Eilenberg Zilber and Kunneth

1. Till now...

Recall that we defined a $\times$-product which takes a pair consisting of a p simplex and a q simplex and gives a $p+q$ simplex. Then we went ahead to prove that this induces a cross product

$$\times : H_p(X,A) \times H_q(Y,B) \to H_{p+q}((X,A)\times(Y,B))$$

where $(X,A)\times(Y,B) := (X\times Y, X\times B \cup A\times Y)$.

Furthermore, we also proved that if X and Y are contractible, then[1] $\Delta_*(X)\otimes\Delta_*(Y)$ is chain contractible[2].

We shall prove that there is a natural map in the other direction too.

2. Eilenberg-Zilber theorem

Theorem 1. *There exists a natural map* $\Theta : \Delta_*(X\times Y) \to \Delta_*(X)\otimes\Delta_*(Y)$*, which in degree* 0 *is* $(x,y)\mapsto x\otimes y$.

The proof is again by using acyclic models. For that we need the Θ to be natural in the following sense. If $f : X\to X'$ and $g : Y \to Y'$ are two maps, then

$$\Theta\circ(f\times g)_\Delta = (f_\Delta\otimes g_\Delta)\circ\Theta$$

where $\otimes$ in $f_\Delta\otimes g_\Delta$ is the tensor of homogenous morphims between graded groups, which repects signs conventions. As before, for the inductive definition to work, we need to choose how Θ behaves with the boundary operator. We assume that,

$$\Theta\circ\partial = \partial\circ\Theta.$$

where the boundary on $\Delta_*(X)\otimes\Delta_*(Y)$ is given by $\partial\otimes\mathrm{id}+\mathrm{id}\otimes\partial$.

Proof. For us the minimal models are the simplices Δ_p and the role of $\iota_p\times\iota_q$ is played by the diagonal $d_p : \Delta_p \to \Delta_p\times\Delta_p$.

We want to define $\Theta : \Delta_k(X\times Y)\to\Delta_k(X)\times\Delta_k(Y)$. The map is already given for $k=0$. Suppose we already have constructed the map till $k=l$. For $k=l+1$, we first construct Θ for $X=Y=\Delta_k$. Note that, $\partial d_k\in\Delta_l(\Delta_k\times\Delta_k)$ and hence $\Theta(\partial(d_k))$ is known by assumption. Now, we shall show that $\Theta(\partial(d_k))$ is itself a boundary:

$$\partial(\Theta(\partial(d_k))) = \Theta(\partial\circ\partial(d_k)) = \Theta(0) = 0.$$

But note that Δ_k is contractible, and hence by the lemma proved last time, $\Delta_*(\Delta_k\times\Delta_k)$ is chain contractible. In particular, $\ker(\partial_k)/\operatorname{im}(\partial_{k-1}) = 0$. Thus, $\Theta(\partial(d_k))$ is itself a boundary. *Choose* $\Theta(d_k)$ to be such that $\partial(\Theta(d_k)) = \Theta(\partial(d_k))$.

Now we extend the definition of Θ be functoriality. Let X and Y be arbitrary topological spaces. Let $\sigma : \Delta_k\to X\times Y$ be a simplex. As before, we will write σ as $f_\Delta d_k$ for some $f : \Delta_k\times\Delta_k \to X\times Y$ to use naturality. With this aim, let $\pi_X : X\times Y\to X$ and $\pi_Y : X\times Y \to Y$ be two projections. Observe that $\sigma = ((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma))\circ d_k$. Define $f = ((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma)) : \Delta_k\times\Delta_k\to X\times Y$. Therefore, $\sigma = f_\Delta d_k$.

Define $\Theta(\sigma) = ((\pi_X\circ\sigma)_\Delta\otimes(\pi_Y\circ\sigma)_\Delta)\circ\Theta(d_k)$.

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[1] with the tensor being that of graded groups, involving signs

[2] in the sense that the identity map is chain homotopic to the augmentation map, which is the usual augmentation map in degree 0 and 0 in other degrees

To check naturality, note that if $f : X \to X'$ and $g : Y \to Y'$ are two maps, then

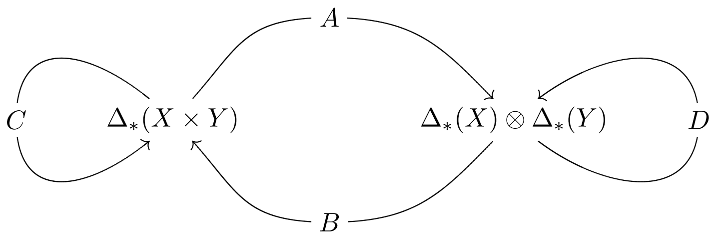
$$\begin{aligned}\Theta((f \times g)_\Delta \sigma) &= ((\pi_{X'} \circ (f \times g)_\Delta \sigma)_\Delta \otimes (\pi_{Y'} \circ (f \times g)_\Delta \sigma)_\Delta) \circ \Theta(d_k) \\ &= ((\pi_{X'} \circ (f \times g) \circ \sigma)_\Delta \otimes (\pi_{Y'} \circ (f \times g) \circ \sigma)_\Delta) \circ \Theta(d_k) \\ &= ((f \circ \pi_X \circ \sigma)_\Delta \otimes (g \circ \pi_Y \circ \sigma)_\Delta) \circ \Theta(d_k) \\ &= \Big((f_\Delta \circ (\pi_X \circ \sigma)_\Delta) \otimes (g_\Delta \circ (\pi_Y \circ \sigma)_\Delta) \Big) \circ \Theta(d_k) \\ &= ((f_\Delta \otimes g_\Delta) \circ ((\pi_X \circ \sigma)_\Delta \otimes (\pi_Y \circ \sigma)_\Delta) \circ \Theta(d_k) \\ &= (f_\Delta \otimes g_\Delta) \circ \Theta(\sigma).\end{aligned}$$

In the above computation the sign did not appear in the second last step because g_Δ has degree 0.

Reading Exercise: Prove that $\partial \circ \Theta = \Theta \circ \partial$. □

Next step is to prove that $\times$ and Θ are homotopy inverses of each other. This is the last crucial step for Eilenberg-Zilber theorem.

Proposition 2. *Consider the diagram*



In the diagram, the letters A, B, C and D denote sets of maps from the chain complex at the source of the arrow to the chain complex at the tip. For example, A is the set of chain complex homomorphisms from $\Delta_*(X \times Y)$ to $\Delta_*(X) \otimes \Delta_*(Y)$. Suppose φ and ψ are chain complex homomrphisms belonging to any one of the above four classes, such that they are canonical isomorphims at degree 0. *Then the two maps are chain homotopic.*

Bredon proves the proposition for φ and ψ belonging to A. I shall prove it for φ and ψ in B, and leave the rest as an exercise.

Proof. The proof is again by method of acyclic models.

Optional reading exercise : Maybe you can dig up acyclic model theorem and read its proof. One reference I know is Spanier's algebraic topology.

In our case the models are $\iota_p \otimes \iota_q \in \Delta_*(\Delta_p) \otimes \Delta_*(\Delta_q)$ and $d_k \in \Delta_*(\Delta_k \times \Delta_k)$. We are given $\varphi, \psi : \Delta_*(X) \otimes \Delta_*(Y) \to \Delta_*(X \times Y)$, and we on degree 0, both these maps are canonical isomorphisms. We have to show that they are chain homotopic. In other words, we have to construct a $D : \big(\Delta_*(X) \otimes \Delta_*(Y)\big)_k \to \big(\Delta_*(X \times Y)\big)_{k+1}$ such that $\varphi - \psi = \partial \circ D + D \circ \partial$.

Let $(f, g) : X \times Y \to X' \times Y'$ be the product of the maps $f : X \to X'$ and $g : Y \to Y'$. The naturality condition we are looking for is $D \circ (f_\Delta \otimes g_\Delta) = (f, g)_\Delta \circ D$.

For $k = 0$, we define $D = 0$, and that is fine as $\varphi_0 - \psi_0 = 0$. Now assume that we have defined D for $k = 0, \ldots, l-1$. We need to define D for $k = l$. As before, we first define D on the models.

Note that $\big(\Delta_*(X)\otimes\Delta_*(Y)\big)_k = \oplus_{p+q=k}\Delta_p(X)\otimes\Delta_q(Y)$. Now take $X=\Delta_p$, $Y=\Delta_q$, p, q such that $p+q=k$. First we define $D(\iota_p\otimes\iota_q)$. For that, note

$$\begin{aligned}\partial(\varphi-\psi-D\circ\partial)(\iota_p\otimes\iota_q) &= \partial\varphi(\iota_p\otimes\iota_q)-\partial\psi(\iota_p\otimes\iota_q)-\partial D\partial(\iota_p\otimes\iota_q)\\ &= \varphi\partial(\iota_p\otimes\iota_q)-\psi\partial(\iota_p\otimes\iota_q)-\partial D(\partial\iota_p\otimes\iota_q+(-1)^p\iota_p\otimes\partial\iota_q)\\ &= \varphi(\partial\iota_p\otimes\iota_q+(-1)^p\iota_p\otimes\partial\iota_q)-\psi(\partial\iota_p\otimes\iota_q+(-1)^p\iota_p\otimes\partial\iota_q)-\\ &\qquad(\partial\circ D)(\partial\iota_p\otimes\iota_q+(-1)^p\iota_p\otimes\partial\iota_q)\\ &= (\varphi-\psi-\partial\circ D)(\partial\iota_p\otimes\iota_q)+(-1)^p\Big((\varphi-\psi-\partial\circ D)(\iota_p\otimes\partial\iota_q)\Big)\\ &= D\partial(\partial\iota_p\otimes\iota_q)+(-1)^pD\partial(\iota_p\otimes\partial\iota_q)\\ &= D(\partial\partial\iota_p\otimes\iota_q)+(-1)^{p-1}D(\partial\iota_p\otimes\partial\iota_q)+(-1)^pD(\partial\iota_p\otimes\partial\iota_q)+\\ &\qquad(-1)^{2p}D(\iota_p\otimes\partial\partial\iota_q)\\ &= 0.\end{aligned}$$

We use the induction hypothesis for the fourth step in the above computation. Therefore, $(\varphi-\psi-D\circ\partial)(\iota_p\otimes\iota_q)$ is a cycle, and since $\Delta_p\times\Delta_q$ is contractible, it is a boundary. *Choose* an element $D(\iota_p\otimes\iota_q)$ of $\Delta_{k+1}(\Delta_q\times\Delta_q)$ such that $\partial D(\iota_p\otimes\iota_q)=(\varphi-\psi-D\circ\partial)(\iota_p\otimes\iota_q)$.

Now for an arbitrary $\sigma\otimes\tau\in\Delta_p(X)\otimes\Delta_q(Y)$, note that $(\sigma_\Delta\otimes\tau_\Delta)(\iota_p\otimes\iota_q)=\sigma\otimes\tau$. Define $D(\sigma\otimes\tau)=(\sigma,\tau)_\Delta(D(\iota_p\times\iota_q))$ and extend linearly.

Exercise : Check that this D gives the required chain homotopy □

Corollary 3 (Eilenberg-Zilber theorem)**.** *$\times$ and Θ are homotopy equivalences between $\Delta_*(X)\otimes\Delta_*(Y)$ and $\Delta_*(X\times Y)$. Therefore,*

$$\begin{aligned}H_p(X\times Y)&\cong H_p(\Delta_*(X)\otimes\Delta_*(Y)),\\ H_p(X\times Y;G)&\cong H_p(\Delta_*(X)\otimes\Delta_*(Y)\otimes G),\\ H^p(X\times Y;G)&\cong H_p(\hom(\Delta_*(X)\otimes\Delta_*(Y),G).\end{aligned}$$