1(H_*(K_*), H_*(L_*))_{n-1} \to 0.$$

□

Now let $X$ and $Y$ be topological spaces. Then setting

$$K_* = \Delta_*(X) \qquad L_* = \Delta_*(Y)$$

gives us the following theorem.

**Theorem 8** (Künneth)**.** *For two spaces $X$ and $Y$, we have a short exact sequence*

$$0 \to (H_*(X) \otimes H_*(Y))_n \to H_n(X \times Y) \to \operatorname{Tor}_1(H_*(X), H_*(Y))_{n-1} \to 0.$$

*Remark* 9. This is an important remark. Though I haven't proved it, **both the short exact sequences above are split, though not canonically**. So for the sake of computation, we can write the homology of the product, as the direct sum of the groups on its either side in the short exact sequences.

[1]The first term in the following is $H_{-1}(B_* \otimes L_*) = 0$ This just reflects the fact that 0-th cohomology is nothing but the kernel of the boundary map.