Degree of a self map of S^n

Degree of a map can be defined in a variety of ways. One way is to use differential topology and do it as in Milnor's "Topology from a differentiable viewpoint". The other way to define it is using homology. The second way leads to a definition, but hardly provides any intuition.

1. Definition via homology

Consider a map $f : S^n \to S^n$. It induces a group homomorphism $f_* : \widetilde{H}^n(S^n) \to \widetilde{H}^n(S^n)$. Recall that $\widetilde{H}^n(S^n) \cong \mathbb{Z}$. This means that f_* is a group homomorphism from $\mathbb{Z}$ to itself.

Definition 1. *Degree* of a map $f : S^n \to S^n$ is defined to be $f_*(1)$ where 1 is the generator of $\widetilde{H}^n(S^n)$.

For S^n it is as simple as that. All the extra complications arise if you want to define a notion of degree for maps between two n dimensional manifolds M and N. We shall come to this more general situation later.

First let us study some properties related to the above definition. In the following f is a map from S^n to S^n.

(1) $\deg(\mathrm{id}_{S^n}) = 1$.

(2) *If f is not surjective, then* $\deg(f) = 0$. To see this, if $x \notin \operatorname{im} f$, note that f factors as

$$S^n \xrightarrow{f'} S^n \setminus \{x\} \xrightarrow{i} S^n$$

and hence f_* factors as

$$\widetilde{H}^n(S^n) \xrightarrow{f'_*} \widetilde{H}^n(S^n \setminus \{x\}) \xrightarrow{i} \widetilde{H}^n(S^n)$$

and hence f_* is zero as $\widetilde{H}^n(S^n \setminus \{x\}) = 0$.

(3) *If f and g are homotopic, then* $\deg(f) = \deg(g)$. This follows from the fact that $f \simeq g$ implies that $f_* = g_*$. Therefore, the degrees are also the same.

(4) $\deg(f \circ g) = \deg(f) \deg(g)$ This follows from the fact that $(f \circ g)_* = f_* \circ g_*$.

(5) *If f is a reflection of S^n, then* $\deg(f) = -1$. This is a bit more subtle. It follows easily if you know simplicial homology is the same as singular homology. In that case one can find a generator for $\widetilde{H}^n(S^n)$ of the form $\Delta_1^n - \Delta_2^n$, which the reflection sends to $\Delta_2^n - \Delta_1^n$.

(6) *The degree of the antipodal map is* $(-1)^{n+1}$. The antipodal map is a composition of $(n+1)$ reflections.

(7) *If f has no fixed points, then* $\deg f = (-1)^{n+1}$. In this case the antipodal map is homotopic to f. To see this, consider the homotopy

$$F(x,t) = \frac{t(-x) + (1-t)f(x)}{\|t(-x) + (1-t)f(x)\|}.$$

2. Computation of degree

Consider a map $f : S^n \to S^n$.

Assumption: There exists a point $y \in S^n$ such that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_m\}$.

Choose neighbourhoods $U_1, \ldots, U_m$ of $x_1, \ldots, x_m$ respectively so that they are pairwise disjoint and such that f maps them into a neighbourhood V of y. Then $f(U_i \setminus x_i) \subset V \setminus y$.

Now, we have a commutative diagram

$$\begin{array}{ccccc}
 & & H_n(U_i, U_i \setminus x_i) & \xrightarrow{(f|_{x_i})_*} & H_n(V, V \setminus y) \\
 & \overset{\cong}{\nearrow} & \Big\downarrow k_i & & \Big\downarrow \cong \\
H_n(S^n, S^n \setminus x_i) & \xleftarrow[p_i]{} & H_n(S^n, S^n \setminus f^{-1}(y)) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus y) \\
 & \underset{\cong}{\searrow} & \Big\uparrow j & & \Big\uparrow \cong \\
 & & H_n(S^n) & \xrightarrow[f_*]{} & H_n(S^n)
\end{array}$$

Isomorphisms in the top half of the diagram come from excision and the isomorphisms in the bottom half come from the long exact sequence of pairs. From this we conclude that $H_n(U_i, U_i \setminus x_i) \cong \mathbb{Z}$ and $H_n(V, V \setminus y) \cong \mathbb{Z}$.

Definition 2. The *local degree of f at x_i*, denoted by $\deg(f|_{x_i})$ is defined to be $(f|_{x_i})_*(1) \in \mathbb{Z}$ under the above identification.

Remark 3. This notion of local degree is particularly useful when f maps each U_i homeomorphically onto V. In that case it is easy to see that $(f|_{x_i})_*$ is an isomorphism of $\mathbb{Z}$ and hence the local degree of f at x_i is just ± 1.

This case happens, for example, when f is differentiable. Then Sard's theorem says that the set $\{f(x) \mid Df_x \text{ is not invertible}\}$ has measure zero, and then one can ensure the above situation holds by inverse function theorem.

Proposition 4. $\deg f = \sum_{i=1}^m \deg f|_{x_i}$.

Proof. $H_n(S^n, S^n \setminus f^{-1}(y)) = H_n(\sqcup U_i, \sqcup(U_i \setminus x_i)) = \oplus_{i=1}^m H_n(U_i, U_i \setminus x_i)$, by excision. Therefore, $H_n(S^n, S^n \setminus f^{-1}(y)) \cong \mathbb{Z}^{\oplus m}$, with k_i being the inclusion of the direct summands and p_i the projection onto the direct summands. Now $p_i \circ j(1) = 1$ and therefore, $j(1) = \sum_i k_i(1)$. Now $f_*(j(1)) = f_*(1) = \deg f$. On the other hand $f_* k_i(1) = (f|_{x_i})_*(1) = \deg f|_{x_i}$. Therefore $j(1) = \sum_i k_i(1)$ implies the proposition. □

3. Construction of maps of degree k

Let $B_1, \dots, B_k$ be pairwise disjoint closed balls in S^n. Let $C = S^n \setminus (\cup_i B_i)$. Let q be the quotient map from S^n to the space obtained by mapping C to a point. Note that the quotient is nothing but $\vee_k S^n$; $q : S^n \to \vee_k S^n$. Now identify all these spheres to a single sphere: $p : \vee_k S^n \to S^n$, such that each p_i, which is p restricted to the i-th spheres in $\vee_k S^n$ is a homeomorphism. Now this means that $\deg p_i = \pm 1$. If $\deg p_i = -1$ for some i replace p_i by $\rho_i \circ p_i$ where ρ_i is a reflection on the i-th sphere. The modified p will have give us a map $f_k := p \circ q : S^n \to S^n$.

Now for any y in S^n, $f^{-1}(y) = x_1, \dots, x_k$ where $x_i \in B_i$, $i = 1, \dots, k$. Now by construction, $\deg f_k|_{x_i} = 1$ (Why?). Therefore $\deg f_k = \sum_{i=1}^k \deg f_k|_{x_i} = k$.

4. Some comments

Here are some facts, which I won't prove.

(1) If $f : S^k \to S^k$ is a map of degree d, then the map $Sf : S^{k+1} \to S^{k+1}$ between their suspensions also have degree d.
(2) If f is a complex polynomial of degree d then the induced map on the Riemann sphere S^2 is of degree d.